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The nonexistence of robust codes for subsets of ω_1

by

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Abstract. Several results are presented concerning the existence or nonexistence, for a subset S of ω_1, of a real r which works as a robust code for S with respect to a given sequence $\langle S_\alpha : \alpha < \omega_1\rangle$ of pairwise disjoint stationary subsets of ω_1, where "robustness" of r as a code may either mean that $S \in L[r, \langle S^*_\alpha : \alpha < \omega_1\rangle]$ whenever each S^*_α is equal to S_α modulo nonstationary changes, or may have the weaker meaning that $S \in L[r, \langle S_\alpha \cap C : \alpha < \omega_1\rangle]$ for every club $C \subseteq \omega_1$. Variants of the above theme are also considered which result when the requirement that S gets exactly coded is replaced by the weaker requirement that some set is coded which is equal to S up to a club, and when sequences of stationary sets are replaced by decoding devices possibly carrying more information (like functions from ω_1 into ω_1).

1. Introduction. To this day, several ways have been found to code a given subset of ω_1 in the presence of forcing axioms. All of the codings I am thinking of here are of the following form: One picks some object $\mathcal{D}$ in H_{ω_2} ($\mathcal{D}$ will be the *decoding device*) in such a way that given any $A \subseteq \omega_1$ there is some $\delta < \omega_2$ and some ω_1-club $(X_i)_{i<\omega_1}$ of $[\delta]^{\aleph_0}$ (or perhaps some real r) such that $A \in M$ whenever M is an inner model such that $\{\mathcal{D}, (X_i)_{i<\omega_1}\} \subseteq M$ (or $\{\mathcal{D}, r\} \subseteq M$). This is illustrated by the following list.

THEOREM 1.1. *Let A be a subset of ω_1.*

(1) [J-So] *If* MA_{ω_1} *holds, then for every sequence $\langle r_\alpha : \alpha < \omega_1\rangle$ of pairwise almost disjoint infinite sets of integers there is a set of integers r such that*

$$\alpha \in A \text{ if and only if } r \cap r_\alpha \text{ is infinite}$$

for all $\alpha < \omega_1$.

(2) ([W, Theorem 5.14]) *If Martin's Maximum* (MM) *holds, then for every stationary and co-stationary $T \subseteq \omega_1$ and every sequence $\langle S_\alpha : \alpha < \omega_1\rangle$ of pairwise disjoint stationary subsets of ω_1 there is an ordinal $\delta < \omega_2$ and*

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an $\subseteq$-increasing and $\subseteq$-continuous sequence $(X_\nu)_{\nu<\omega_1}$ of countable subsets of δ such that $\bigcup_\nu X_\nu = \delta$ and such that ([1])

$$\mathrm{ot}(X_\nu) \in T \textit{ for every } \nu \in S_\alpha \textit{ whenever } \alpha \in A$$

and

$$\mathrm{ot}(X_\nu) \notin T \textit{ for every } \nu \in S_\alpha \textit{ whenever } \alpha \notin A.$$

(3) ([W, Theorem 5.9]) *If* MM *holds, then for all sequences $\langle S_\alpha : \alpha<\omega_1\rangle$ and $\langle T_\alpha : \alpha < \omega_1\rangle$ of pairwise disjoint stationary subsets of ω_1 there is an ordinal $\delta < \omega_2$ of cofinality ω_1 and there is a strictly increasing and continuous $f:\omega_1\to\delta$ cofinal in δ such that $f``T_\alpha \subseteq \widetilde{S}_{h(\alpha)}$ for all $\alpha<\omega_1$, where $h:\omega_1\to\omega_1$ is the enumerating function of A, and $\widetilde{B}$ is, given a set $B\subseteq\omega_1$, the set of all γ between ω_1 and ω_2 such that, given any surjection $\pi:\omega_1\to\gamma$, there is a club $C\subseteq\omega_1$ such that $\mathrm{ot}(\pi``\nu)\in B$ for all $\nu\in C$ (equivalently, such that $\Vdash_{\mathcal{P}(\omega_1)/\mathrm{NS}_{\omega_1}} \gamma \in j(B)$, where NS_{ω_1} is the nonstationary ideal on ω_1 and j denotes the generic elementary embedding obtained from forcing with $\mathcal{P}(\omega_1)/\mathrm{NS}_{\omega_1}$* ([2])*).*

(4) ([T]) *If Bounded Martin's Maximum* (BMM) *holds, then for every sequence $\langle r_\alpha : \alpha<\omega_1\rangle$ of pairwise distinct members of 2^ω and every sequence $\langle S_\alpha : \alpha<\omega_1\rangle$ of pairwise disjoint stationary subsets of ω_1 there are ordinals $\beta<\gamma<\delta<\omega_2$ and an $\subseteq$-increasing and $\subseteq$-continuous sequence $(X_\nu)_{\nu<\omega_1}$ of countable subsets of δ such that $\bigcup_\nu X_\nu=\delta$ and*

(a) $\max \Delta(r_{\mathrm{ot}(X_\nu\cap\beta)}, r_{\mathrm{ot}(X_\nu\cap\gamma)}, r_{\mathrm{ot}(X_\nu)}) = \Delta(r_{\mathrm{ot}(X_\nu\cap\beta)}, r_{\mathrm{ot}(X_\nu\cap\gamma)})$ *for all $\nu\in S_\alpha$ whenever $\alpha\in A$,*

(b) $\min \Delta(r_{\mathrm{ot}(X_\nu\cap\beta)}, r_{\mathrm{ot}(X_\nu\cap\gamma)}, r_{\mathrm{ot}(X_\nu)}) = \Delta(r_{\mathrm{ot}(X_\nu\cap\beta)}, r_{\mathrm{ot}(X_\nu\cap\gamma)})$ *for all $\nu\in S_\alpha$ whenever $\alpha\notin A$,*

where $\Delta(r_0,r_1)=\min\{m : r_0(m)\neq r_1(m)\}$ for all distinct reals r_0, r_1, and $\Delta(r_0,r_1,r_2)$ is the two-element set $\{\Delta(r_0,r_1),\Delta(r_0,r_2),\Delta(r_1,r_2)\}$ for all distinct reals $r_0,r_1,r_2\in 2^\omega$.

(5) ([M]) *If the Bounded Proper Forcing Axiom holds, then given any ladder system $\langle C_\xi : \xi\in\mathrm{Lim}(\omega_1)\rangle$* ([3]) *and any sequence $\langle S_\alpha : \alpha<\omega_1\rangle$ of pairwise disjoint stationary subsets of ω_1 there is an ordinal $\delta<\omega_2$ and an $\subseteq$-increasing and $\subseteq$-continuous sequence $(X_\nu)_{\nu<\omega_1}$ of countable subsets of δ such that $\bigcup_\nu X_\nu=\delta$ and such that for every limit $\nu<\omega_1$ there is $\nu_0<\nu$*

([1]) Given a set of ordinals X, $\mathrm{ot}(X)$ will denote the order type of X.

([2]) Given an ordinal $\gamma<\omega_2$ and a surjection $\pi:\omega_1\to\gamma$, the function on ω_1 sending every ν to $\mathrm{ot}(\pi``\nu)$ represents, in the generic ultrapower derived from forcing with $\mathcal{P}(\omega_1)/\mathrm{NS}_{\omega_1}$, an ordinal of order type γ. We call such a function a *canonical function* for γ.

([3]) That is, each C_ξ is a subset of ξ of order type ω and with supremum ξ.

such that for all ξ, $\nu_0 < \xi < \nu$,

$$X_\nu \cap \omega_1 \in \bigcup_{\alpha \in A} S_\alpha \text{ if and only if } w(X_\xi \cap \omega_1, X_\nu \cap \omega_1) < w(X_\xi, X_\nu)$$

where, given two sets $X \subseteq Y$ *of countable ordinals,* $w(X, Y)$ *denotes the cardinality of* $\sup(X) \cap \pi_Y^{-1}``C_{\mathrm{ot}(Y)}$ *(and* π_Y *is the transitive collapse of* Y*).*

A well known theorem of H. Woodin ([W, Theorems 3.16, 3.17]) says that if NS_{ω_1} is saturated and $\mathcal{P}(\omega_1)^\sharp$ exists, then every club of ω_1 contains a club which is constructible from a real, and thus the second uniform indiscernible u_2 is ω_2 (and, as a consequence, for every ordinal $\delta < \omega_2$ there is a real r such that $|\delta|^{L[r]}$ is at most the ω_1 of V). On the other hand, by a result of R. Schindler [S], the existence of the sharp of every set of ordinals follows already from BMM. Hence, (1), (4) and (5) from Theorem 1.1 are instances of the following fact (since the relevant extra decoding objects, like the fixed ladder system in (5), can certainly be coded by a stationary and co-stationary subset of ω_1):

- If MM holds (more generally, if BMM holds and NS_{ω_1} is saturated), then there is a sequence $\langle S_\alpha : \alpha < \omega_1 \rangle$ of pairwise disjoint stationary subsets of ω_1 such that given any subset A of ω_1 there is a real r with $A \in L[r, \langle S_\alpha : \alpha < \omega_1 \rangle]$.

Likewise, (2) and (3) from Theorem 1.1 take obviously the following stronger general form:

- If MM holds ([4]), then given any sequence $\langle S_\alpha : \alpha < \omega_1 \rangle$ of pairwise disjoint stationary subsets of ω_1 and any subset A of ω_1 there is a real r such that $A \in L[r, \langle S_\alpha : \alpha < \omega_1 \rangle]$.

Thus, sufficiently strong forcing axioms imply the existence, given an arbitrary subset A of ω_1 and an arbitrary sequence of length ω_1 of mutually disjoint stationary subsets of ω_1, of a real which works as a code for A with respect to our sequence of stationary sets. This is also true, by the same considerations, about Woodin's $\mathbb{P}_{\max}$ axiom $(*)$ ([5]): Both (2) and (3) in Theorem 1.1 are true under $(*)$ (by [W], Lemma 5.18, and [W, Lemma 5.5], respectively), and not in a vacuous way (that is, $(*)$ does imply the existence of ω_1-sequences of mutually disjoint stationary subsets of ω_1 ([6])), and so is the fact that every club of ω_1 contains a club constructible from a real ([7]).

([4]) For (2) already BMM plus the saturation of NS_{ω_1} suffice.

([5]) Namely that AD holds in $L(\mathbb{R})$ and that $L(\mathcal{P}(\omega_1))$ is a $\mathbb{P}_{\max}$ extension of $L(\mathbb{R})$ (for the definition of $\mathbb{P}_{\max}$ forcing and its basic theory the reader is referred to Chapter 4 of [W]); $\mathbb{P}_{\max}$ is a partial order in $L(\mathbb{R})$, which, under $\mathrm{AD}^{L(\mathbb{R})}$, forces over $L(\mathbb{R})$ the Axiom of Choice and that $(*)$ holds (by [W, Lemma 4.38, Theorem 4.50 and Lemma 5.5]).

([6]) By [W, Theorem 4.50].

([7]) Again by [W, Theorem 4.50].

Although in this paper I will focus mainly on decoding devices consisting of an ω_1-sequence of mutually disjoint stationary subsets of ω_1, it is worth remarking at this point that the simplest possible decoding device one may think of—namely a single subset of ω_1 which is both stationary and co-stationary—does do its job under strong enough forcing axioms or under $(*)$:

Observation 1.1. *Suppose that* BMM *holds and* NS_{ω_1} *is saturated, or else that the* $\mathbb{P}_{\max}$ *axiom* $(*)$ *holds. Then, given any stationary and co-stationary* $S \subseteq \omega_1$ *and any* $A \subseteq \omega_1$ *there is a real* r *such that* $A \in L[r, S]$.

Proof. Under the first hypothesis, this is a consequence of the following corollary to a result due, independently, to P. Larson and H. Woodin (see the proof of Theorem 5.14 in [W]):

Fact 1.1. *If* NS_{ω_1} *is saturated, then the set of* $X \in [\omega_2]^{\aleph_0}$ *such that* $X \cap \omega_1 \in S$ *and such that* $\mathrm{ot}(X) \in T$ *if and only if* $X \cap \omega_1 \in A$ *is a stationary subset of* $[\omega_2]^{\aleph_0}$ *whenever* S, T *and* $\omega_1 \setminus T$ *are stationary subsets of* ω_1 *and* A *is a subset of* ω_1.

Fix a sequence $\langle r_\alpha : \alpha < \omega_1\rangle$, where each r_α is a set of integers coding $\chi_A \restriction \alpha$ and $\chi_A : \omega_1 \to \{0, 1\}$ is the characteristic function of A. Fix also a partition $\langle S_\alpha : \alpha < \omega_1\rangle$ of ω_1 into stationary sets. Pick any $n < \omega$. By Fact 1.1, the standard forcing notion $\mathcal{P}$ for shooting, with countable conditions, an ω_1-club inside $\{X \in [\omega_2]^{\aleph_0} : \mathrm{ot}(X) \in S \text{ iff } X \cap \omega_1 \in \bigcup\{S_\alpha : \alpha < \omega_1, n \in r_\alpha\}\}$ (8) preserves stationary subsets of ω_1. Hence, by BMM there is some $\alpha_n < \omega_2$ such that, given any surjection $\pi : \omega_1 \to \alpha_n$, there is a club $C \subseteq \omega_1$ such that

$$\{\nu \in C : \mathrm{ot}(\pi``\nu) \in S\} = C \cap \bigcup\{S_\alpha : \alpha < \omega_1,\, n \in r_\alpha\}.$$

Since NS_{ω_1} is saturated and the sharp of every set exists, every club of ω_1 contains a club constructible from a real and u_2 is ω_2. Therefore we may find a real r coding the sequence $(\alpha_n)_n$ as well as clubs witnessing the above equality for surjections from ω_1^V to the α_n's which are constructible from reals. Thus, we may assume that, for some club C of ω_1 in $L[r, S]$, the sequence indexed by $n < \omega$ of the pairs

$$\Big\langle \bigcup\{S_\alpha \cap C : \alpha < \omega_1,\, n \in r_\alpha\},\, \bigcup\{S_\alpha \cap C : \alpha < \omega_1,\, n \notin r_\alpha\}\Big\rangle$$

belongs to $L[r, S]$. Now pick any $\xi \in C$ and let s_ξ be the set of integers n such that ξ is in the first component of the nth member of the above ω-sequence, that is, such that ξ is in $\bigcup\{S_\alpha \cap C : \alpha < \omega_1,\, n \in r_\alpha\}$. Then $\langle s_\xi : \xi \in C\rangle$ is clearly constructible from r together with S and, given $\xi \in C$ and $n < \omega$, we have $n \in s_\xi$ exactly when $n \in r_\alpha$, where $\alpha < \omega_1$ is unique

(8) $\mathcal{P}$ consists of initial segments of such a club of length a successor countable ordinal ordered by extension.

such that $\xi \in S_\alpha$. It follows that the unordered set $Y = \{r_\alpha : \alpha < \omega_1\}$, being equal to $\{s_\xi : \xi \in C\}$, belongs to $L[r, S]$. Finally, consider the map on Y sending every s_ξ to the unique ordinal α such that s_ξ codes a function from α into $\{0, 1\}$. This function, which can be read off from Y, sends s_ξ to α exactly when ξ is in S_α. Therefore the sequence $\langle r_\alpha : \alpha < \omega_1\rangle$ belongs to $L[r, S]$, and so in particular does A.

Under the assumption that $(*)$ holds this is proved exactly in the same way using Woodin's results on $\mathbb{P}_{\max}$ mentioned before. ■

All codings expressed in Theorem 1.1 may be sensitive to nonstationary changes in the decoding device $\langle S_\alpha : \alpha < \omega_1\rangle$. In (2), for instance, the fact that $S_\alpha \Vdash_{\mathcal{P}(\omega_1)/\mathrm{NS}_{\omega_1}} \delta \in j(T)$ need *not* imply that $S_\alpha \Vdash_{\mathcal{P}(\omega_1)/\mathrm{NS}_{\omega_1}} \delta \in j(T^*)$ if $T^* \subseteq \omega_1$ is such that the symmetric difference $T \triangle T^*$ is nonstationary. However, in the present context it seems natural to ask for the existence of robust codings for subsets of ω_1, where "robustness" of a real as a code for a given set $A \subseteq \omega_1$ with respect to a sequence $\langle S_\alpha : \alpha < \omega_1\rangle$ of pairwise disjoint stationary sets means that the fact that A gets coded by r and $\langle S_\alpha : \alpha < \omega_1\rangle$, that is, that

$$A \in L[r, \langle S_\alpha : \alpha < \omega_1\rangle],$$

does not depend on any alteration of the members of the decoding device $\langle S_\alpha : \alpha < \omega_1\rangle$ by a nonstationary set, but only on the class of the S_α's in $\mathcal{P}(\omega_1)/\mathrm{NS}_{\omega_1}$ ([9]). This question, as well as variants of it resulting from changing the meaning of the expressions "robust" and "coding" in some slight way (and from considering decoding devices in H_{ω_2} possibly more powerful than the above sequences), are the questions being addressed in Sections 2 and 3.

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2. Slightly perturbing a decoding device. Theorem 2.1 below shows, under the assumption that the sharp of every real exists, that robust codes—in the sense of the previous section—for stationary and co-stationary stationary subsets of ω_1 simply do not exist.

A few words on the history of Theorem 2.1 may be in order here. In [L] Larson forced, over a model with a supercompact limit of supercompact cardinals, to obtain a model in which (a strong form of) MM holds and $(*)$

([9]) The codings expressed in (2) and (3) certainly are independent of changes in the S_α's modulo countable sets, and the decoding devices used in (1), (4) and (5) can of course be chosen so that this is also the case with them.

fails. I observed that in Larson's model there is an ω_1-sequence of stationary subsets of ω_1 which admits no robust code with respect to any ω_1-sequence of mutually disjoint stationary subsets of ω_1. This was the starting point of the present paper. Eventually I proved, using some theory of $\mathbb{P}_{\max}$ forcing, that a relatively strong large cardinal assumption ([10]) implies, in the universe, the nonexistence of robust codes for any stationary and co-stationary subset of ω_1. Then Woodin provided a different argument for deriving the same conclusion from just the existence of a Woodin cardinal below a measurable. Moreover, in his proof the conclusion applied actually to any subset of ω_1 not constructible from any real. I noticed that essentially the same argument could be applied just assuming that $x^\dagger$ exists for every real x, and finally Woodin pointed out that the same kind of argument should work in the context of just the existence of sharps for reals. This, of course, is the optimal hypothesis for dealing with stationary and co-stationary subsets of ω_1 (Corollary 2.1).

THEOREM 2.1 (Woodin). *Assume $x^\sharp$ exists for every real x. Then, given any $A \subseteq \omega_1$, any sequence $\langle S_\alpha : \alpha < \omega_1\rangle$ of pairwise disjoint subsets of ω_1 and any real r, if A is not constructible from a real, then there are sets S^*_α for $\alpha < \omega_1$ such that $S_\alpha \triangle S^*_\alpha$ is nonstationary for all α, and yet $A \notin L[r, \langle S^*_\alpha : \alpha < \omega_1\rangle]$.*

Proof. Assume that the sharp of every real exists and suppose that A, $\langle S_\alpha : \alpha < \omega_1\rangle$ and r witness the failure of the theorem. By enlarging S_0 and shrinking the S_α's for $\alpha \neq 0$ slightly, adding one member to the above sequence, and/or switching the indices of two of its members if necessary, we may assume that $S_\alpha \cap (\alpha+1)$ is empty for all $\alpha > 1$ and that $\{S_\alpha : \alpha < \omega_1\}$ is a partition of ω_1. We shall see that A is constructible from a real, which will be a contradiction. Let $\langle \gamma_\xi : \xi \in \mathrm{Ord}\rangle$ be the increasing enumeration of the class of Silver indiscernibles for $L[r]$. Given any ordinal γ let $\mathcal{P}_{<\gamma}$ be the partial order, ordered by extension, consisting of all finite functions $p \subseteq \gamma \times \gamma$ such that $p(\nu) < \nu$ for every ν in the domain of p. We will construct a certain sequence $\langle g_\xi : \xi < \omega_1\rangle$ such that

(a) each g_ξ is generic for $\mathcal{P}_{<\gamma_\xi}$ over $L[r]$,
(b) $g_\xi = g_{\xi'} \cap \mathcal{P}_{<\gamma_\xi}$ for all $\xi < \xi' < \omega_1$,
(c) $G := \bigcup_{\xi<\omega_1} g_\xi$ is generic for $\mathcal{P}_{<\omega_1^V}$ over $L[r]$.

Since each $\mathcal{P}(\gamma_\xi)^{L[r]}$ is countable, the construction above can be carried out. Also, (c) follows automatically from (a) and (b) since $\mathcal{P}_{<\omega_1^V}$ has the ω_1^V-chain condition in $L[r]$. This sequence will be built by recursion on $\xi < \omega_1$

([10]) Specifically, the assumption that $\mathrm{AD}^{L(\mathbb{R})}$ holds, together with the existence of a Woodin cardinal below a measurable and with the invariance of the theory of $L(\mathbb{R})$ under small ccc forcing.

using some fixed surjection $\sigma : \omega_1 \to \omega_1 \times \omega$ with the property that $\sigma(\xi) = \langle\alpha, n\rangle$ only if $\alpha \le \xi$. For each $\xi < \omega_1$, $\langle b^\xi_n : n < \omega\rangle$ will be some enumeration of all subsets of γ_ξ in $L[r][g_\xi]$.

Note that, given any $\xi < \xi' < \omega_1$ and any filters g and g', generic for $\mathcal{P}_{<\gamma_\xi}$ and $\mathcal{P}_{<\gamma_{\xi'}}$, respectively, satisfying the coherence property expressed in (b), the map sending the interpretation in $L[r]$ of a Skolem term applied to a strictly increasing sequence

$$\langle\gamma_{\xi_0}, \dots, \gamma_{\xi_n}, \gamma_\xi, \gamma_{\xi+i_0}, \dots, \gamma_{\xi+i_m}\rangle$$

of Silver indiscernibles for $L[r]$ to the interpretation of the same term applied to the sequence

$$\langle\gamma_{\xi_0}, \dots, \gamma_{\xi_n}, \gamma_{\xi'}, \gamma_{\xi'+i_0}, \dots, \gamma_{\xi'+i_m}\rangle$$

extends to a unique elementary embedding from $L[r][g]$ into $L[r][g']$. We shall denote this embedding by $j_{g,g'}$. It follows that if $\langle g_\xi : \xi < \omega_1\rangle$ satisfies (a) and (b) above, then $\langle L[r'][g_\xi], j_{g_\xi, g_{\xi'}} : \xi \le \xi' < \omega_1\rangle$ is a directed system. Then we let $j_{g_\xi, G} : L[r][g_\xi] \to L[r][G]$ denote, for any given ξ, the limit map from $L[r][g_\xi]$ into the limit object $L[r][G]$ (G will obviously be $\bigcup_{i<\omega_1} g_i$) corresponding to this directed system.

Pick g_0 arbitrarily and let $\langle s_\beta : \beta < \gamma_0\rangle$ be the sequence in $L[r][g_0]$ consisting of pairwise disjoint stationary subsets of γ_0 ($= \omega_1^{L[r][g_0]}$) defined by setting $s_\beta = (\bigcup g_0)^{-1}(\beta)$ for every $\beta < \gamma_0$. Now pick $\xi < \omega_1$ and suppose g_ξ has been built. Then choose $g_{\xi+1}$ in such a way that the following two conditions are met:

(1) If $\alpha < \omega_1$ is such that $\gamma_\xi \in S_\alpha$, then $\gamma_\xi \in j_{g_0, g_{\xi+1}}(\langle s_\beta : \beta < \gamma_0\rangle)(\alpha)$.
(2) If there is some generic filter g for $\mathcal{P}_{<\gamma_{\xi+1}}$ over $L[r]$ such that $g \cap \mathcal{P}_{<\gamma_\xi} = g_\xi$, such that $j_{g_0,g}$ satisfies the requirement expressed in (1) and, moreover, such that $j_{g_\alpha, g}(b^\alpha_n) \ne A \cap \gamma_{\xi+1}$ (where $\sigma(\xi) = \langle\alpha, n\rangle$), then we let $g_{\xi+1}$ be such a filter.

These conditions can obviously be satisfied by the definition of $\langle s_\beta : \beta < \gamma_0\rangle$ in $L[r][g_0]$. Let S^*_α be, for each α, $j_{g_0,G}(\langle s_\beta : \beta < \gamma_0\rangle)(\alpha)$. Clearly, for each ξ and each $\alpha < \omega_1$, $\gamma_\xi \in S^*_\alpha$ if and only if $\gamma_\xi \in S_\alpha$. Now suppose $A \in L[r, \langle S^*_\alpha : \alpha < \omega_1\rangle]$. Then $A = j_{g_\alpha, G}(b)$ for some subset b of γ_α belonging to $L[r][g_\alpha]$ and some α. Let n be such that $b = b^\alpha_n$, let $\xi \ge \alpha$ be such that $\sigma(\xi) = \langle\alpha, n\rangle$, and let $\beta < \omega_1$ be such that $\gamma_\xi \in S_\beta$. The theorem will be proved once we show that A is in $L[a]$ whenever a is a real coding r and g_ξ.

Take such an a and consider, in $L[a]$, the set B consisting of all those ordinals $\gamma < \omega_1^V$ for which there is some condition p in $\mathcal{P}_{<\omega_1}$ such that $p \cap \mathcal{P}_{<\gamma_\xi} \in g_\xi$, $p(\gamma_\xi) = \beta$, and such that $p\restriction[\gamma_\xi, \omega_1)$ forces in $\mathcal{P}_{[\gamma_\xi, \omega_1)}$ (11)

(11) This notation obviously represents the suborder of $\mathcal{P}_{<\omega_1}$ consisting of those functions with domain disjoint from γ_ξ.

over $L[r][g_\xi]$ that $\gamma \in j_{g_\alpha,\dot{G}}(b)$, where $\dot{G}$ is a name for the generic object for $\mathcal{P}_{<\omega_1^V}$ which is the union of g_ξ and the generic object for $\mathcal{P}_{[\gamma_\xi,\omega_1)}$. It suffices to see that B is A. Suppose there is some $\gamma \in A \setminus B$ (the case that γ is in $B \setminus A$ is argued for symmetrically). Then there is a condition p in $\mathcal{P}_{<\omega_1}$ compatible with g_ξ which sends γ_ξ to β and such that $p\restriction[\gamma_\xi,\omega_1)$ forces in $\mathcal{P}_{[\gamma_\xi,\omega_1)}$ over $L[r][g_\xi]$ that γ is not in $j_{g_\alpha,\dot{G}}(b)$. As γ is in A $(= j_{g_\alpha,G}(b))$, there is also a condition p' in G sending γ_ξ to β and forcing that γ is in $j_{g_\alpha,\dot{G}}(b)$. Now, by indiscernibility, there is an ordinal $\gamma^* < \gamma_{\xi+1}$ and there are conditions q and q' in $\mathcal{P}_{<\gamma_{\xi+1}}$ sending γ_ξ to β and such that $q \cap \mathcal{P}_{<\gamma_\xi}$ and $q' \cap \mathcal{P}_{<\gamma_\xi}$ are both in g_ξ and $q\restriction[\gamma_\xi,\gamma_{\xi+1})$ forces in $\mathcal{P}_{[\gamma_\xi,\gamma_{\xi+1})}$ over $L[r]$ that γ^* is not in $j_{g_\alpha,\dot{g}_{\xi+1}}(b)$ ([12]), whereas $q'\restriction[\gamma_\xi,\gamma_{\xi+1})$ forces that γ^* is in $j_{g_\alpha,\dot{g}_{\xi+1}}(\dot{b})$. It follows that we have been able to choose $g_{\xi+1}$ in such a way that $\gamma^* \notin j_{g_\alpha,G}(b)$ if $\gamma^* \in A$ and $\gamma^* \in j_{g_\alpha,G}(b)$ if $\gamma^* \notin A$, and this contradicts the fact that $j_{g_\alpha,G}(b)$ was supposed to be A. ∎

Next I will consider the situation in which we are given a set $A \subseteq \omega_1$ and a real r which is meant to be a code for (an approximation of) A with respect to a sequence $\mathcal{S} = \langle S_\alpha : \alpha < \omega_1\rangle$ of stationary sets, and we look at the weaker version of robustness for r arising when we just ask that the class of A in $\mathcal{P}(\omega_1)/\mathrm{NS}_{\omega_1}$ (rather than A itself) be coded; this is the version of robustness, that is, that results when the demand that A itself be constructible from r together with any decoding device $\mathcal{S}^*$ almost equal to $\mathcal{S}$ is weakened to the requirement that, for every such $\mathcal{S}^*$, there is some set in $L[r,\mathcal{S}^*]$ which is equal to A up to a club. This kind of variant of the original situation will be taken up again in the next section. Note that if we are to obtain a negative result along the lines of Theorem 2.1, some restriction on A and on the S_α's will have to be taken into account: If A is (almost equal to) $\bigcup_{\alpha\in X} S_\alpha$ for some $X \in L$, then any small variation on the S_α's will construct a set almost equal to A.

THEOREM 2.2. *Assume $x^\sharp$ exists for every real x. Then, given any $S \subseteq \omega_1$, any sequence $\langle S_\alpha : \alpha < \omega_1\rangle$ of mutually disjoint subsets of ω_1 and any real r, if there is some α_0 such that $S \cap S_{\alpha_0}$ and $S_{\alpha_0} \setminus S$ are both stationary, then there are sets S^*_α $(\alpha < \omega_1)$ such that*

(a) *$S_\alpha \triangle S^*_\alpha$ is nonstationary for all α,*
(b) *$S^* \notin L[r, \langle S^*_\alpha : \alpha < \omega_1\rangle]$ whenever $S \triangle S^*$ is nonstationary.*

Proof. Suppose the theorem fails. Again we may assume that $S_\alpha \cap (\alpha+1)$ is empty for all $\alpha > 1$ and that $\bigcup_\alpha S_\alpha = \omega_1$. Fix a sequence

([12]) Again $\mathcal{P}_{[\gamma_\xi,\gamma_{\xi+1})}$ has the natural meaning and $\dot{g}_{\xi+1}$ is a $\mathcal{P}_{[\gamma_\xi,\gamma_{\xi+1})}$-term for the generic filter for $\mathcal{P}_{<\gamma_{\xi+1}}$ which is the union of g_ξ and the generic filter for $\mathcal{P}_{[\gamma_\xi,\gamma_{\xi+1})}$.

$\langle T_n^\alpha : \alpha < \omega_1, n < \omega\rangle$ of mutually disjoint stationary subsets of S_{α_0} with the property that all $T_n^\alpha \cap S$ and all $T_n^\alpha \setminus S$ are stationary.

Letting all undefined notions here stand for corresponding notions in the construction in the proof of Theorem 2.1, this time we build a sequence $\langle g_\xi : \xi < \omega_1\rangle$ satisfying properties (a)–(c) in that proof in such a way that, given any $\xi < \omega_1$ for which g_ξ has already been built, $g_{\xi+1}$ is such that the requirement corresponding to (1) in the proof of Theorem 2.1 is met and, if possible, such that $\gamma_\xi \in j_{g_\alpha, g_{\xi+1}}(d_n^\alpha)$ if $\gamma_\xi \in T_n^\alpha$ and $\alpha_0 < \alpha \leq \xi$ (where, given any α, $\langle d_n^\alpha : n < \omega\rangle$ is now some fixed enumeration of all subsets of $j_{g_0, g_\alpha}(\langle s_\beta : \beta < \gamma_0\rangle)(\alpha_0)$ in $L[r][g_\alpha]$).

Again we let S_α^* be, for each α, $j_{g_0, G}(\langle s_\beta : \beta < \gamma_0\rangle)(\alpha)$. As in Theorem 2.1, for all ξ and $\alpha < \omega_1$, $\gamma_\xi \in S_\alpha$ if and only if $\gamma_\xi \in S_\alpha^*$. Now suppose S^* is some set in $L[r, \langle S_\alpha^* : \alpha < \omega_1\rangle]$ almost equal, modulo NS_{ω_1}, to S. Then $S^* \cap S_{\alpha_0}^*$ has to be stationary. Let $n < \omega$ and $\alpha_0 < \alpha < \omega_1$ be such that $S^* \cap S_{\alpha_0}^* = j_{g_\alpha, G}(d_n^\alpha)$, and let $\xi < \xi'$ be arbitrarily chosen so that $\alpha < \xi$, $\gamma_\xi \in T_n^\alpha$, and $\gamma_{\xi'} \in S^* \cap S_{\alpha_0}^*$. Then, since γ_ξ and $\gamma_{\xi'}$ are Silver indiscernibles for $L[r]$ and since $S^* \cap S_{\alpha_0}^* \cap \gamma_{\xi'+1} = j_{g_\alpha, g_{\xi'+1}}(d_n^\alpha)$, there is a condition p in $\mathcal{P}_{[\gamma_\xi, \gamma_{\xi+1})}$ forcing over $L[r][g_\xi]$, for a name $\dot{g}_{\xi+1}$ for the generic filter for $\mathcal{P}_{<\gamma_{\xi+1}}$ which is the union of g_ξ and of the generic filter for $\mathcal{P}_{<\gamma_\xi}$, that $\dot{g}_{\xi+1} \cap \mathcal{P}_{<\gamma_\xi}$ is g_ξ, that γ_ξ is in $j_{g_0, \dot{g}_{\xi+1}}(\langle s_\beta : \beta < \gamma_0\rangle)(\alpha_0)$, and that γ_ξ is in $j_{g_\alpha, \dot{g}_{\xi+1}}(d_n^\alpha)$. This shows that $g_{\xi+1}$ will have been chosen in such a way that $\gamma_\xi \in j_{g_\alpha, g_{\xi+1}}(d_n^\alpha)$. But $j_{g_\alpha, g_{\xi+1}}(d_n^\alpha)$ is contained in S^*. Since $S_{\alpha_0}^* \cap S^* \cap \{\gamma_\xi : \xi < \omega_1\}$ is unbounded in ω_1, this shows that T_n^α is almost contained, modulo NS_{ω_1}, in S^*. Now, $T_n^\alpha \setminus S$ was chosen to be stationary, so that $S \triangle S^*$ is stationary after all. This contradiction finishes the proof. ∎

Given a real x, the existence of $x^\sharp$ is equivalent to the fact that for every $X \subseteq \omega_1$ in $L[x]$ there is a club $C \subseteq \omega_1$ such that either $C \subseteq X$ or else $X \cap C = \emptyset$; in other words, the nonexistence of $x^\sharp$ is equivalent to the fact that some stationary and co-stationary subset of ω_1 is constructible from x. Hence, the hypothesis used in Theorems 2.1 and 2.2 is optimal:

Corollary 2.1. *The following conditions are equivalent*:

(1) $x^\sharp$ *exists for every real* x.

(2) *Given any stationary and co-stationary* $S \subseteq \omega_1$, *any real* r *and any sequence* $\langle S_\alpha : \alpha < \omega_1\rangle$ *of pairwise disjoint subsets of* ω_1 *there are sets* S_α^* $(\alpha < \omega_1)$ *such that each* S_α *is almost equal to* S_α^* *modulo* NS_{ω_1} *and such that* $S \notin L[r, \langle S_\alpha^* : \alpha < \omega_1\rangle]$.

(3) *Given any real* r, *any sequence* $\langle S_\alpha : \alpha < \omega_1\rangle$ *of pairwise disjoint subsets of* ω_1, *and any* $S \subseteq \omega_1$ *such that* $S \cap S_{\alpha_0}$ *and* $S_{\alpha_0} \setminus S$ *are stationary for some* α_0 *there are sets* S_α^* $(\alpha < \omega_1)$ *such that each* S_α *is almost equal to* S_α^* *modulo* NS_{ω_1} *and such that* $S \triangle S^*$ *is stationary for every* $S^* \in L[r, \langle S_\alpha^* : \alpha < \omega_1\rangle]$.

There are models of ZFC in which some subset A of ω_1 admits, in a nontrivial way, a robust code with respect to some sequence $\langle S_\alpha : \alpha < \omega_1\rangle$ of pairwise disjoint stationary sets (in other words, A is not constructible from any real and yet there is a real which codes A in a robust way with respect to some sequence of pairwise disjoint stationary sets): Fix, in L, two collections $\mathcal{S}$ and $\mathcal{T}$ of size $\aleph_1$ of stationary subsets of ω_1 such that all distinct members of $\mathcal{S} \cup \mathcal{T}$ are disjoint. Consider any extension over L by any ccc forcing iteration $\mathcal{P}$ such that the final generic object can be coded by a subset A of ω_1 and such that every real in $L^{\mathcal{P}}$ appears at some initial stage of the iteration. In $L^{\mathcal{P}}$ build $\langle S_\alpha : \alpha < \omega_1\rangle$ by picking each S_α in $\mathcal{S}$ or in $\mathcal{T}$, and in such a way that the map from ω_1 to $\{\mathcal{S}, \mathcal{T}\}$ sending each α to the collection to which S_α belongs codes A. It is clear that no real in $L^{\mathcal{P}}$ constructs A. Further, if $\langle S^*_\alpha : \alpha < \omega_1\rangle \in L^{\mathcal{P}}$ is such that each $S_\alpha \mathbin{\triangle} S^*_\alpha$ is nonstationary in $L^{\mathcal{P}}$, then for every α there is a unique member S of $\mathcal{S} \cup \mathcal{T}$ such that $S^*_\alpha \cap S$ is stationary in $L[\langle S^*_\alpha : \alpha < \omega_1\rangle]$, and this set S is S_α (since, for any $S' \in \mathcal{S} \cup \mathcal{T}$ different from S_α, every club in $L^{\mathcal{P}}$ witnessing that the intersection of S^*_α with S' is nonstationary contains a club in L). It follows that A is constructible from $\langle S^*_\alpha : \alpha < \omega_1\rangle$.

This construction can be easily modified in such a way that the coding of the set A becomes nontrivial, in the sense that the real r playing the role of code is necessarily nonconstructible: Let $\mathcal{P}$ be the standard forcing for adding $\aleph_1$ Cohen reals viewed as the partial order of finite functions contained in $(\omega_1 \times \omega) \times \{0, 1\}$. In $V^{\mathcal{P}}$ let A code the generic filter G and, letting $\vec{c} := (\bigcup G){\restriction}(\omega_1 \setminus \{0\} \times \omega)$, pick the sequence $\langle S_\alpha : \alpha < \omega_1\rangle$ in $L[\vec{c}]$ coding $\vec{c}$ in the same way as in the previous paragraph. Then, by the same argument as before, $c := (\bigcup G){\restriction}\{0\} \times \omega$ is a robust code for A with respect to $\langle S_\alpha : \alpha < \omega_1\rangle$ and, as c is Cohen generic over $L[\vec{c}]$, $A \notin L[\langle S_\alpha : \alpha < \omega_1\rangle]$.

3. Restricting a decoding device to a club. One may ask whether the conclusion of Corollary 2.1(2) can be strengthened to yield the statement resulting from adding to the requirement that every $S_\alpha \mathbin{\triangle} S^*_\alpha$ be nonstationary the extra requirement that each S^*_α be in fact completely contained in S_α or even the stronger requirement that, for some club $C \subseteq \omega_1$, $S^*_\alpha = S_\alpha \cap C$ for all α. These questions can be best addressed after introducing a certain distinction between sequences of subsets of ω_1. We say that a sequence $\langle S_\alpha : \alpha < \omega_1\rangle$ of subsets of ω_1 is *regressive* if $S_\alpha \cap (\alpha + 1)$ is empty for all $\alpha > 0$ ([13]), and otherwise we say that it is *nonregressive* ([14]).

([13]) In other words, if the function sending each nonzero $\nu < \omega_1$ to the unique (if any) α such that $\nu \in S_\alpha$ is regressive on its domain.

([14]) Note that the first manipulations in the given sequences of stationary sets in the proofs of Theorems 2.1 and 2.2 yield regressive sequences of stationary sets.

Given any sequence of subsets of ω_1 we can obviously shrink its members by countable segments so as to obtain a regressive sequence, so that the answer to the first of the above questions reduces to considering only regressive sequences. Concerning the second question, the following observation and Corollary 3.1 below show that regressive and nonregressive sequences may behave differently as to the existence of robust codes—in the present sense—for a given subset of ω_1.

OBSERVATION 3.1. *Given any subset A of ω_1 there is a nonregressive sequence $\langle S_\alpha : \alpha < \omega_1\rangle$ of pairwise disjoint stationary subsets of ω_1 such that $A \in L[\langle S_\alpha \cap C : \alpha < \omega_1\rangle]$ for every club $C \subseteq \omega_1$.*

Proof. Let $\mathcal{S} = \langle \overline{S}_\xi : \xi < \omega_1\rangle$ and $\langle T_\alpha : \alpha < \omega_1\rangle$ be sequences of mutually disjoint stationary subsets of ω_1 such that $\mathcal{S}$ is regressive and $S_\alpha \cap T_\beta = \emptyset$ for all $\alpha, \beta < \omega_1$, and let $\langle X_\alpha : \alpha < \omega_1\rangle$ be a constructible partition of ω_1 into unbounded sets. Fix also, for every α, a one-to-one map $f_\alpha : T_\alpha \to X_\alpha$ such that $f_\alpha(\nu) > \nu$ for all $\nu \in T_\alpha$. We build a third sequence $\langle S_\alpha : \alpha < \omega_1\rangle$ of pairwise disjoint stationary sets by setting $S_{f_\alpha(\nu)} = \{\nu\} \cup \overline{S}_{f_\alpha(\nu)}$ for each $\alpha \in A$ and each $\nu \in T_\alpha$, and by setting $S_\xi = \overline{S}_\xi$ for all $\xi \in \omega_1 \setminus \bigcup_{\alpha\in A} \mathrm{range}(f_\alpha)$. By construction, $\langle S_\alpha : \alpha < \omega_1\rangle$ consists of pairwise disjoint stationary sets and, since all T_α's are stationary, it is easy to see that, given any club $C \subseteq \omega_1$, A is the set of α's for which there is some nonzero $\xi \in X_\alpha$ such that $S_\xi \cap C \cap (\xi+1)$ is nonempty. Hence, $A \in L[\langle S_\alpha \cap C : \alpha < \omega_1\rangle]$ for every club $C \subseteq \omega_1$. ■

I do not see how to adapt the present proof of Theorem 2.1 in order to deal with the second question above, for regressive sequences of stationary subsets of ω_1, in general (under the assumption that the sharp of every real exists). Nevertheless, as Corollary 3.1 shows, the corresponding form of Corollary 2.1(2) does hold, under $\mathrm{AD}^{L(\mathbb{R})}$, in any $\mathbb{P}_{\max}$ extension of $L(\mathbb{R})$ ([15]).

In an earlier version of this paper I proved Corollary 3.1, via the absoluteness Theorem 4.65 from [W], from the stronger assumption that AD holds in $L(\mathbb{R})$ and that there is a Woodin cardinal with a measurable cardinal above. Using the essence of the proof of the old version of (the first part of) Corollary 3.1 together with standard facts of the $\mathbb{P}_{\max}$ theory under $\mathrm{AD}^{L(\mathbb{R})}$, an anonymous referee found a simplified argument for proving that result from just $\mathrm{AD}^{L(\mathbb{R})}$. The proof of the more general Theorem 3.1, or at least the proof of its first part, arises quite naturally out of the referee's argument.

([15]) Remember that, under $\mathrm{AD}^{L(\mathbb{R})}$, any $\mathbb{P}_{\max}$ extension of $L(\mathbb{R})$ is a model of ZFC in which $\mathrm{AD}^{L(\mathbb{R})}$ holds.

Theorem 3.1. *Under* $\mathrm{AD}^{L(\mathbb{R})}$, *given any formula* $\varphi(x,y)$ *of set theory with parameters in* $L(\mathbb{R})$ *and any* $\mathbb{P}_{\max}$ *generic filter* G *over* $L(\mathbb{R})$, *the following statement about* $\varphi(x,y)$ *holds in* $L(\mathbb{R})[G]$:

Suppose there is a stationary and co-stationary $S \subseteq \omega_1$ *and there is a regressive sequence* $\mathcal{S} = \langle S_\alpha : \alpha < \omega_1\rangle$ *of pairwise disjoint stationary subsets of* ω_1 *such that* $\varphi(S, \mathcal{S}\restriction C)$ *holds for every club* $C \subseteq \omega_1$, *where* $\mathcal{S}\restriction C$ *denotes the sequence* $\langle S_\alpha \cap C : \alpha < \omega_1\rangle$. *Then there is a subset* $\mathcal{X}$ *of* $\mathcal{P}(\omega_1)$ *of size* $2^{\aleph_1}$ *and there is a club* $C \subseteq \omega_1$ *such that* $\varphi(A, \mathcal{S}\restriction C)$ *holds for all* $A \in \mathcal{X}$. *Moreover, if there is some* α_0 *for which both* $S \cap S_{\alpha_0}$ *and* $S_{\alpha_0} \setminus S$ *are stationary, then* $\mathcal{X}$ *can be taken so that* $A \triangle B$ *is stationary for all distinct* A *and* B *in* $\mathcal{X}$.

Corollary 3.1. *Under the* $\mathbb{P}_{\max}$ *axiom* $(*)$, *for every stationary and co-stationary* $S \subseteq \omega_1$, *every regressive sequence* $\langle S_\alpha : \alpha < \omega_1\rangle$ *of pairwise disjoint stationary subsets of* ω_1 *and every real* r *there is a club* $C \subseteq \omega_1$ *such that* $S \notin L[r, \langle S_\alpha \cap C : \alpha < \omega_1\rangle]$. *Furthermore, if there is some* α_0 *such that* $S \cap S_{\alpha_0}$ *and* $S_{\alpha_0} \setminus S$ *are stationary, then there is a club* $C \subseteq \omega_1$ *with the property that* $S \triangle S'$ *is stationary for every subset* S' *of* ω_1 *in* $L[r, \langle S_\alpha \cap C : \alpha < \omega_1\rangle]$.

Proof. By taking $\varphi(x,y)$ in Theorem 3.1 to be $x \in L[r,y]$ it follows that, under $(*)$, there cannot be any stationary and co-stationary $S \subseteq \omega_1$ and any regressive sequence $\langle S_\alpha : \alpha < \omega_1\rangle$ of mutually disjoint stationary subsets of ω_1 such that $S \in L[r, \langle S_\alpha \cap C : \alpha < \omega_1\rangle]$ for every club $C \subseteq \omega_1$. The reason for this is that otherwise, by the conclusion of the theorem, $\mathcal{P}(\omega_1)^{L[r,\langle S_\alpha \cap C : \alpha<\omega_1\rangle]}$ would be of size $2^{\aleph_1}$ for some club $C \subseteq \omega_1$. However, under $(*)$ the sharp of every subset A of ω_1 exists (and therefore in $L[A]$ there are only $\aleph_1$ subsets of ω_1).

For the second conclusion let $\varphi(x,y)$ be a formula with parameter r saying that there is some $x' \subseteq \omega_1$ in $L[r,y]$ with $x \triangle x'$ nonstationary. Assume $(*)$ and suppose the conclusion fails. Let $\langle S_\alpha : \alpha < \omega_1\rangle$ be a regressive sequence of mutually disjoint stationary subsets of ω_1, let α_0 be a countable ordinal and let $S \subseteq \omega_1$ be such that $S \cap S_{\alpha_0}$ and $S_{\alpha_0} \setminus S$ are stationary, and suppose $H_{\omega_2} \models \varphi(S, \langle S_\alpha \cap C : \alpha < \omega_1\rangle)$ holds for every club C. Then there is a club $C \subseteq \omega_1$ and there is a set $\mathcal{X}$ of size $2^{\aleph_1}$ of subsets of ω_1 such that $A \triangle B$ is stationary for all distinct $A, B \in \mathcal{X}$ and such that, for every given $A \in \mathcal{X}$, there is some $A' \subseteq \omega_1$ with $A \triangle A'$ nonstationary and $A' \in L[r, \langle S_\alpha \cap C : \alpha < \omega_1\rangle]$. But the map sending $A \in \mathcal{X}$ to A' is then one-to-one, so that $\mathcal{P}(\omega_1)^{L[r,\langle S_\alpha\cap C:\alpha<\omega_1\rangle]}$ is of size $2^{\aleph_1}$, which again is a contradiction. ∎

Note that the conclusion of Corollary 3.1 cannot be strengthened to say that for every real r, every stationary and co-stationary $S \subseteq \omega_1$ and every regressive sequence $\langle S_\alpha : \alpha < \omega_1\rangle$ of mutually disjoint stationary subsets of ω_1 there is a club $C \subseteq \omega_1$ such that $S \notin L[r, \langle S_\alpha \cap D : \alpha < \omega_1\rangle]$ for every club $D \subseteq C$. The reason is that, given any $A \subseteq \omega_1$ and any sequence $\langle S_\alpha : \alpha < \omega_1\rangle$ as above, one can always find a club $D \subseteq \omega_1$ such that $\langle S_\alpha \cap D : \alpha < \omega_1\rangle$ codes A: one may pick D for example in such a way that, for each $\alpha < \omega_1$, the αth nonlimit point of $\bigcup_{\xi<\omega_1} S_\xi \cap D$ is in S_α if and only if α is in A.

Note also that the conclusion of Theorem 3.1 does not hold if we replace regressive sequences of mutually disjoint stationary sets by regressive functions from ω_1 into ω_1: Given any function $\mathcal{F} \subseteq \omega_1 \times \omega_1$ there is exactly one set $A \subseteq \omega_1$ such that, given any club $C \subseteq \omega_1$, $A = \{\alpha < \omega_1 : (\mathcal{F}{\restriction}C)^{-1}(\alpha) \text{ is stationary}\}$.

Proof of Theorem 3.1. Fix a formula $\varphi(x, y)$ with parameters in $L(\mathbb{R})$ for which $\mathbb{P}_{\max}$ forces the antecedent of the statement about $\varphi(x, y)$ expressed in the theorem ([16]). By changing $\varphi(x, y)$ slightly if necessary we may assume that the sequence $\mathcal{S}$ witnessing the above hypothesis is forced to define a partition of ω_1.

By [W, Theorem 4.50], there is a $\mathbb{P}_{\max}$ condition $\langle (M, I), a\rangle$ and there are an I-positive $s \in M$ such that $\omega_1 \setminus s$ is also I-positive and a regressive partition $\langle s_i : i < \omega_1^M\rangle \in M$ of ω_1^M into I-positive sets such that $\langle (M, I), a\rangle$ forces that, letting $j_{\dot G} : (M, I) \to (M^*, I^*)$ be the unique iteration of (M, I) sending a to $A_{\dot G}$ ([17]), (where $\dot G$ is a term for the generic filter) and letting $\mathcal{S}_{\dot G}$ be $j_{\dot G}(\langle s_i : i < \omega_1^M\rangle)$, $\varphi(j_{\dot G}(s), \mathcal{S}_{\dot G}{\restriction}C)$ holds for all clubs $C \subseteq \omega_1$.

Let x be a real coding (M, I). Let G be generic for $\mathbb{P}_{\max}$ over $L(\mathbb{R})$ containing $\langle (M, I), a\rangle$. In order to finish the proof (of the theorem without its 'Moreover' part) it suffices to show that for every function $F : \omega_1 \to \omega_1$ in $L(\mathbb{R})[G]$ there is, in $L(\mathbb{R})[G]$, some $A \subseteq \omega_1$ such that, for all $\nu < \omega_1$, the νth successor Silver indiscernible for $L[x]$ is in A if and only if $F(\nu) = 1$ and such that $\varphi(A, \mathcal{S}_G{\restriction}C)$ holds, where C is the club of limit Silver indiscernibles for $L[x]$ below ω_1.

Let $\langle (N, J), b\rangle$ be any condition in G extending $\langle (M, I), a\rangle$. Again by [W, Theorem 4.50] we may assume, towards a contradiction, that there is a function $f : \omega_1^N \to \omega_1^N$ in N such that $\langle (N, J), b\rangle$ forces that, letting $\mathcal{X}$ be the set of $A \subseteq \omega_1$ such that, for all ν, the νth successor indiscernible for $L[x]$ is in A if and only if $j_{\dot G}(f)(\nu) = 1$, there is no A in $\mathcal{X}$ such that $\varphi(A, \mathcal{S}_{\dot G}{\restriction}C)$.

Since every Silver indiscernible for $L[x]$ is in the critical sequence of every iteration of (M, I), in N we can construct an iteration $k : (M, I) \to (M^*, I^*)$ of length ω_1^N such that

(1) $\mathcal{P}^{M^*}(\omega_1^N) \cap J = I^*$,

([16]) By homogeneity of $\mathbb{P}_{\max}$ under $\mathrm{AD}^{L(\mathbb{R})}$, if some condition forces that hypothesis about φ, then so does every condition.

([17]) Given a filter G of $\mathbb{P}_{\max}$, A_G will denote the union of all b such that $\langle (N, J), b\rangle \in G$ for some N and J.

(2) for all $\xi, \nu < \omega_1^N$, if ξ is the νth successor Silver indiscernible for $L[x]$, then $\xi \in k(s)$ if and only if $f(\nu) = 1$,

(3) letting j be the unique iteration of (M, I) sending a to b, for every limit Silver indiscernible $\xi < \omega_1^N$ for $L[x]$ and every $\gamma < \omega_1^N$, $\xi \in k(\langle s_i : \xi < \omega_1^M \rangle)(\gamma)$ if and only if $\xi \in j(\langle s_i : \xi < \omega_1^M \rangle)(\gamma)$.

The above construction can be carried out with the help of a bookkeeping argument very much as in [W, Lemma 4.36]: We start fixing a partition $\langle B_{\alpha,\beta,\gamma} : \alpha, \beta, \gamma < \omega_1^N \rangle$ in N consisting of J-positive sets such that, for all α, β and γ, $B_{\alpha,\beta,\gamma} \cap (\alpha + 1) = \emptyset$ and such that, for all α and β, $B_{\alpha,\beta,\gamma}$ is contained in $j(\langle s_i : i < \omega_1^M \rangle)(\gamma)$. The iteration k will be of the form $\langle (M_\alpha, I_\alpha), G_{\alpha'}, k_{\alpha',\alpha} : \alpha' < \alpha < \omega_1^N \rangle$. Given some α for which (M_α, I_α) has already been defined, we fix a sequence $\langle X^\alpha_{\beta,\gamma} : \beta, \gamma < \omega_1^{M_\alpha} \rangle$ such that, given any $\gamma < \omega_1^{M_\alpha}$, $\langle X^\alpha_{\beta,\gamma} : \beta < \omega_1^M \rangle$ enumerates all I_α-positive subsets of $k_{0,\alpha}(\langle s_i : i < \omega_1^M \rangle)(\gamma)$ in M_α. Then, given any Silver indiscernible ξ for $L[x]$,

(a) if ξ is the νth successor indiscernible, we put $k_{0,\xi}(s)$ in G_ξ if and only if $j(f)(\nu) = 1$,

(b) if ξ is a limit indiscernible and ξ is in $B_{\alpha,\beta,\gamma}$, then we put $k_{\alpha,\xi}(X^\alpha_{\beta,\gamma})$ in G_ξ.

Now let H be, in $L(\mathbb{R})[G]$, the set of all $\mathbb{P}_{\max}$ conditions $\langle (M', I'), a' \rangle$ for which there is a (unique) iteration $j' : (M', I') \to (M'', I'')$ of (M', I') sending a' to $j_G(k(a))$ and such that $I'' = \mathcal{P}(\omega_1)^{M''} \cap \mathrm{NS}_{\omega_1}$. By the proof of Theorem 4.60 in [W] and by Lemma 4.56 in [W], H is $\mathbb{P}_{\max}$ generic over $L(\mathbb{R})$ and $L(\mathbb{R})[G] = L(\mathbb{R})[H]$. Moreover, H contains $\langle (N, J), k(a) \rangle$, which, by the construction of k, is a condition extending $\langle (M, I), a \rangle$ as witnessed by k, and thus forcing that $\varphi(j_{\dot{G}}(k(s)), \langle j_{\dot{G}}(\langle s_i : i < \omega_1^M \rangle)(\alpha) \cap D : \alpha < \omega_1 \rangle)$ holds for every club D of ω_1, and in particular for C. Note that $j_H \restriction N$ and $j_G \restriction N$ send $k(a)$ to the same set $j_G(k(a))$, and therefore $j_G : (N, J) \to (N', J')$ and $j_H : (N, J) \to (N', J')$ are the same iteration of (N, J). Finally, again by the construction of k,

(i) for every $\nu < \omega_1^N$, the νth successor Silver indiscernible for $L[x]$ is in $k(s)$ if and only if $f(\nu) = 1$, and therefore, for every $\nu < \omega_1$, the νth successor Silver indiscernible for $L[x]$ is in $j_H(k(s))$ $(= j_G(k(s)))$ if and only if $j_G(f)(\nu) = 1$,

(ii) $j(\langle s_i : i < \omega_1^M \rangle)(\alpha) \cap C \cap \omega_1^N = k(\langle s_i : i < \omega_1^M \rangle)(\alpha) \cap C \cap \omega_1^N$ for every $\alpha < \omega_1^N$, and therefore $j_G(j(\langle s_i : i < \omega_1^M \rangle))(\alpha) \cap C$ is equal to $j_H(k(\langle s_i : i < \omega_1^M \rangle))(\alpha) \cap C$ $(= j_G(k(\langle s_i : i < \omega_1^M \rangle))(\alpha) \cap C)$ for every $\alpha < \omega_1$.

Hence, on the one hand we see that $j_H(k(s))$ is a member of $\mathcal{X}$, and on the other hand $\varphi(j_H(k(s)), \langle j_G(j(\langle s_i : i < \omega_1^M \rangle))(\alpha) \cap C : \alpha < \omega_1 \rangle)$ holds, which contradicts the choice of $\langle (N, J), b \rangle$ since $\langle (N, J), b \rangle$ is in G.

Suppose now that $\mathbb{P}_{\max}$ forces the hypothesis about $\varphi(x,y)$ expressed in the 'Moreover' part of the theorem. Fix corresponding $\langle (M,I),a\rangle \in \mathbb{P}_{\max}$ and s, $\langle s_i : i<\omega_1^M\rangle \in M$ as in the proof of the first part of the theorem. Fix also an ordinal $\alpha<\omega_1^M$ such that $s_\alpha \cap s$ and $s_\alpha \setminus s$ are both I-positive and again let x be a real coding $\langle (M,I),a\rangle$. Let $\langle (N,J),b\rangle$ be a condition extending $\langle (M,I),a\rangle$ such that $x\in N$ and let $\langle t_i : i<\omega_1^N\rangle$ be a partition in N of $j(s_\alpha)$ into J-positive sets, where j is the iteration of (M,I) sending a to b. This time it will suffice to show that for every $\mathbb{P}_{\max}$ generic filter G over $L(\mathbb{R})$ containing $\langle (N,J),b\rangle$ and every proper subset B of ω_1 in $L(\mathbb{R})[G]$ there is, in $L(\mathbb{R})[G]$, some $A\subseteq\omega_1$ such that

(a) $j_G(\langle t_i : i<\omega_1^N\rangle)(\beta)\setminus A$ is nonstationary for all $\beta\in B$ and $j_G(\langle t_i : i<\omega_1^N\rangle)(\beta)\cap A$ is nonstationary for all $\beta\in\omega_1\setminus B$,

(b) $\varphi(A,\mathcal{S}_G\restriction C)$ holds, where C is the club of countable Silver indiscernibles for $L[x]$.

Fix therefore a condition $\langle (N',J'),b'\rangle\in G$ extending $\langle (N,J),b\rangle$ and a proper subset $\overline{b}$ of $\omega_1^{N'}$ in N' and assume, towards a final contradiction, that $\langle (N',J'),b'\rangle$ forces that there is no $A\subseteq\omega_1$ such that (a) and (b) above (with $\dot{G}$ instead of G and $j_{\dot{G}}(\overline{b})$ instead of B) hold. In N' there is then an iteration $k:(M,I)\to(M^*,I^*)$ of (M,I) such that, letting j_0 be the iteration of (M,I) sending a to b' and letting j_1 be the iteration of (N,J) sending b to b',

(1) $\mathcal{P}(\omega_1^{N'})^{M^*}\cap J' = I^*$,

(2) given any Silver indiscernible $\xi<\omega_1^{N'}$ for $L[x]$, if $\gamma<\omega_1^{N'}$ is such that $\xi\in j_1(\langle t_i : i<\omega_1^N\rangle)(\gamma)$, then $\xi\in k(s)$ if and only if $\gamma\in\overline{b}$,

(3) for every Silver indiscernible $\xi<\omega_1^{N'}$ for $L[x]$ and every $\gamma<\omega_1^{N'}$, $\xi\in k(\langle s_i : i<\omega_1^M\rangle)(\gamma)$ if and only if $\xi\in j_0(\langle s_i : i<\omega_1^M\rangle)(\gamma)$.

Such an iteration k can be easily constructed in N' using a bookkeeping argument as in the first part of the proof. We are able to make (2) work because $s_\alpha\cap s$ and $s_\alpha\setminus s$ are both I-positive. The rest of the argument is now as in the first part of the theorem: We let H be the set of all $\mathbb{P}_{\max}$ conditions $\langle (M',I'),a'\rangle$ for which there is a (unique) iteration $j':(M',I')\to(M'',I'')$ of (M',I') sending a' to $j_G(k(a))$ and such that $I''=\mathcal{P}(\omega_1)^{M''}\cap \mathrm{NS}_{\omega_1}$. Again, H is $\mathbb{P}_{\max}$ generic over $L(\mathbb{R})$, $L(\mathbb{R})[G]=L(\mathbb{R})[H]$, and H contains $\langle (N,J),k(a)\rangle$, which is a condition extending $\langle (M,I),a\rangle$ as witnessed by k. Hence, on the one hand $\varphi(j_H(k(s)),\langle j_H(k(\langle s_i : i<\omega_1^M\rangle))(\alpha)\cap C : \alpha<\omega_1\rangle)$ holds—and therefore, by (3) of the construction, so does $\varphi(j_H(k(s)),\langle j_G(\langle s_i : i<\omega_1^M\rangle)(\alpha)\cap C : \alpha<\omega_1\rangle)$—and on the other hand $j_H(k(s))$ has the property that $j_G(\langle t_i : i<\omega_1^N\rangle)(\beta)\setminus j_H(k(s))$ is nonstationary for all $\beta\in j_G(\overline{b})$ and $j_G(\langle t_i : i<\omega_1^N\rangle)(\beta)\cap j_H(k(s))$ is nonstationary for all $\beta\in\omega_1\setminus j_G(\overline{b})$ (18).

(18) Because $j_G(x)=j_H(x)$ for all $x\in N'$.

But that is a contradiction to the choice of $\langle (N', J'), b' \rangle$ since $\langle (N', J'), b' \rangle$ is in G. ∎

Every function with domain ω_1 and finite sequences of countable ordinals as values can obviously be coded by a function from ω_1 into ω_1, so that the simplest decoding devices—from the point of view of the information they carry—that one may consider beyond the functions from ω_1 into ω_1 are the functions from ω_1 into the set of countable subsets of ω_1. I will finish the paper with a simple observation concerning decoding devices $\mathcal{F}$ of this last kind, which shows the existence, under MA_{ω_1}, of codes $r \in \mathbb{R}$ having the stronger kind of robustness that ensures that the coding of a given set be realizable also in the presence of every function $\mathcal{F}^* \subseteq \mathcal{F}$ such that the domain of $\mathcal{F}^*$ is merely an unbounded subset of the domain of $\mathcal{F}$.

OBSERVATION 3.2. (MA_{ω_1}) *Given any set X of size $\aleph_1$ of members of the Baire space and any $A \subseteq \omega_1$ there is a real r such that $A \in L[r, Y]$ whenever Y is an uncountable subset of X.*

Proof. Let $\langle r_\alpha : \alpha < \omega_1 \rangle$ be a one-to-one enumeration of X. Fix a recursive list $\langle \sigma_n : n < \omega \rangle$ of the finite sequences of integers and also a recursive partition $\langle e_i : i < \omega \rangle$ of ω into infinite sets. Define $\langle a_i : i < \omega \rangle$ by letting a_i be, for each i, the set of $n < \omega$ such that the length of σ_n is in e_i. Then we define sets of integers a_i^α ($i < \omega$, $\alpha < \omega_1$) by letting a_i^α be the set of $n \in a_i$ such that σ_n is an initial segment of r_α. Note that, for every $i < \omega$ and for all distinct $\alpha, \beta < \omega_1$, a_i^α and a_i^β are almost disjoint subsets of a_i and that each r_α is recursively equivalent to $(a_i^\alpha : i < \omega)$. Let A_α be, for each $\alpha < \omega_1$, a set of integers coding $A \cap \alpha$. For each i we can find, by MA_{ω_1}, $x_i \subseteq \omega$ such that $x_i \cap a_i^\alpha$ is infinite exactly when $i \in A_\alpha$. Letting now x be a real coding $(x_i)_i$, we deduce that A_α, and therefore also $A \cap \alpha$, can be decoded from x and r_α for each α. Now the desired result follows trivially since, given any uncountable $Y \subseteq X$, A is the union of the sets decoded by x together with some real in Y. ∎

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