LONG REALS

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Abstract. The familiar continuum $\mathbb{R}$ of real numbers is obtained by a well-known procedure which, starting with the set of natural numbers $\mathbb{N} = \omega$, produces in a canonical fashion the field of rationals $\mathbb{Q}$ and, then, the field $\mathbb{R}$ as the completion of $\mathbb{Q}$ under Cauchy sequences (or, equivalently, using Dedekind cuts). In this article, we replace ω by any infinite suitably closed ordinal κ in the above construction and, using the natural (Hessenberg) ordinal operations, we obtain the corresponding field κ–$\mathbb{R}$, which we call the field of the *κ–reals*. Subsequently, we study the properties of the various fields κ–$\mathbb{R}$ and develop their general theory, mainly from the set-theoretic perspective. For example, we investigate their connection with standard themes such as forcing and descriptive set theory.

1. Introduction

One can hardly exaggerate on the importance of the *continuum* $\mathbb{R}$ of real numbers, it being arguably one of the paramount objects of interest in mathematics. The study of the real numbers and of their properties has been practiced for centuries, thus giving rise to entire fields of knowledge and to several indispensable tools, many of which have eventually become fundamental part of the mathematical edifice.

Notwithstanding, and mainly due to the non-availability of the formal method in earlier times, it was not before the late 19th century that the rigorous construction of $\mathbb{R}$ saw the light.[1]

The most renowned formal treatments of the continuum were given by Dedekind and, independently, by Cantor. Dedekind, who announced his ideas in 1858, constructed the reals via (what are now called) *cuts* of rationals; on the other hand, Cantor worked with *Cauchy sequences* of rational numbers. Both approaches were published in 1872 (cf. [4], [9])

Date: January 20, 2018. The finalized published version of the article will appear in the **Journal of Logic and Analysis**.

2010 *Mathematics Subject Classification.* 12L15, 12J15, 12L99, 03E10, 03E15.

Key words and phrases. Real numbers, Hessenberg operations, Ordered fields, Forcing, Descriptive set theory.

[1] It should be noted that attempts to construct the real numbers start to appear as early as the late 18th century, for instance in work of Bolzano. In the 19th century, other attempts were made by Weierstrass, Méray, Heine, et al; see Epple [14] for an extensive presentation of the historical details.

and were subsequently shown to be equivalent, producing the familiar *complete*[2] ordered field $\langle \mathbb{R}, 0, 1, +, \cdot, \leqslant \rangle$.

Either of these methods is now referred to as the standard way to construct the reals, starting with the set of natural numbers $\mathbb{N} = \omega$. Our initial motivation is the following natural question.

Main Question. What happens if we replace ω in the aforementioned construction(s) by an infinite cardinal $\kappa > \omega$?

The previous question appears to be elementary, or perhaps naïve. It could certainly have been asked by Cantor himself, or by any other knowledgeable mathematician of the late 19th century. Nevertheless, and in comparison with the vast field of knowledge which has been produced in practically all surrounding themes, it is only marginally that this question has drawn attention, or that a systematic and comprehensive answer has been given to it.

At the outset, we should note that there have been many sources dealing with the surrounding theory of generalized metric spaces, generalized convergence, measure, category, etc. For instance, Cohen and Goffman already studied notions of *transfinite* convergence in 1949 (cf. [6]), whereas Stevenson and Thron worked with general $\aleph_\mu$–metric spaces in 1969 (cf. [35]); more related material may also be found in the work of Sikorski [34]. Moreover, other generalized transfinite fields that have appeared in the literature (see, for example, Ehrlich [12]), have occasional affinity with our constructions but do not tackle our motivational question directly. The well-known field of the *surreal numbers*, introduced by Conway (cf. [7], [8]), is distinct from the long reals that we study here, as is a more recent work by Galeotti (cf. [16]).

The only works that seem to be intimately related to our results appear around 1950 and 1960. First, Sikorski constructed transfinite integer and rational numbers in 1948 (cf. [33]), while about a decade later, Klaua considered transfinite real numbers (cf. [24], [25]) using a method similar to the one that we present here.[3] However, our present construction gives rise to a (complete ordered) field, whereas Klaua's construction does not.[4]

Interestingly, both Sikorski and Klaua focus solely on the case of cardinals κ of uncountable cofinality (and mainly on regular κ). On

[2] This term may either refer to *Cauchy completeness*, or to completeness with respect to the *ordering* of the field. The second notion is stronger, but they are both equivalent in the presence of the *Archimedean property*.

[3] All these cited sources came to our attention only after the majority of the present work had been completed. This was partly due to the fact that sources such as [24], [25] and [33] are difficult to access. For a more accessible overview of these early tries, the interested reader may consult the more recent [26] and [27] by Klaua.

[4] See also Cantini [3] for some algebraic and topological properties of Klaua's construction.

the other hand, and as will hopefully become clear in what follows (see Theorems 4.6 and 4.7 below), it is the case of countable cofinality that gives the most fruitful theory from a set-theoretic perspective, which is the one that we mainly adopt in this work.

With the present article, we would like to revive the interest in the construction of transfinite (or long) reals. In particular, we would like to draw attention both on the sometimes surprising results emerging in this area, and on the remarkably wide variety of open problems and lines for further research that are consequently revealed. Recent work, by various people, on generalized Baire spaces and related set-theoretic issues (see, for instance, [15] and [23]) should be thought as complementary in this direction, possibly with important underlying connections waiting to surface.

The structure of the article is as follows. We first give the necessary preliminaries in the following section; in particular, and for the reader's convenience, we also recall the ordinary construction of the real numbers. In Section 3, we review the *natural operations* on the ordinals, reflect on their naturalness in our context, and draw some negative results regarding transfinite exponentiation with respect to these operations. Subsequently, we define the ordered field of the κ*–rationals*, for any non-zero ordinal κ closed under the natural operations.

In Section 4, we proceed with the anticipated construction of the κ*–reals* and we start the study of their basic properties. A brief account of κ–calculus, that is, the calculus of the κ–reals, is given in Section 5.

In Section 6, and in a more set-theoretic vein, we look at the κ–reals from the perspective of *forcing*, as well as in terms of *category*. We then continue in Section 7 where we look at sets of long reals from a descriptive set-theoretic point of view.

Finally, in Section 8, we conclude with some open questions and some related thoughts for further investigation.

Acknowledgments. The authors would like to thank the anonymous referees for their comments and suggestions.

This research was carried out during the second author's visit to the University of East Anglia (UEA), in the spring of 2013. For this research stay, the second author was supported by the European Science Foundation (ESF) under the INFTY exchange grant No. 4143, within the framework of the ESF activity "New Frontiers of Infinity: Mathematical, Philosophical and Computational Prospects". The second author wishes to express his gratitude to the members of the School of Mathematics of UEA, and in particular to Prof. Mirna Džamonja, for their very warm hospitality.

The first author acknowledges support of EPSRC Grant EP/N032160/1.

2. Preliminaries

2.1. **Notation.** Our notation and terminology are mostly standard.[5] ZFC stands for the usual first-order axiomatization of Zermelo–Fraenkel set theory, together with the Axiom of Choice. For any set X, we write $|X|$ for the cardinality of X.

The class of ordinal numbers will be denoted by **ON**. Lower case Greek letters stand for ordinals, with the letters κ, λ and μ typically used in the case of ordinals closed under natural multiplication (see Definition 3.1). Ordinal intervals are readily comprehensible; for example, given $\alpha < \beta$, we write (α, β) for the set of ordinals which lie strictly between α and β. Given a set X of ordinals, $\mathrm{ot}(X)$ denotes the order type of X. If α is a limit ordinal, then $\mathrm{cf}(\alpha)$ is its cofinality. Given an ordinal α, we write $\aleph_\alpha$ for the α–th infinite cardinal; $\aleph_0 = \omega = \mathbb{N}$ stands for (the cardinality of) the set of natural numbers. If λ is an infinite cardinal, we let H_λ be the collection of all sets whose transitive closure has size less than λ. Given sets X and Y, we write ${}^X Y$ for the set of all functions f with $\mathrm{dom}(f) = X$ and $\mathrm{ran}(f) \subseteq Y$; if $|X| = \lambda$ and $|Y| = \kappa$, then $|{}^X Y| = \kappa^\lambda$.

For any linear order $\langle L, <_L \rangle$ and any $A, B \subseteq L$, we write $A <_L B$ to mean that $x <_L y$ for all $x \in A$ and all $y \in B$. In the same context, given $X \subseteq L$ we let $\mathrm{cf}(X)$ be the cofinality of X; that is, the least (regular) cardinal λ for which there is a strictly $<_L$–increasing sequence $(a_\xi)_{\xi<\lambda}$ in X that is cofinal in $\langle X, <_L \cap X \times X \rangle$. In a similar fashion we define $\mathrm{coin}(X)$, the coinitiality of X. Also, we will say that $A \subseteq L$ is bounded if and only if there are x, $y \in L$ such that $x \leqslant_L a \leqslant_L y$ for all $a \in A$. Of course, we say that a sequence of members of L is bounded if and only if its range is bounded.

Partial orders (aka posets) that are employed in forcing constructions will be denoted by capital letters such as P, Q and R. We shall write $q < p$ to mean that q is stronger than p or, equivalently, that q properly extends p. We denote the greatest element of a poset by $\mathbb{1}$; in particular, we always assume that forcing posets are non-empty. Given a poset P, the P–names are indicated by "dots" and "checks" as usual; we sometimes supress these in order to ease readability, with the intended meaning being clear from the context. The universe of P–names will be denoted by V^P. If $\dot{x}$ is a P–name and G is a P–generic filter (over the relevant model), then $\dot{x}_G$ stands for the interpretation of the name by the filter.

Trees are special cases of posets.[6] A *branch* through a tree T is a function b with domain the height of T and such that all initial

[5] See Jech [19] or Kanamori [20] for an account of all undefined set-theoretic notions.

[6] Namely, the posets in which the set of predecessors of every given member is well-ordered.

segments of b are pairwise comparable nodes of T. The *body* of a tree T is denoted by $[T]$ and stands for the collection of the branches of T. We shall be mainly interested in trees T of the form $T = {}^{<\kappa}X$, where κ is a non-zero naturally closed ordinal and X is a set (typically ω or 2). Given such a tree T and given $s \in T$, we write $\mathrm{supp}(s)$ for the support of s, that is, the collection $\{\alpha \in \mathrm{dom}(s) : s(\alpha) \neq 0\}$; similarly, we write $\mathrm{supp}(b)$ to denote the support of b whenever b is a branch of T. Moreover, in the same context and for any elements $s, t \in T$, we write $s \sqsubseteq t$ to mean that s is an initial segment of t; that is, $t \restriction \mathrm{dom}(s) = s$. Two conditions $s, t \in T$ are *incompatible*, denoted by $s \perp t$, if there is some $\alpha < \mathrm{dom}(s) \cap \mathrm{dom}(t)$ such that $s(\alpha) \neq t(\alpha)$. For any $s \in T$ we denote by T_s the set $\{u \in T : s \sqsubseteq u \vee u \sqsubseteq s\}$, that is, the set of predecessors of s together with the *cone* above it. Finally, a tree T is called *splitting* if each $s \in T$ splits into two incompatible conditions: that is, there are $u, v \in T$ such that $s \sqsubseteq u$, $s \sqsubseteq v$ and $u \perp v$.

2.2. **Constructing $\mathbb{R}$.** For completeness, and in order to appreciate the obstacles and surprises that arise when trying to generalize the ordinary construction of the real numbers, let us briefly recall it here. It should be mentioned that, although there are different ways in which this procedure may be carried out, the essential idea remains the same and the resulting objects are isomorphic.

Starting with the set ω, the first step is to construct the ordered field of rational numbers by defining the following equivalence relation $\sim$ on the set of triples $A = \{(n, m, k) : n, m, k \in \omega \wedge k \neq 0\}$:

$$(n, m, k) \sim (n', m', k') \iff n \cdot k' + m' \cdot k = n' \cdot k + m \cdot k'.$$

Intuitively, the intended meaning is that

$$(n, m, k) \sim (n', m', k') \iff \frac{n - m}{k} = \frac{n' - m'}{k'},$$

that is, two triples are equivalent if they represent the same rational. We then let $\mathbb{Q}$ be the set of equivalence classes $A/\!\sim$ and define the identity elements of the field as $0_{\mathbb{Q}} = [(0, 0, 1)]_\sim$ and $1_{\mathbb{Q}} = [(1, 0, 1)]_\sim$.

We define the operations of addition $+_{\mathbb{Q}}$ and multiplication $\cdot_{\mathbb{Q}}$ in the obvious way:

$$[(n, m, k)]_\sim +_{\mathbb{Q}} [(n', m', k')]_\sim = [(n \cdot k' + n' \cdot k, m \cdot k' + m' \cdot k, k \cdot k')]_\sim$$

and

$$[(n, m, k)]_\sim \cdot_{\mathbb{Q}} [(n', m', k')]_\sim = [(n \cdot n' + m \cdot m', n \cdot m' + m \cdot n', k \cdot k')]_\sim.$$

Finally, we define the ordering $\leqslant_{\mathbb{Q}}$ by letting:

$$[(n, m, k)]_\sim \leqslant_{\mathbb{Q}} [(n', m', k')]_\sim \iff n \cdot k' + m' \cdot k \leqslant n' \cdot k + m \cdot k'.$$

It is now easily checked that the above are well-defined and that the resulting structure $\langle\, \mathbb{Q}\,, 0_{\mathbb{Q}}\,, 1_{\mathbb{Q}}\,, +_{\mathbb{Q}}\,, \cdot_{\mathbb{Q}}\,, \leqslant_{\mathbb{Q}} \rangle$ is an ordered field in which the natural numbers can be embedded in a straightforward way.[7] Moreover, by its construction, this field is a *system of rational numbers*[8] and, as such, it is unique up to isomorphism. Therefore, we may call $\mathbb{Q}$ *the* field of rational numbers.

Notation. In order to avoid unnecessary formalistic complications, we will consistently drop the subscript "$\mathbb{Q}$" from the operations, the ordering and the identity elements of the field $\mathbb{Q}$; moreover, we will also drop any reference to classes of $\sim$ and write $\dfrac{n-m}{k}$, or even $k^{-1}\cdot(n-m)$, instead of $[(n,m,k)]_{\sim}$. Expressions of the form "$a-b$" and "a^{-1}" are understood in the context of the field and have the intended meaning.

Having constructed the field of rational numbers, and as already mentioned, there are two (equivalent) ways that produce the real numbers: via Cauchy sequences or via Dedekind cuts. We now briefly recall some of the relevant details.

Definition 2.1. A sequence $(a_n)_{n\in\omega}$ of rationals is called *Cauchy* if, for every $m \in \omega$, there exists some $n_0 \in \omega$ such that, for all $n, n' \geqslant n_0$,

$$|a_n - a_{n'}| < \frac{1}{m+1}\,.$$

Now define an equivalence relation $\approx$ on the set of Cauchy sequences by letting:

$$(a_n) \approx (b_n) \iff \lim_{n<\omega}(a_n - b_n) = 0.$$

Given this definition, we let $\mathbb{R}$ be the quotient set of the space of Cauchy sequences of rationals modulo the relation $\approx$. The field operations $+_{\mathbb{R}}$ and $\cdot_{\mathbb{R}}$ on (equivalence classes of) Cauchy sequences are defined coordinate-wise in the obvious way, with the corresponding identity elements being $0_{\mathbb{R}} = [(0,0,0,\ldots)]_{\approx}$ and $1_{\mathbb{R}} = [(1,1,1,\ldots)]_{\approx}$. Finally, we define the ordering on $\mathbb{R}$ by letting:

$$[(a_n)]_{\approx} \leqslant_{\mathbb{R}} [(b_n)]_{\approx} \iff (\exists\, n_0 \in \omega)\,(\forall\, n \geqslant n_0)\,(b_n - a_n \geqslant 0).$$

[7] Some authors start by first constructing the ring of integers $\mathbb{Z}$, and then proceed with the field $\mathbb{Q}$. Note that our construction of $\mathbb{Q}$ can easily account for the integers as well: for any $x \in \mathbb{Q}$, we clearly have that $x \in \mathbb{Z}$ if and only if there exist $n, m \in \omega$ such that $x = [(n,m,1)]_{\sim}$. Alternatively, one may define $\mathbb{Z}$ directly from the natural numbers, as the quotient of $\omega \times \omega$ modulo the equivalence relation $\sim_{\mathbb{Z}}$ defined by: $(n,m) \sim_{\mathbb{Z}} (n',m') \iff n + m' = n' + m$. Although these extra steps are not necessary, it is worth mentioning them since we shall refer to generalized (or long) integers below.

[8] See, for instance, Appendix A in Moschovakis [30], or Rudin [31] for more details and related background material.

One then checks that the resulting structure $\langle \mathbb{R}, 0_{\mathbb{R}}, 1_{\mathbb{R}}, +_{\mathbb{R}}, \cdot_{\mathbb{R}}, \leqslant_{\mathbb{R}} \rangle$ is a complete[9] ordered field in which the rationals can be embedded in a natural way.

Alternatively, one may follow Dedekind's method and work with the so-called *Dedekind cuts*.

Definition 2.2. A set $A \subseteq \mathbb{Q}$ is called a *Dedekind cut* if the following hold:

(a) $A \neq \varnothing$ and $A \neq \mathbb{Q}$.
(b) If $x \in A$ and $y < x$ then $y \in A$.
(c) For every $x \in A$ there is $y \in A$ such that $x < y$.

The real numbers can then be defined as:

$$\mathbb{R} = \{A \subseteq \mathbb{Q} : A \text{ is a Dedekind cut}\}.$$

The corresponding field operations $+_{\mathbb{R}}$ and $\cdot_{\mathbb{R}}$ are defined in a straightforward (but rather tedious) manner – we omit these details which may be found in an abundance of sources (for example, see the Appendix of Chapter 1 in the classical [31]). The identity elements of the field are given by $0_{\mathbb{R}} = \{x \in \mathbb{Q} : x < 0\}$ and $1_{\mathbb{R}} = \{x \in \mathbb{Q} : x < 1\}$, and the ordering is defined by letting $A \leqslant_{\mathbb{R}} B \iff A \subseteq B$. Finally, one again checks that the resulting structure is a complete ordered field.

Since there is a unique – up to isomorphism – complete[10] ordered field, both constructions are equivalent in the sense that they lead to isomorphic structures. Hence, from now on, we are justified in calling $\mathbb{R}$ *the* field of real numbers, and this is what we actually do.

3. Natural operations and exponentiation

Returning to our initial motivating question, we would like to generalize the previously described constructions in order to account for any cardinal $\kappa > \omega$. We are aiming at producing nice algebraic structures with these constructions; specifically, the corresponding version of the integers should be a commutative ring, and the corresponding versions of the rationals and the reals should be fields.

In this setting, the first thing to notice is that the ordinary operations on the ordinals (that is, addition and multiplication) fail to be even commutative,[11] which makes them unsuitable for our task. One solution to this problem is to use the *Hessenberg* ordinal operations $\oplus$ and $\otimes$ (also called *natural* operations), which were originally introduced by Hessenberg in [18]; see also Carruth [5]. These are defined as follows, for any ordinals α and β.

[9] Let us underline that *complete* here should perhaps be called ω–complete, stressing the fact that the cofinality (resp. coinitiality) of bounded sets for which suprema (resp. infima) may be found is ω.

[10] That is, ω–complete.

[11] For example, $1 + \omega = \omega < \omega + 1$.

For the natural sum, we let

$$\alpha \oplus \beta = \max\{\alpha, \beta\} + \min\{\alpha, \beta\}.$$

Equivalently, $\alpha \oplus \beta$ is the order-type of the *longest* well-order extending the disjoint union of α and β.

To define $\otimes$, we use the *Cantor normal form* and write (uniquely) the ordinals α and β as polynomials (in ω):

$$\alpha = p_\alpha(\omega) = \omega^{\alpha_0} \cdot n_0 + \omega^{\alpha_1} \cdot n_1 + \ldots + \omega^{\alpha_k} \cdot n_k,$$

where $k \in \omega$, $\alpha \geqslant \alpha_0 > \alpha_1 > \ldots > \alpha_k$ and $n_i \in \omega \setminus \{0\}$ for all $i < k+1$, and

$$\beta = p_\beta(\omega) = \omega^{\beta_0} \cdot m_0 + \omega^{\beta_1} \cdot m_1 + \ldots + \omega^{\beta_l} \cdot m_l,$$

where $l \in \omega$, $\beta \geqslant \beta_0 > \beta_1 > \ldots > \beta_l$ and $m_i \in \omega \setminus \{0\}$ for all $i < l+1$. We then let

$$\alpha \otimes \beta = p_\alpha(\omega) \cdot p_\beta(\omega),$$

where, for the latter operation, we compute the formal polynomial product of $p_\alpha(\omega)$ and $p_\beta(\omega)$, using $\oplus$ for all relevant additions. Equivalently, $\alpha \otimes \beta$ is the order-type of the *longest* well-order extending the product order on $\alpha \times \beta$.

Clearly, $\oplus$ and $\otimes$ are commutative and associative, 0 is the identity for $\oplus$, and 1 is the identity for $\otimes$. Moreover, the distributive law holds on both sides. These are of course the minimal requirements on any pair of operations on the ordinals relative to which our task can be carried out.

At this point it seems reasonable to query to which extent Hessenberg addition and Hessenberg multiplication constitute the only reasonable choice of operations to be considered in our setting. In this respect, note that if $\boxplus$ and $\boxtimes$ are a pair of operations satisfying the above minimal algebraic requirements, extending the usual addition and multiplication on ω, and such that $\alpha \boxplus \alpha = \alpha + \alpha = \alpha \cdot 2$ for every limit ordinal α (this last one looks like a reasonable "simplicity assumption"), then necessarily $\boxplus = \oplus$. Furthermore, $\boxtimes = \otimes$ if, in addition, we require that $\omega^\alpha \boxtimes \omega^\beta = \omega^{\alpha \boxplus \beta}$ for all ordinals α, β. This seems like a convenient extra simplicity condition on $\boxplus$ and $\boxtimes$, given that every ordinal can be uniquely expressed in Cantor normal form. This reduces the calculation of the product $\gamma \boxtimes \delta$, for any ordinals γ and δ, to a straightforward computation involving ultimately only addition and product of natural numbers. This extra simplicity condition seems all the more natural given the following additional consideration.

We could choose to represent ordinals in normal form choosing a (possibly) different ordinal η closed under ordinary multiplication (for example $\eta = \omega_1$); we would indeed have that for every ordinal α there are unique ordinals $\alpha_0 > \ldots > \alpha_n$ and $\beta_0, \ldots, \beta_n$, where $\beta_i \in \eta \setminus \{0\}$ for all i, such that $\alpha = \eta^{\alpha_0} \cdot \beta_0 + \ldots + \eta^{\alpha_n} \cdot \beta_n$. The above simplicity condition on $\boxplus$ and $\boxtimes$ expressed relative to normal forms in η–basis would yield that the product of any two ordinals can be reduced in a simple way to computations of additions of the form $\alpha \boxplus \beta$, for ordinals α, $\beta < \eta$, which we know how to perform as necessarily $\boxplus = \oplus$, together with computations of products $\alpha \boxtimes \beta$, again for α, $\beta < \eta$. But this would beg the question of finding $\alpha \boxtimes \beta$ for these choices of α, β. If we are to apply our simplicity condition to this product, we will need to choose a new basis $\eta_0 < \eta$, if $\omega < \eta$, relative to which to represent α and β in normal form; using those normal forms we will then be able to compute $\alpha \boxtimes \beta$. Iterating this construction we can easily check that if we impose this extra simplicity condition on $\boxplus$ and $\boxtimes$, then necessarily $\gamma \boxtimes \delta = \gamma \otimes \delta$ for all ordinals γ, δ.

Let us now define the class of ordinals to which the relevant construction in this article will apply.

Definition 3.1. An ordinal κ is called *naturally closed* if $\alpha \oplus \beta < \kappa$ and $\alpha \otimes \beta < \kappa$, for all ordinals α, $\beta < \kappa$.

Equivalently, an ordinal is naturally closed if and only if it is closed under ordinary ordinal addition and multiplication. In standard set-theoretic terminology, this means that an ordinal is naturally closed if and only if it is both additively and multiplicatively indecomposable. The naturally closed ordinals are 0 and the ordinals of the form $\omega^{(\omega^\alpha)}$, for some $\alpha \in \mathbf{ON}$. In particular, the first non-zero naturally closed ordinal is ω, and the second one is ω^ω.

Now, let κ be some fixed non-zero naturally closed ordinal and consider the commutative ordered semiring $\langle\, \kappa\,, 0\,, 1\,, \oplus\,, \otimes\,, <\rangle$.[12] It should be mentioned, at this point, that such ordered semirings (of all ordinals less than an infinite naturally closed ordinal with sums and products defined *à la* Hessenberg) were introduced by Ehrlich in [13], in connection with a generalization of the Archimedean condition; they were employed again by van den Dries and Ehrlich in [10].

Using the exact same procedures that produce $\mathbb{Z}$ and $\mathbb{Q}$ from ω, we may construct the ordered ring of the "κ–integers" (denoted by κ–$\mathbb{Z}$) and, then, the ordered field of the "κ–rationals" (denoted by κ–$\mathbb{Q}$). In this terminology, ω–$\mathbb{Z}$ and ω–$\mathbb{Q}$ are just the standard integer and rational numbers, respectively.

Obviously, we have the natural inclusion κ–$\mathbb{Z} \subseteq \kappa$–$\mathbb{Q}$, for all κ. Furthermore, it is also clear that, for any $\kappa < \lambda$, we have natural inclusions κ–$\mathbb{Z} \subseteq \lambda$–$\mathbb{Z}$ and κ–$\mathbb{Q} \subseteq \lambda$–$\mathbb{Q}$.

Notation. From now on, and in order to ease readability, we will write $+$ and $\cdot$ instead of $\oplus$ and $\otimes$, although we will be exclusively using the

[12] Our initial motivation was to perform these constructions with any cardinal $\kappa > \omega$. Being a cardinal is of course not the issue when we choose to use natural operations in the constructions, but rather the weaker condition of being closed under these operations.

latter operations, as they were defined above (unless otherwise mentioned). Moreover, we shall sometimes drop the multiplication symbol altogether and write $\alpha\beta$ instead of $\alpha \otimes \beta$. Once again, expressions of the form $-\alpha$ and $\frac{1}{\alpha}$ have the intended meaning in the context of a field.

Note that, in the typical case of interest in which $\kappa > \omega$ is a naturally closed ordinal, the field κ–$\mathbb{Q}$ contains elements of the form $\omega - 3$, $\omega^3 - 1$, $\frac{1}{\omega^2}$, $\omega^{\omega-2}$ (where this is meant to stand for $\frac{\omega^\omega}{\omega^2}$), etc. It is important to keep in mind that exponentiation is always understood as *ordinary ordinal exponentiation.* Even though one may try to define an appropriate notion of "natural exponentiation", one that behaves in the expected way with respect to the natural product, it turns out that this is impossible, as highlighted by the next two results.[13]

Theorem 3.2. *There is no function $f : \omega \longrightarrow \mathbf{ON}$ such that:*

- (i) $f(2) \geqslant \omega$.
- (ii) *For all* $n, m \in \omega$, *if* $n < m$ *then* $f(n) \leqslant f(m)$.
- (iii) *For all* $n, m \in \omega$, $f(n \cdot m) = f(n) \cdot f(m)$.

Proof. Suppose, towards a contradiction, that $f : \omega \longrightarrow \mathbf{ON}$ is such a function. For every $n \geqslant 2$, we have that

$$f(n) = \omega^{\beta_n} \cdot b_n + p_n(\omega),$$

where $\beta_n, b_n > 0$ and $p_n(\omega)$ is a polynomial in ω of degree smaller than β_n. In turn, we can also write the (non-zero) ordinal β_n as:

$$\beta_n = \omega^{\alpha_n} \cdot a_n + q_n(\omega),$$

where $a_n > 0$ and $q_n(\omega)$ is a polynomial in ω of degree smaller than α_n. We call the coefficient $a_n \in \omega$ the f–order of n and write:

$$a_n = o_f(n).$$

Note that, for all $n \geqslant 2$, we have that $o_f(n) \geqslant 1$. Also, for every $k > 0$, condition (iii) gives that $o_f(2^k) = k \cdot o_f(2)$, where in fact

$$f(2^k) = (\omega^{k \cdot o_f(2)\omega^{\alpha_2} + q}) \cdot (b_2)^k + p,$$

where q and p are polynomials in ω of degrees smaller than α_2 and $k \cdot o_f(2)\omega^{\alpha_2} + q$ respectively.

Clearly, for every $n \geqslant 2$ there is a $k > 0$ such that $2^k \leqslant n < 2^{k+1}$. From the latter and the monotonicity condition (ii), it then follows that, for such n and k:

$$f(n) = \omega^{\beta_n} \cdot b_n + p_n,$$

where $\beta_n = o_f(n) \cdot \omega^{\alpha_2} + q_n$, and q_n, p_n are polynomials in ω of small degrees, and then

$$k \cdot o_f(2) \leqslant o_f(n) \leqslant (k+1) \cdot o_f(2).$$

[13] A similar result has recently appeared in Altman [2].

Note that, for all $n \geqslant 2$, the exponent α_n must be fixed since, otherwise, condition (ii) would be violated. In other words, for all $n \geqslant 2$, we indeed have that $\alpha_n = \alpha_2$. From this observation it additionally follows that, for all $n, m \geqslant 2$, the f–order satisfies the following properties:

(1) $o_f(n \cdot m) = o_f(n) + o_f(m)$;
(2) if $n < m$ then $o_f(n) \leqslant o_f(m)$.

Therefore, there exists some $k > 0$ (in fact, $k = o_f(2)$ works) and some $n \in [2^k, 2^{k+1})$ such that $o_f(n) = o_f(n+1)$.

Now fix some $s \in \omega$ such that $s > \dfrac{\ln 2}{\ln\left(1+\frac{1}{n}\right)}$ and note that, by choice of s, we have that $2n^s < (n+1)^s$ which moreover implies that

$$o_f(n^s) < o_f(n^s) + o_f(2) = o_f(2n^s) \leqslant o_f((n+1)^s),$$

that is, $o_f(n^s) < o_f((n+1)^s)$. But this contradicts property (1) of the f–order and the fact that $o_f(n) = o_f(n+1)$. $\square$

The functional properties in the statement of Theorem 3.2 are a natural set of properties that the function sending $n \in \omega$ to $\exp(n, \omega)$ should surely satisfy, for any reasonable choice of a generalized exponentiation $\exp(\alpha, \beta)$.

Given that the existence of such a function is ruled out, we can therefore conclude that there can be no such generalized "natural exponentiation", as stated below.

Corollary 3.3. *Let $\kappa > \omega$ be an ordinal. Then, there is no function* $\exp : \kappa \times \kappa \longrightarrow \boldsymbol{ON}$ *such that:*

(i) *For all $n, m \in \omega$,* $\exp(n \cdot m, \omega) = \exp(n, \omega) \cdot \exp(m, \omega)$.
(ii) *For all $n, m \in \omega$, if $n < m$ then* $\exp(n, \omega) \leqslant \exp(m, \omega)$.
(iii) *For all $n \in \omega$ and for all $\alpha < \beta < \kappa$,* $\exp(n, \alpha) \leqslant \exp(n, \beta)$.

Proof. If such a function existed, then we would easily produce a counterexample to Theorem 3.2 by considering the function $f : \omega \longrightarrow \mathbf{ON}$ defined by: $f(n) = \exp(n, \omega)$. $\square$

If K is an ordered field, the absolute value function on K has the obvious meaning: if $a \in K$, then we write $|a|$ to denote a if $a \geqslant 0$, and to denote $-a$ if $a < 0$.

The following theorem provides us with a very useful representation of elements in κ–$\mathbb{Q}$.

Theorem 3.4. (Representation theorem)
Let $\kappa > \omega$ be a naturally closed ordinal and let λ be such that $\kappa = \omega^\lambda$. Let $x \in \kappa$–$\mathbb{Q}$ and fix some $0 < \alpha < \lambda$ such that $|x| > \frac{1}{\omega^\alpha}$. Then, there exist (unique) $n \in \omega$, $q_0, \ldots, q_n \in \omega$–$\mathbb{Q} \setminus \{0\}$, $\beta_0, \ldots, \beta_n \in \lambda$–$\mathbb{Z}$ with $\beta_0 < \beta_1 < \ldots < \beta_n \leqslant \alpha$, and $r_\alpha \in \kappa$–$\mathbb{Q}$ such that

$$x = q_0 \cdot \frac{1}{\omega^{\beta_0}} + q_1 \cdot \frac{1}{\omega^{\beta_1}} + \ldots + q_n \cdot \frac{1}{\omega^{\beta_n}} + r_\alpha,$$

and $|r_\alpha| < \dfrac{1}{\omega^\alpha}$.

Proof. The conclusion follows from a straightforward application of the Euclidean algorithm for polynomial division, once we have expressed the given $x = \frac{\alpha}{\beta}$ using the Cantor normal forms of its numerator and denominator. □

Remark 3.5. Note the slight abuse of notation in the statement of Theorem 3.4 when referring to members of λ–$\mathbb{Z}$ in cases when $\lambda < \kappa$ and λ–$\mathbb{Z}$ may not be defined according to our official restriction to naturally closed ordinals.[14] This is for notational convenience. Strictly speaking, we should say that there exist $n, m \in \omega$ with $\max\{n, m\} > 0$, $p_0, \ldots, p_{m-1}, q_0, \ldots, q_{n-1} \in \omega$–$\mathbb{Q} \setminus \{0\}$, $\beta_0, \ldots, \beta_{n-1}, \gamma_0, \ldots, \gamma_{m-1} \in \lambda$ such that $\gamma_0 < \ldots < \gamma_{m-1}$ and $\beta_0 < \beta_1 < \ldots < \beta_{n-1} \leqslant \alpha$, and $r_\alpha \in \kappa$–$\mathbb{Q}$ such that

$$x = p_0 \cdot \omega^{\gamma_0} + \ldots + p_{m-1} \cdot \omega^{\gamma_{m-1}} + \frac{q_0}{\omega^{\beta_0}} + \ldots + \frac{q_{n-1}}{\omega^{\beta_{n-1}}} + r_\alpha,$$

and $|r_\alpha| < \dfrac{1}{\omega^\alpha}$.

Throughout, we will typically indulge in this type of abuse of notation when using Theorem 3.4.

Theorem 3.4 says that we may (uniquely) approximate any κ–rational up to any desired degree of "precision", where the latter is given by the (possibly infinitesimal) integer power $\frac{1}{\omega^\alpha}$. In this context, we will refer to the quantity $x - r_\alpha$ as the *α–approximation* of x, to α as the *order* of the approximation, and to r_α as the *remainder*.

Given the so far constructed fields κ–$\mathbb{Q}$ (for various infinite κ), and having sorted out issues regarding basic arithmetic and representation, we may now proceed to the next step which will bring us to the core of our present study.

4. The κ–reals

Let $\kappa > \omega$ be a fixed naturally closed ordinal and let us consider the field κ–$\mathbb{Q}$. It is only natural to wonder whether one may "complete" this field in order to produce the corresponding "κ–reals". For this, we first define what it means for a sequence of κ–rationals to be Cauchy, as follows (cf. Definition 2.1).

Definition 4.1. Let κ be a non-zero naturally closed ordinal and let $(a_\xi)_{\xi<\lambda}$ be a sequence of κ–rationals, for some $\lambda \leqslant \kappa$. We say that the

[14] Incidentally, note that we could nevertheless still define λ–$\mathbb{Z}$ also in this case, which would be closed under addition, although it might not be closed under multiplication.

sequence is *Cauchy* if for every $\alpha < \kappa$ there exists some $\xi_0 < \lambda$ such that, for all $\xi, \xi' > \xi_0$,

$$|a_\xi - a_{\xi'}| < \frac{1}{\alpha + 1} .$$

Of course, for any such κ, the basic example is the Cauchy sequence

$$\left(\frac{1}{\xi + 1}\right)_{\xi < \kappa} .$$

On the other hand, the basic non-example (for $\kappa > \omega$) is the sequence

$$\left(\frac{1}{n + 1}\right)_{n < \omega} ,$$

which – despite the fact that it is strictly decreasing and bounded – is not Cauchy because, for all $0 < n, n' < \omega$, we clearly have that

$$\left|\frac{1}{n} - \frac{1}{n'}\right| > \frac{1}{\omega} .$$

Note that if $\kappa > \omega$ and $\mathrm{cf}(\kappa) = \omega$, then the above non-example can be turned into one of length κ: let $(\alpha_n)_{n<\omega}$ be a strictly increasing sequence converging to κ and consider

$$\left(\frac{1}{n_\xi + 1}\right)_{\xi < \kappa}$$

where, for each ξ, n_ξ is the least $n < \omega$ such that $\xi \leqslant \alpha_n$.

As a related comment, note that, for any $\kappa > \omega$, the field κ–$\mathbb{Q}$ is *not* Archimedean. Nevertheless, it satisfies a generalized *κ–Archimedean property*: namely, for every $x \in \kappa$–$\mathbb{Q}$, there exists some $\alpha < \kappa$ such that $x < \alpha$.[15] This property is what makes Definition 4.1 the reasonable generalization of the standard case.

For any κ, Cauchy sequences in κ–$\mathbb{Q}$ are necessarily bounded. Although, as we just saw, the converse may fail even for monotone sequences of length κ, it is true for regular cardinals κ (when restricting to monotone sequences of length κ).

Proposition 4.2. *Suppose that $\kappa \geqslant \omega$ is a regular cardinal and let $(a_\xi)_{\xi<\kappa}$ be an increasing (or a decreasing) sequence of κ–rationals. Then, $(a_\xi)_{\xi<\kappa}$ is Cauchy if and only if it is bounded.*

Proof. It is easy to see that the forward direction is true in general. For the converse, fix some regular cardinal $\kappa > \omega$ and suppose that the sequence $(a_\xi)_{\xi<\kappa}$ is increasing and bounded in κ–$\mathbb{Q}$; that is, there is

[15] Moreover, it can be shown that an ordered field of cofinality κ is order complete (in its cofinality) if and only if it is Cauchy-complete and has the κ–Archimedean property. For the proof, one proceeds by a direct modification of the standard arguments (see, for example, Appendix A in Moschovakis [30]). See also Ehrlich [11] for some results on generalized Archimedean properties.

some $x \in \kappa$–$\mathbb{Q}$ such that, for all $\xi < \kappa$, we have that $a_\xi < x$. Without loss of generality, suppose that $a_\xi \geqslant 0$ for all $\xi < \kappa$.

Towards a contradiction, suppose that the sequence is not Cauchy. Thus, there is some $\alpha < \kappa$ such that, for all $\xi_0 < \kappa$, there are $\xi, \xi' > \xi_0$ with

$$|a_\xi - a_{\xi'}| \geqslant \frac{1}{\alpha + 1} = \varepsilon.$$

In other words, if we consider the distances between various terms of the sequence, we have that unboundedly often in length κ we encounter (disjoint) intervals each of which has length at least ε.

Now, by the κ–Archimedean property of κ–$\mathbb{Q}$, if x is an upper bound for the sequence, then there is some $\beta < \kappa$ such that $x < \varepsilon \cdot \beta$. But this implies that there is some $\xi < \kappa$ such that $x \leqslant a_\xi$, and this contradicts the fact that x is an upper bound of the sequence, since there will be some $\xi' > \xi$ such that $x < a_{\xi'}$.

Clearly, a similar argument works in the case of a decreasing and bounded κ–sequence in κ–$\mathbb{Q}$. □

By a straightforward adaptation of our basic non-example, it easily follows that the above proposition characterizes infinite regular cardinals as exactly those non-zero naturally closed ordinals κ (regardless of their cofinality) such that bounded monotone κ–sequences in κ–$\mathbb{Q}$ are necessarily Cauchy.

Given the notion of a Cauchy sequence, we may now consider the completion of the κ–rationals with respect to such sequences, appealing to the usual construction that produces $\mathbb{R}$ from $\mathbb{Q}$. This results in the complete ordered field of the κ–reals, which we denote by κ–$\mathbb{R}$, and in which κ–$\mathbb{Q}$ can be embedded in a natural way. In this terminology, ω–$\mathbb{R}$ denotes the field of ordinary real numbers.

Evidently, for any non-zero naturally closed ordinals $\kappa \leqslant \lambda$ we have the inclusion κ–$\mathbb{Q} \subseteq \lambda$–$\mathbb{R}$. However, we shall show below (see the remarks after Corollary 4.10) that the inclusion κ–$\mathbb{R} \subseteq \lambda$–$\mathbb{R}$ does not hold in general; in fact, for any $\kappa > \omega$ we have that ω–$\mathbb{R} \nsubseteq \kappa$–$\mathbb{R}$.

Notation. In order to avoid ambiguities, given $x < y$ in κ–$\mathbb{R}$, we shall denote the corresponding (open) interval of the field κ–$\mathbb{R}$ by $(x, y)_\kappa$. As we just mentioned, it is not true in general that $(x, y)_\omega \subseteq (x, y)_\kappa$, for $x < y$ in ω–$\mathbb{Q}$.

An alternative construction of the field κ–$\mathbb{R}$ can be done via Dedekind cuts. The problem with such a construction is that one has to be careful with the sort of cuts that are chosen; for instance, the "cut"

$$\left\{ x \in \kappa\text{–}\mathbb{Q} : \, (\forall n \in \omega) \, \left(x < \frac{1}{n+1} \right) \right\}$$

would fail to define a κ–real, highlighting the fact that additional restrictions should be imposed. Quite naturally, it is enough to consider only the cuts that have some Cauchy sequence which is unbounded in them. Then, Dedekind's construction goes through and the resulting object is isomorphic to the one obtained via generalized Cauchy sequences – we omit the details. From now on, we will write κ–$\mathbb{R}$ to mean the object obtained by either of these two methods.

The field κ–$\mathbb{R}$, despite its (Cauchy) completeness, remains incomplete in various ways: note that a sequence converges to a point in the field only if it has cofinality $\mathrm{cf}(\kappa)$. Similarly, if $X \subseteq \kappa$–$\mathbb{R}$ has $\mathrm{cf}(X) < \mathrm{cf}(\kappa)$, then X does not have a least upper bound in the field. In fact, κ–$\mathbb{R}$ is far from being a *continuum* since it has many "holes". For instance:

Proposition 4.3. *For every $\kappa > \omega$, there is no $x \in \kappa$–$\mathbb{Q}$ such that*

$$\left\{\frac{n}{\omega} : n \in \omega\right\} < \{x\} < \left\{\frac{1}{n+1} : n \in \omega\right\}.$$

Proof. Suppose, towards a contradiction, that there are infinite ordinals $\alpha, \beta \in \kappa$ such that, for all $n \in \omega$,

$$\frac{n}{\omega} < \frac{\alpha}{\beta} < \frac{1}{n+1}.$$

For the purposes of this proof, it is important to distinguish between the ordinary and the natural ordinal operations; hence, we temporarily return to the initial notation according to which we use $+$ and $\cdot$ for the former, and $\oplus$ and $\otimes$ for the latter operations.

Now, the above inequalities can be equivalently stated as:

$$\beta \cdot n = \beta \otimes n < \alpha \otimes \omega$$

and

$$\alpha \cdot n = \alpha \otimes n < \beta,$$

for every $n \in \omega$. In turn, these imply that

$$\beta \cdot \omega \leqslant \alpha \otimes \omega$$

and

$$\alpha \cdot \omega \leqslant \beta.$$

We did not get a contradiction yet, since $\alpha \cdot \omega \leqslant \alpha \otimes \omega$ in general. We may nevertheless argue as follows.

Consider the Cantor normal form $\alpha = \omega^{\alpha_0} \cdot n_0 + \ldots + \omega^{\alpha_k} \cdot n_k$, where $k \in \omega$, $\alpha \geqslant \alpha_0 > \alpha_1 > \ldots > \alpha_k$ and $n_i \in \omega$ for all $i < k+1$, and note that

$$\alpha \otimes \omega = \omega^{\alpha_0+1} \cdot n_0 + \ldots + \omega^{\alpha_k+1} \cdot n_k,$$

while

$$\alpha \cdot \omega = \omega^{\alpha_0+1}.$$

Hence, since $\alpha \cdot \omega \leqslant \beta$, we get that $\omega^{\alpha_0+2} \leqslant \beta \cdot \omega$ which, combined with the inequality $\beta \cdot \omega \leqslant \alpha \otimes \omega$, consequently gives

$$\omega^{\alpha_0+2} \leqslant \omega^{\alpha_0+1} \cdot n_0 + \ldots + \omega^{\alpha_k+1} \cdot n_k,$$

which is the desired contradiction. □

Recall that a linear order $\langle L, <_L \rangle$ is called *κ–saturated*, for some infinite cardinal κ, if for every $A, B \subseteq L$ with $|A|+|B| < \kappa$ and $A <_L B$ there exists some $x \in L$ such that $A <_L \{x\} <_L B$. Such linear orders were first considered by Hausdorff in the early 20th century (cf. [17]), and have been extensively studied ever since. For instance, Alling's work from 1962 on κ–saturated real closed fields is also relevant (cf. [1]).

In this context, and as an immediate corollary of the previous proposition, we have the following:

Corollary 4.4. *For every $\kappa > \omega$, κ–$\mathbb{R}$ is never $\aleph_1$–saturated as a linear order.*

In spite of this deficit, and for regular κ, the κ–order completeness of κ–$\mathbb{R}$ does imply a weak form of κ^+–saturation: for any $A, B \subseteq \kappa$–$\mathbb{R}$ with $A < B$ and such that $\mathrm{cf}(A) = \mathrm{coin}(B) = \kappa$, there exists some $x \in \kappa$–$\mathbb{R}$ such that $A < \{x\} < B$.

Moreover, κ–$\mathbb{R}$ is also incomplete (or, better, not closed) from an algebraic point of view. Of course, in the case of $\kappa = \omega$, the standard reals are certainly not algebraically closed, but they do include, for example, various n–th roots of their elements. On the other hand, when $\kappa > \omega$, things are different in this respect as we will see below (see the discussion after Corollary 4.10).

The following lemma (whose easy proof we omit) says that every κ–irrational is the limit of a strictly increasing sequence of κ–rationals, of length $\mathrm{cf}(\kappa)$.

Lemma 4.5. *Let κ be a non-zero naturally closed ordinal and fix some $x \in \kappa$–$\mathbb{R} \setminus \kappa$–$\mathbb{Q}$. Then, there is a strictly increasing sequence $(a_\xi)_{\xi<\mathrm{cf}(\kappa)}$ in κ–$\mathbb{Q}$ such that $x = \sup_{\xi<\mathrm{cf}(\kappa)} a_\xi$.*

Clearly, for every non-zero naturally closed ordinal κ we have that κ–$\mathbb{Q} \subseteq \kappa$–$\mathbb{R}$. It is perhaps natural to expect that this inclusion be strict, as is the case for the standard reals ω–$\mathbb{R}$. Perhaps surprisingly, this is not true in general. In fact, we have the following characterization:

Theorem 4.6. *For any non-zero naturally closed ordinal κ, we have that $\mathrm{cf}(\kappa) > \omega$ if and only if κ–$\mathbb{Q} = \kappa$–$\mathbb{R}$.*

Proof. If $\kappa > \omega$ and $\mathrm{cf}(\kappa) = \omega$, then there are κ–irrational numbers in κ–$\mathbb{R}$. To see this, let us first fix a strictly increasing sequence of infinite ordinals $\langle \alpha_n : n \in \omega \rangle$, with $\sup_{n\in\omega} \alpha_n = \kappa$. Now consider the sequence $(a_n)_{n\in\omega}$ where, for every $n \in \omega$,

$$a_n = \sum_{i\leqslant n} \frac{1}{\alpha_i}\,.$$

Note that a_n is a well-defined element of κ–$\mathbb{Q}$. It is immediate to see that the sequence $(a_n)_{n\in\omega}$ is Cauchy and, thus, it converges to some $x \in \kappa$–$\mathbb{R}$ which we naturally denote by:

$$x = \lim_{n<\omega} a_n =_{\text{def.}} \sum_{i<\omega} \frac{1}{\alpha_i}\,.$$

One easily checks that $x \neq \frac{\alpha}{\beta}$ for all $\alpha, \beta < \kappa$, and therefore we have that $x \in \kappa$–$\mathbb{R} \setminus \kappa$–$\mathbb{Q}$ as desired.

For the other direction, suppose that $\mathrm{cf}(\kappa) > \omega$, and let us focus on the unit interval $(0,1)_\kappa$ of κ–$\mathbb{R}$, for which we shall see that $(0,1)_\kappa \subseteq \kappa$–$\mathbb{Q}$. We leave it to the reader to verify that a similar argument shows that $(x,y)_\kappa \subseteq \kappa$–$\mathbb{Q}$ for every $x < y$ in κ–$\mathbb{Q}$.

So, let $(a_\xi)_{\xi<\mathrm{cf}(\kappa)}$ be a Cauchy sequence of κ–rationals in $(0,1)_\kappa \cap \kappa$–$\mathbb{Q}$ converging to some $x \in (0,1)_\kappa$, with $x \notin \kappa$–$\mathbb{Q}$. We may assume, by Lemma 4.5, that $(a_\xi)_{\xi<\mathrm{cf}(\kappa)}$ is strictly increasing. In particular, we have that $a_\xi < x$ for all $\xi < \mathrm{cf}(\kappa)$.

Let λ be such that $\kappa = \omega^\lambda$. Since the sequence $(a_\xi)_{\xi<\mathrm{cf}(\kappa)}$ does not converge to 0, there exists some $\alpha_0 < \lambda$ such that for all $\xi < \mathrm{cf}(\kappa)$ there is a further index $\xi' > \xi$ with

$$a_{\xi'} \geqslant \frac{1}{\omega^{\alpha_0}}\,.$$

Now let $\xi_0 < \mathrm{cf}(\kappa)$ be such that

$$|x - a_\xi| = x - a_\xi < \frac{1}{\omega^{\alpha_0}}$$

for all $\xi > \xi_0$.

Recalling the representation given by Theorem 3.4, it follows that, for all $\xi > \xi_0$, the α_0-approximation of the rational a_ξ agrees with the real x up to order α_0; that is, for all $\xi > \xi_0$,

$$a_\xi = q_0 \cdot \frac{1}{\omega^{\beta_0}} + q_1 \cdot \frac{1}{\omega^{\beta_1}} + \ldots + q_n \cdot \frac{1}{\omega^{\beta_n}} + r^{(\xi)}_{\alpha_0},$$

for some fixed $n \in \omega$, $q_0, \ldots, q_n \in \omega$–$\mathbb{Q} \setminus \{0\}$, $\beta_1, \ldots, \beta_n \in \lambda$–$\mathbb{Z}$ with $\beta_0 < \beta_1 < \ldots < \beta_n \leqslant \alpha_0$, and remainders $r^{(\xi)}_{\alpha_0} < \dfrac{1}{\omega^{\alpha_0}}$ which depend on the index ξ only. In other words, for indices greater than ξ_0 we have that the α_0–approximation of the elements of the sequence is "frozen" and, moreover, the same is true for the real x:

$$x = q_0 \cdot \frac{1}{\omega^{\beta_0}} + q_1 \cdot \frac{1}{\omega^{\beta_1}} + \ldots + q_n \cdot \frac{1}{\omega^{\beta_n}} + s_{\alpha_0},$$

for some $s_{\alpha_0} < \dfrac{1}{\omega^{\alpha_0}}$, with $s_{\alpha_0} \notin \kappa$–$\mathbb{Q}$ by the assumption $x \notin \kappa$–$\mathbb{Q}$.

Starting with α_0 and ξ_0, we may construct in a similar manner, recursively for $i < \omega$, a strictly increasing sequence of (non-zero) ordinals $\{\alpha_i : i \in \omega\} \subseteq \lambda$, along with a corresponding sequence of ordinals

$\{\xi_i : i \in \omega\} \subseteq \mathrm{cf}(\kappa)$ so that, for each $i \in \omega$ and for all indices $\xi > \xi_i$, we have that

$$x - a_\xi < \frac{1}{\omega^{\alpha_i}},$$

and, moreover, the α_i–approximation of the rational a_ξ agrees with the real x up to order α_i. In other words, we successively "freeze" longer and longer approximations of the real x, according to the orders of approximation α_i. The assumption $x \notin \kappa$–$\mathbb{Q}$ ensures that the construction does not stabilize after finitely many steps.

Given the sequences $(\alpha_i)_{i\in\omega}$ and $(\xi_i)_{i\in\omega}$, we now use the fact that $\mathrm{cf}(\kappa) > \omega$ and pick some index $\eta < \kappa$ so that $\eta > \sup_{i<\omega} \xi_i$. We furthermore let $\beta = \sup_{i<\omega} \alpha_i < \lambda$ and consider the β–approximation of the rational element a_η. The desired contradiction now follows from the fact that the latter approximation must agree on all of its initial segments with the α_i–approximations of the rationals a_{ξ_i}, for every $i \in \omega$. But this is impossible, since the β–approximation of a_η must be a finitary object. □

By the previous theorem, it follows that if κ has uncountable cofinality, then the κ–rationals are already (Cauchy) complete[16] and, therefore, $|\kappa\text{–}\mathbb{Q}| = |\kappa\text{–}\mathbb{R}| = \kappa$.

Theorem 4.6 makes the study of κ–$\mathbb{R}$ for $\mathrm{cf}(\kappa) > \omega$ uninteresting, and in fact it implies that this is a degenerate case of the theory. In what follows, and except for some general results in Section 6 (see Theorem 6.1 and Corollary 6.3), we will mainly focus on the case of countable cofinality, where things become more interesting from a set-theoretic point of view.

As we have already seen, for non-zero naturally closed ordinals κ of countable cofinality there exist κ–irrational numbers; we will now in fact determine the cardinality of κ–$\mathbb{R} \setminus \kappa$–$\mathbb{Q}$.

Theorem 4.7. *For any naturally closed ordinal $\kappa > \omega$ with $\mathrm{cf}(\kappa) = \omega$, $|\kappa\text{–}\mathbb{R}| = \kappa^{\aleph_0}$. In fact, if λ is such that $\kappa = \omega^\lambda$, then every $x \in \kappa$–$\mathbb{R}$ is represented (uniquely) by an expression of the form:*

$$x = p_0 \cdot \omega^{\alpha_0} + \ldots + p_n \cdot \omega^{\alpha_n} + \sum_{i<\mu} \frac{q_i}{\omega^{\beta_i}},$$

where $\mu \leqslant \omega$, $n \in \omega$, $p_0, \ldots, p_n, q_i \in \omega$–$\mathbb{Q} \setminus \{0\}$, $\alpha_j, \beta_i \in \lambda$ with $\alpha_0 > \ldots > \alpha_n$, and where $(\beta_i)_{i<\mu}$ is a strictly increasing sequence of ordinals in λ which, in addition, is cofinal in λ if $\mu = \omega$. Moreover, every such expression determines a member of κ–$\mathbb{R}$.

Proof. Fix some naturally closed ordinal κ with $\mathrm{cf}(\kappa) = \omega$ and some $x \in \kappa$–$\mathbb{R}$. It is enough to argue for the displayed representation of x

[16] But, obviously, this completeness is not an indication of "richness" of the field; quite the opposite. A similar result, alas for uncountable regular cardinals, was obtained by Sikorski; see [33].

(and its uniqueness), from which the equality $|\kappa\text{–}\mathbb{R}| = \kappa^{\aleph_0}$ will easily follow since every such expression clearly determines a κ–real.

For this, we shall evidently use Theorem 3.4 and some relevant ideas from the proof of Theorem 4.6. Note that if $x \in \kappa\text{–}\mathbb{Q}$, then the desired representation is clear from Theorem 3.4, where in such a case $\mu < \omega$.

If $x \in \kappa\text{–}\mathbb{R} \setminus \kappa\text{–}\mathbb{Q}$, we fix a strictly increasing sequence of infinite ordinals $\langle \alpha_m : m \in \omega \rangle$, with $\sup_{m<\omega} \alpha_m = \lambda$. Furthermore, we fix some strictly increasing Cauchy sequence $(a_m)_{m<\omega}$ converging to x and such that, for every $m \in \omega$, we have that

$$|x - a_m| < \frac{1}{\alpha_m} .$$

We then consider, for each $m \in \omega$, the α_m–approximation of a_m, which we denote by A_m. Note that this does not depend on a_m since, by the choice of the sequence, for any $k > m$ we have that the α_m–approximation of a_k agrees with A_m.[17] For each $m \in \omega$, we may thus write the approximation A_m as:

$$A_m = p_0 \cdot \omega^{\beta_0} + \ldots + p_n \cdot \omega^{\beta_n} + \sum_{i \leqslant k_m} \frac{q_i}{\omega^{\gamma_i}} ,$$

for appropriate parameters and some $k_m \in \omega$. But now the desired representation of x follows, since by definition of convergence we have that $\lim_{m<\omega} A_m = x$.

For elements in $\kappa\text{–}\mathbb{Q}$, the uniqueness of the representation follows from Theorem 3.4. To argue for the κ–irrationals, it is enough to notice that distinct representations must already differ for some least order of approximation, which implies that, from this point on, they cannot converge to the same real $x \in \kappa\text{–}\mathbb{R}$; we leave the details to the reader. □

Note how the above theorem generalizes, both in terms of cardinality and in terms of representation, the standard case of $\omega\text{–}\mathbb{R}$ to any naturally closed ordinal of countable cofinality.

Moreover, again when $\mathrm{cf}(\kappa) = \omega$, it allows us to represent the elements of $(0,1)_\kappa$ naturally as branches of an appropriate subtree of ${}^{<\kappa}\omega$.

Definition 4.8. Given a naturally closed ordinal $\kappa > \omega$ with $\mathrm{cf}(\kappa) = \omega$, if λ is such that $\kappa = \omega^\lambda$, then we define the tree $\kappa\text{–}\mathbb{T}$ by letting:

$$s \in \kappa\text{–}\mathbb{T} \iff s \in {}^{<\lambda}\omega \ \wedge \ |\mathrm{supp}(s)| < \aleph_0.$$

[17] Recall that a similar argument was used in the proof of Theorem 4.6. In this sense, the consideration of the Cauchy sequence is only an auxiliary step which may be by-passed; in fact, we could have considered directly the α–approximations of the κ–reals, and not just those of the κ–rationals, although this does not make much difference after all.

Clearly, κ–$\mathbb{T}$ is an ω–branching tree of height λ.

Let us fix a bijection $\pi : \omega \longrightarrow \omega$–$\mathbb{Q}$ such that $\pi(0) = 0$. Then, the representation given in Theorem 4.7 yields that branches of κ–$\mathbb{T}$ naturally correspond to reals in $(0,1)_\kappa$, and vice versa.[18] This correspondence, in one direction, is given by the map $c_\kappa : [\kappa\text{–}\mathbb{T}] \longrightarrow (0,1)_\kappa$ defined by:

$$c_\kappa(b) = \sum_{\alpha \in \operatorname{supp}(b)} \frac{\pi(b(\alpha))}{\omega^\alpha},$$

which is clearly a bijection between $[\kappa\text{–}\mathbb{T}]$ and $(0,1)_\kappa$. Given $X \subseteq [\kappa\text{–}\mathbb{T}]$, we will denote the set $\{c_\kappa(b) : b \in X\}$ by $c_\kappa(X)$.

The following notion, together with the aforementioned tree representation, will be very useful in Section 7.

Definition 4.9. Let $\kappa > \omega$ be a naturally closed ordinal with $\operatorname{cf}(\kappa) = \omega$, let λ be such that $\kappa = \omega^\lambda$, and let $\langle \alpha_n : n \in \omega \rangle$ be an increasing sequence of ordinals with $\alpha_0 = 0$ and $\sup_{n<\omega} \alpha_n = \lambda$. Then, for any non-empty $s \in \kappa$–$\mathbb{T}$, we denote by $o(s)$ the *order* of s in κ–$\mathbb{T}$ (with respect to $\langle \alpha_n : n \in \omega \rangle$), which is defined as the unique $n \in \omega$ such that $\max(\operatorname{supp}(s)) \in [\alpha_n, \alpha_{n+1})$.

The following basic result should be clear by now, but let us stress it by stating it as a corollary to Theorems 4.6 and 4.7.

Corollary 4.10. *For every infinite κ, κ–$\mathbb{Q}$ is dense in κ–$\mathbb{R}$.*

Towards closing the current section, and returning to issues related to various forms of incompleteness of the κ–reals, it follows from Theorems 4.6 and 4.7 that, for any $\kappa > \omega$, ordinary square roots such as $\sqrt{2}$ do not exist in the field κ–$\mathbb{R}$. In other words, κ–$\mathbb{R}$ is indeed far from being algebraically closed. On the other hand, we do have plenty of (non-ordinary) square roots, such as, for example,

$$\sum_{n<\omega} \frac{1}{\aleph_n} = \sqrt{\sum_{n \neq m < \omega} \left(\frac{1}{\aleph_n^2} + \frac{2}{\aleph_n \cdot \aleph_m} \right)}$$

in $\aleph_\omega$–$\mathbb{R}$. We are confident that the reader can come up with more examples of this sort. Just before concluding, let us briefly look at some classical theorems from real analysis and how they (do not) apply in the case of the κ–reals.

As our first example, we consider the Heine–Borel theorem, one direction of which is still true by essentially the same proof: for every infinite κ, if $X \subseteq \kappa$–$\mathbb{R}$ is compact then it is closed and bounded. However, for the other direction, note that if $\kappa > \omega$, then the set $X = \{\frac{1}{n+1} : n < \omega\}$ is a bounded closed subset of κ–$\mathbb{R}$ which is not

[18] In fact, the elements in κ–$\mathbb{T}$ are in 1–1 correspondence with the κ–rationals in $(0,1)_\kappa \cap \kappa$–$\mathbb{Q}$.

compact: indeed, $\{(\frac{1}{n+1}, 2)_\kappa : n < \omega\}$ is an open cover of X without any finite subcover.

The main obstacle in generalizing the proof of the converse of the Heine–Borel theorem, and also behind the failures of other well-known theorems (see Section 5 as well), is the fact that, when $\kappa > \omega$, κ–$\mathbb{R}$ satisfies only the κ–order completeness. Another such failure concerns the Intermediate Value Theorem: it is easy to see that Example 5.2 in Section 5 below gives a counterexample to this theorem. On the other hand, Cantor's Intersection Theorem clearly continues to hold, as already noted by Sikorski in [34].

Finally, regarding the Bolzano–Weierstrass theorem, it is again known by the work of Sikorski (see [33]) that a generalized version of the property holds for the fields κ–$\mathbb{Q}$ when κ is a regular cardinal (this, by Theorem 4.6, can be generalized to any naturally closed ordinal of uncountable cofinality). Note that this is never true for κ–$\mathbb{R}$ when $\kappa > \omega$ has countable cofinality, as the example of the sequence $(\frac{1}{n+1})_{n<\omega}$ shows. The interested reader may further consult Cohen and Goffman [6], Keisler and Schmerl [22], Schmerl [32], Sikorski [34] and Stevenson and Thron [35], where a breadth of related general results can be found.

Given the constructed κ–reals and their basic properties, we now move on to further study their behavior in different mathematical contexts.

5. A few words on κ–calculus

Not surprisingly, and in the light of the previous remarks regarding the failures of basic results from real analysis, the differential calculus of the κ–reals turns out to be quite pathological.

Although one may define continuity and differentiation of functions $f : \kappa\text{–}\mathbb{R} \longrightarrow \kappa\text{–}\mathbb{R}$ in a natural fashion, many well-known theorems from standard calculus do not go through, except for special cases. In some sense, it seems like pure coincidence that these theorems actually hold in the case of ω–$\mathbb{R}$. Our motto here is the following:
Standard reals are too crude to notice and affect the fine distinctions that make all the difference in the infinitesimal world.

Of course, the interesting setting is when $\kappa > \omega$ has countable cofinality. Let us now see some examples, where every reference to the derivative of a function f at a point x_0 refers of course to the limit

$$f'(x_0) = \lim_{|x_0 - x| \longrightarrow 0} \frac{|f(x_0) - f(x)|}{|x_0 - x|},$$

whenever this limit exists.

Example 5.1. Let $\kappa > \omega$ be a naturally closed ordinal such that $\mathrm{cf}(\kappa) = \omega$, let λ be such that $\kappa = \omega^\lambda$, and let $\pi : \omega\text{–}\mathbb{Q} \longrightarrow \omega\text{–}\mathbb{Q}$ be an order-preserving bijection with $\pi(0) = 0$. We now define the map

$f_\pi : \kappa\text{–}\mathbb{R} \longrightarrow \kappa\text{–}\mathbb{R}$ by sending the element

$$x = p_0 \cdot \omega^{\alpha_0} + \ldots + p_n \cdot \omega^{\alpha_n} + \sum_{i<\mu} \frac{q_i}{\omega^{\beta_i}},$$

for $\mu \leqslant \omega$ and for suitable parameters p_j, q_i, α_j, β_i, to

$$f_\pi(x) = p_0 \cdot \omega^{\alpha_0} + \ldots + p_n \cdot \omega^{\alpha_n} + \sum_{i<\mu} \frac{\pi(q_i)}{\omega^{\beta_i}}.$$

It is easily checked that f_π is a continuous function. Moreover, f_π satisfies the Intermediate Value Theorem: for every $a < b$ in κ–$\mathbb{R}$ and every $y \in (f_\pi(a), f_\pi(b))$, there exists some $c \in (a, b)$ such that $f_\pi(c) = y$.

In addition, f_π is differentiable if and only if π is linear. In that case, if $a \in \omega$–$\mathbb{Q}$ is such that $\pi(q) = aq$ for all $q \in \omega$–$\mathbb{Q}$, then $(f_\pi)'(x) = a$, for all $x \in \kappa$–$\mathbb{R}$. We may call such a function f_π a "local bijection".

Example 5.2. Again, let $\kappa > \omega$ be a naturally closed ordinal such that $\mathrm{cf}(\kappa) = \omega$. Let λ be such that $\kappa = \omega^\lambda$. Let $(\alpha_n)_{n<\omega}$ be a strictly increasing sequence of non-zero additively indecomposable ordinals[19] converging to λ. Note that, for every $\alpha \in \lambda$ such that $\alpha \geqslant \alpha_0$, there exist unique $n \in \omega$ and $\beta < \alpha_{n+1}$ such that $\alpha = \alpha_n + \beta$. Let $\rho : \kappa \longrightarrow \kappa$ be the map which, for all $\alpha \in \kappa$, is given by:

$$\rho(\alpha) = \begin{cases} \alpha_0 + \alpha & \text{if } \alpha < \alpha_0, \\ \alpha_{n+1} + \beta & \text{if } \alpha = \alpha_n + \beta. \end{cases}$$

Now define the function $g_\rho : \kappa\text{–}\mathbb{R} \longrightarrow \kappa\text{–}\mathbb{R}$ by sending the element

$$x = p_0 \cdot \omega^{\alpha_0} + \ldots + p_n \cdot \omega^{\alpha_n} + \sum_{i<\mu} \frac{q_i}{\omega^{\beta_i}},$$

for $\mu \leqslant \omega$ and for suitable parameters p_j, q_i, α_j, β_i, to

$$g_\rho(x) = p_0 \cdot \omega^{\rho(\alpha_0)} + \ldots + p_n \cdot \omega^{\rho(\alpha_n)} + \sum_{i<\mu} \frac{q_i}{\omega^{\rho(\beta_i)}}.$$

It is not difficult to see that g_ρ is continuous and differentiable with $(g_\rho)' = 0$ (the proof that $(g_\rho)' = 0$ uses the fact that the α_n's are additively indecomposable), although it is clearly not a constant function. Additionally, for all distinct x, $y \in \kappa$–$\mathbb{R}$,

$$|g_\rho(x) - g_\rho(y)| > |x - y|$$

whenever $|x - y| \geqslant \frac{1}{n}$ for some $n \in \omega$; on the other hand, if $|x - y| < \frac{1}{n}$ for all $n \in \omega$, then

$$|g_\rho(x) - g_\rho(y)| < |x - y|.$$

In other words, g_ρ has the following curious (segregating) behavior: it separates even further apart elements which are already "far" from

[19] Meaning that, for all n and all α, $\beta < \alpha_n$, $\alpha + \beta < \alpha_n$.

each other, whereas it brings even closer together elements which are already "close" to one another.

A closely related example is given by a rank-into-rank embedding I1 (see Chapter 5 in Kanamori [20]), one of the strongest large cardinal axioms not known to be inconsistent with ZFC set theory.

Example 5.3. Let $j : V_{\lambda+1} \longrightarrow V_{\lambda+1}$ be a non-trivial elementary embedding with critical point[20] $\mathrm{cp}(j) = \kappa$ and $\lambda = \sup_{n<\omega} \kappa_n$, where $\kappa_0 = \kappa$ and, for each $n \in \omega$, we let $\kappa_{n+1} = j(\kappa_n)$.

Clearly, $\mathrm{cf}(\lambda) = \omega$ and λ–$\mathbb{R} \subseteq V_{\lambda+1}$. Let us now consider the map $f = j \restriction (\lambda\text{–}\mathbb{R}) : \lambda\text{–}\mathbb{R} \longrightarrow \lambda\text{–}\mathbb{R}$, which, just as in the previous example, can be easily seen to be continuous and differentiable with derivative $f' = 0$. Note that f (that is, j) behaves similarly to the function g_ρ described above. In particular, it has a similar segregating behavior.

Having brought set theory into the discussion this way, let us now move on to the study of the long reals from a more set-theoretic point of view.

6. Forcing and category

We begin by noting that, for every non-zero naturally closed ordinal κ, the construction of the field κ–$\mathbb{Q}$ is absolute for transitive models of ZFC. However, in the non-trivial case in which $\mathrm{cf}(\kappa) = \omega$, the construction of κ–$\mathbb{R}$ does not seem at all to be absolute, unless the relevant models have the same ω–sequences of ordinals in κ.

The natural question, then, is whether we can add a new κ–real by forcing. The obvious candidate for a poset achieving this is to consider the κ–real line itself as a forcing notion; that is, we let $P^{(\kappa)}$ be the poset consisting of the non-empty κ–rational intervals, ordered by inclusion. When $\kappa = \omega$, $P^{(\omega)}$ is just ordinary Cohen forcing, the most basic example of a forcing notion and a very well understood one, which adds a new ω–real. More generally:

Theorem 6.1. *For any non-zero naturally closed ordinal κ, $P^{(\kappa)}$ is forcing-equivalent to* $\mathrm{Col}(\omega, \kappa)$ (*the collapse of κ to ω using finite conditions*).[21]

Proof. Fix some $\kappa > \omega$, let λ be such that $\kappa = \omega^\lambda$, and fix some partition $\{X_\xi : \xi < \lambda\}$ of λ consisting of sets unbounded in λ. Let $G \subseteq P^{(\kappa)}$ be generic over V. We show that, in $V[G]$, there exists a surjection $f : \omega \longrightarrow \lambda$. This will be enough, since $\mathrm{Col}(\omega, \kappa)$ is the

[20] By *critical point* we mean the least ordinal moved by the embedding; see [20] for more details.

[21] Of course, the poset that adds one Cohen subset to ω via finite partial functions of the form $p : n \longrightarrow \{0, 1\}$ is forcing-equivalent to $\mathrm{Col}(\omega, \omega)$, the "collapse" of ω to itself via finite partial functions from ω to itself.

unique, up to forcing-equivalence, poset of size κ that adds such a surjection (as of course $|\lambda| = |\kappa|$).

We work in $V[G]$. For every interval $I \in G$, let $\ell(I)$ be its left endpoint.

Claim 6.2. *There exists a sequence $\langle I_n : n \in \omega\rangle$ of nested intervals from G such that $V[G] = V[\langle I_n : n \in \omega\rangle]$ and, for every $n \in \omega$,*

$$\ell(I_n) = z_n + \sum_{i \leqslant k_n} \frac{q_i^n}{\omega^{\beta_i^n}},$$

for some $z_n \in \kappa\text{–}\mathbb{Z}$, $k_n < \omega$, $q_i^n \in \omega\text{–}\mathbb{Q} \setminus \{0\}$ $(i \leqslant k_n)$, and $(\beta_i^n)_{i \leqslant k_n}$ a strictly increasing sequence of ordinals in λ. Moreover, we may assume that $(k_n)_{n<\omega}$ is strictly increasing, and that $(\beta_{k_n}^n)_{n<\omega}$ is a strictly increasing sequence converging to λ.

Proof. It is clear, by a density argument, that this can be done if $\mathrm{cf}^{V[G]}(\lambda) = \omega$. It thus suffices to show that $\mathrm{cf}^{V[G]}(\lambda) = \omega$.

For this, consider the function g sending $n < \omega$ to the least $\beta < \kappa$ such that

$$(z - \sum_{i \leqslant l_n} \frac{q_i}{\omega^{\beta_i}},\, z + \sum_{i \leqslant l_n} \frac{q_i}{\omega^{\beta_i}}) \in G,$$

where $l_n < \omega$, $l_n \geqslant n$, $z \in \kappa\text{–}\mathbb{Z}$, $q_i \in \omega\text{–}\mathbb{Q} \setminus \{0\}$ for all i, $(\beta_i)_{i \leqslant l_n}$ is a strictly increasing sequence of ordinals in λ, and $\beta_{l_n} = \beta$. Now note that g has range cofinal in λ, by a standard density argument. □

Next, for every $I \in P^{(\kappa)}$ and every $\xi < \lambda$, we claim that there exists some $n \in \omega$ such that $I_n \subseteq I$ and

$$\ell(I_n) = z_n + \sum_{i \leqslant k_n} \frac{q_i^n}{\omega^{\beta_i^n}},$$

with $\beta_i^n \in X_\xi$ for some i. To see this, let $I' \subseteq I$ be an interval of the form

$$I' = \Big(\ell(I) + \frac{1}{\omega^\beta}\,,\, \ell(I) + \frac{2}{\omega^\beta}\Big),$$

for some large enough ordinal $\beta \in X_\xi$, and apply a density argument.

We may now define a function $f : \omega \longrightarrow [\lambda]^{<\omega}$ with $\bigcup \mathrm{ran}(f) = \lambda$ by sending each $n \in \omega$ to the set of $\xi < \lambda$ such that $\beta_i^n \in X_\xi$ for some $i \leqslant k_n$, where β_i^n is determined by $\ell(I_n)$ as above, with respect to the fixed sequence $\langle I_n : n \in \omega\rangle \in V[G]$. It follows that λ is countable in $V[G]$, which finishes the proof. □

We can now state the following corollary, which asserts the failure of a natural generalization of the Baire category theorem for $\kappa \geqslant \omega_1$:[22]

[22] There are various related results appearing in the literature; see, for instance, Cohen and Goffman [6], Sikorski [34] and Stevenson and Thron [35].

Corollary 6.3. *For any naturally closed ordinal $\kappa \geqslant \omega_1$ there are $\aleph_1$–many open dense subsets of κ–$\mathbb{R}$ whose intersection is empty. In particular, κ–$\mathbb{R}$ is the union of $\aleph_1$–many nowhere dense sets.*

Proof. We argue in terms of $P^{(\kappa)}$. Given $\kappa \geqslant \omega_1^V$, the forcing $P^{(\kappa)}$ adds a surjection $\dot{f} : \omega \longrightarrow \omega_1^V$, by the previous theorem. Now notice that, for $\alpha < \omega_1^V$, the sets $D_\alpha = \{p \in P^{(\kappa)} : (\exists\, n)\,(p \Vdash \dot{f}(n) = \alpha)\}$ are open dense in $P^{(\kappa)}$. But of course $\bigcap\{D_\alpha : \alpha < \omega_1^V\} \neq \varnothing$, as otherwise there would be a surjection $f : \omega \longrightarrow \omega_1$. □

Note that, when $\mathrm{cf}(\kappa) > \omega$, even the classical Baire category theorem fails: for each $n \in \omega$, let E_n be the set of all $I \in P^{(\kappa)}$ of the form

$$(z - \sum_{i<n} \frac{q_i}{\omega^{\beta_i}},\, z + \sum_{i<n} \frac{q_i}{\omega^{\beta_i}}),$$

where $z \in \kappa$–$\mathbb{Z}$, $q_i \in \omega$–$\mathbb{Q} \setminus \{0\}$ for all i, and $(\beta_i)_{i<n}$ is a strictly increasing sequence of ordinals. Then, each E_n is an open dense subset of κ–$\mathbb{Q}$ ($= \kappa$–$\mathbb{R}$), but of course $\bigcap_{n<\omega} E_n$ is empty. In fact, in this case the failure is quite strong since there are countably many open dense sets whose intersection is empty.

Nevertheless, following closely the proof of the standard case we can still rescue the classical theorem in the case of countable cofinality.

Proposition 6.4. *If κ is a naturally closed ordinal with $\mathrm{cf}(\kappa) = \omega$, then the intersection of countably many open dense subsets of κ–$\mathbb{R}$ is dense in κ–$\mathbb{R}$.*

Proof. Let $\{D_n : n \in \omega\}$ be a collection of open dense subsets of κ–$\mathbb{R}$, and fix some non-empty open rational interval $I \subseteq \kappa$–$\mathbb{R}$. Moreover, fix a strictly increasing sequence of ordinals $\langle\, \alpha_n : n \in \omega \,\rangle$, with $\sup_{n<\omega} \alpha_n = \kappa$. Since D_0 is dense, there is some $x_0 \in \kappa$–$\mathbb{R}$ and some $0 < \varepsilon_0 < \frac{1}{\alpha_0}$ such that

$$I_0 = [x_0 - \varepsilon_0,\, x_0 + \varepsilon_0] \subseteq D_0 \cap I.$$

We now use the density of each D_n in order to build, recursively for $n \geqslant 1$, a sequence $(x_n)_{n<\omega}$ and a sequence $(\varepsilon_n)_{n<\omega}$ such that, for every $n \geqslant 1$, $0 < \varepsilon_n < \frac{1}{\alpha_n}$ and

$$I_n = [x_n - \varepsilon_n, x_n + \varepsilon_n] \subseteq (x_{n-1} - \varepsilon_{n-1}, x_{n-1} + \varepsilon_{n-1}) \cap D_n.$$

Note that the sequence $(x_n)_{n<\omega}$ is Cauchy and therefore there exists some $x \in \kappa$–$\mathbb{R}$ such that $x = \lim_{n<\omega} x_n$. Finally, $x \in I_{n+1}$ for each $n \in \omega$ and, thus, we get that $x \in I \cap \bigcap_{n<\omega} D_n$ as desired. □

Observe that the previous proof does not go through when $\mathrm{cf}(\kappa) > \omega$, since in this case no countable (non-eventually constant) sequence converges; in fact, we know that κ–$\mathbb{Q} = \kappa$–$\mathbb{R}$ by Theorem 4.6.

Corollary 6.5. *The Baire category theorem holds in* κ–$\mathbb{R}$ *if and only if* $\mathrm{cf}(\kappa) = \omega$.

Let us now turn to the descriptive set-theoretic context where, as we shall soon see, the classical techniques[23] tie nicely with the κ–reals.

7. Descriptive set theory

Throughout this section, we assume that κ is a naturally closed ordinal of countable cofinality. Moreover, we frequently refer to the appropriate tree representation of the κ–reals in $(0,1)_\kappa$, as described in Section 4; recall Definitions 4.8 and 4.9, and the related remarks.

Following the usual arguments of the standard case, we can see that a set $X \subseteq (0,\,1)_\kappa$ is *closed* if and only if X is (coded by) the body of a tree: that is, $X = c_\kappa([T])$ for some tree $T \subseteq \kappa$–$\mathbb{T}$. Now recall that a set $X \subseteq \kappa$–$\mathbb{R}$ is *perfect* if it is closed and has no isolated points. If X is a subset of $(0,\,1)_\kappa$, this is equivalent to being coded by the body of a splitting tree. It is easy to see by the standard arguments (cf. the proof of Proposition 7.1), that if $\mathrm{cf}(\kappa) = \omega$ and $X \subseteq \kappa$–$\mathbb{R}$ is a non-empty perfect set, then $|X| \geqslant 2^{\aleph_0}$. Moreover, we can also prove the analogue of the Cantor–Bendixson theorem.

Proposition 7.1. *If* $X \subseteq \kappa$–$\mathbb{R}$ *is closed, then there exists* $S \subseteq X$ *with* $|S| \leqslant \kappa$ *and such that* $X \setminus S$ *is perfect.*

Proof. We may assume $\kappa > \omega$. Also, given $A \subseteq \kappa$–$\mathbb{R}$, a translate of A (that is, a set of the form $x + A = \{x + a \,:\, a \in A\}$), is closed if and only if A is closed. We may therefore assume that X is a closed subset of the unit interval. In particular, there is a tree T with $X = c_\kappa([T])$. Namely,

$$T = \{s \in \kappa\text{–}\mathbb{T} :\, (\exists x \in X)\,(s \sqsubseteq c_\kappa^{-1}(x))\}.$$

We then let

$$S = \bigcup\{[T_u] : u \in T \,\wedge\, \big|[T_u]\big| \leqslant \kappa\}$$

and note that $c_\kappa(S) \subseteq X$ with $|S| \leqslant \kappa$. It remains to see that $X \setminus c_\kappa(S)$ is perfect. For this, we consider the (non-empty) subtree

$$T_0 = \{u \in T : \big|[T_u]\big| > \kappa\}$$

and, after simple computations, we obtain that

$$x \in X \setminus c_\kappa(S) \Longleftrightarrow x \in [T_0].$$

From the latter, the desired conclusion will follow once we have shown that T_0 is a splitting tree. Towards a contradiction, assume that some

[23] See Kanamori [20], Kechris [21], Moschovakis [29], or even Chapter 10 in Moschovakis [30] for more details on the basics of classical descriptive set theory.

$u \in T_0$ does not split; in particular, all extensions of u in T_0 are comparable and hence they define a unique branch $x \in [T_0]$. It now follows that, in the initial tree T,

$$[T_u] = \{x\} \cup \bigcup \{[T_w] : u \sqsubseteq w \wedge \big|[T_w]\big| \leqslant \kappa\},$$

from which we obtain $\big|[T_u]\big| \leqslant \kappa$, contradicting the fact that $u \in T_0$. $\square$

It is easily seen that the aforementioned splitting of X (when X is a closed subset of the unit interval) is unique; we call S the *scattered part* of X and $X \setminus S$ the *kernel* of X.

Also, note that the bound $|S| \leqslant \kappa$ is the best possible: given any closed and bounded set $X \subseteq \kappa$–$\mathbb{R}$, we may attach the (closed) set κ–$\mathbb{Z}$ to it, and then $X \cup \kappa$–$\mathbb{Z}$ will be a closed set whose isolated points have cardinality at least κ.

As far as the possible sizes of perfect sets are concerned, the following is a direct consequence of our previous discussion.

Corollary 7.2. *Suppose that $2^{\aleph_0} < \kappa$. Then, for every cardinal μ with $2^{\aleph_0} \leqslant \mu \leqslant \kappa$, there exists a perfect set $P \subseteq \kappa$–$\mathbb{R}$ such that $|P| = \mu$.*

Towards a more general setting, we now give the following direct variant of the well-known perfect set property:

Definition 7.3. A set $X \subseteq \kappa$–$\mathbb{R}$ is said to have the *κ–perfect set property* if either $|X| \leqslant \kappa$ or X contains a perfect set.

We may generalize Proposition 7.1 in order to account for the usual σ–algebra of Borel subsets of κ–$\mathbb{R}$. As a matter of notation, if λ is such that $\kappa = \omega^\lambda$, we let κ–$\mathbb{T}_2$ be the binary subtree of κ–$\mathbb{T}$ which has height λ and consists exactly of the $\{0,1\}$–sequences s of length $<\lambda$ with finite support such that $\mathrm{ot}([\alpha_0, \alpha_1)) \geqslant \omega$ for all distinct $\alpha_0 < \alpha_1$ in $\mathrm{supp}(s)$.

Now suppose $\kappa > \omega$, let λ be such that $\kappa = \omega^\lambda$, let $(\xi_i)_{i<\lambda}$ be the strictly increasing enumeration of the limit ordinals in λ, and let $c^*_\kappa : [\kappa\text{–}\mathbb{T}_2] \longrightarrow (0,1)_\kappa$ be the map given by

$$c^*_\kappa(b) = \sum_{\substack{\xi_i + n \,\in\, \mathrm{supp}(b) \\ n<\omega}} \frac{\pi(n)}{\omega^{1+i}},$$

where $\pi : \omega \longrightarrow \omega$–$\mathbb{Q}$ is a bijection with $\pi(0) = 0$ (cf. Definition 4.8 and its subsequent remarks). Then, c^*_κ is clearly a bijection between $[\kappa$–$\mathbb{T}_2]$ and $(0,1)_\kappa$. Given $X \subseteq [\kappa$–$\mathbb{T}_2]$, we will denote the set $\{c^*_\kappa(b) : b \in X\}$ by $c^*_\kappa(X)$.

Definition 7.4. Suppose $\kappa > \omega$ and let $A \subseteq (0,1)_\kappa$. We define a two-player game $G(A)$ in which, for each $i \in \omega$, Player I chooses an element $s_i \in \kappa$–$\mathbb{T}_2$, while Player II chooses some $k_i \in \{0,1\}$ as shown below:

$$\begin{array}{c|ccccc} I & s_0 & & s_1 & & \dots \\ \hline II & & k_0 & & k_1 & \dots \end{array}$$

Given a complete play of the game and letting $x = s_0^\frown\langle k_0\rangle^\frown s_1^\frown\langle k_1\rangle^\frown\dots$ be the concatenation of the alternating moves, we say that Player I wins if $x \in [\kappa\text{–}\mathbb{T}_2]$ and $c^*_\kappa(x) \in A$; otherwise, Player II wins.

The following theorem is reminiscent of the corresponding result in the classical descriptive set-theoretic context.

Theorem 7.5. *Suppose $\kappa > \omega$, let $A \subseteq (0,1)_\kappa$ and consider the game $G(A)$. Then:*

- (i) *If Player II has a winning strategy, then $|A| \leqslant \kappa$.*
- (ii) *Player I has a winning strategy if and only if there exists a perfect $P \subseteq A$.*

Proof. Let $\langle\, \alpha_n : n \in \omega \,\rangle$ be an increasing sequence of ordinals with $\alpha_0 = 0$ and $\sup_{n<\omega} \alpha_n = \kappa$. Fix some $A \subseteq (0,1)_\kappa$ and consider $G(A)$.

For (i), let τ be a winning strategy for Player II and recall Definition 4.9 regarding the order $o(p)$ of any element $p \in \kappa\text{–}\mathbb{T}_2$, with respect to $(\alpha_n)_{n<\omega}$. Then, for each partial play $p_* = \langle\, s_0, k_0, \dots, s_n, k_n \,\rangle$ of the game $G(A)$, if $p = s_0^\frown\langle k_0\rangle^\frown \dots {}^\frown s_n^\frown\langle k_n\rangle$ is in $\kappa\text{–}\mathbb{T}_2$, then consider the set D_p consisting of those $x \in [\kappa\text{–}\mathbb{T}_2]$ such that

$$p \sqsubseteq x \longrightarrow (\exists\, u \in \kappa\text{–}\mathbb{T}_2)\,(p \sqsubseteq u \,\wedge\, o(u) > o(p) \,\wedge\, u^\frown\tau(u) \sqsubseteq x).$$

But now note that, if $x \in D_p$ for all p as above, then one can use the strategy τ and the definition of the D_p's to express x as the (concatenation of) a complete play of the game $x = s_0^\frown\langle k_0\rangle^\frown s_1^\frown\langle k_1\rangle^\frown\dots$, so that $x \notin A$. Observe that the requirement "$o(u) > o(p)$" ensures that the sequence of partial plays converges in $\kappa\text{–}\mathbb{R}$. Therefore, we obtain

$$\bigcap_p c^*_\kappa(D_p) \subseteq (0,1)_\kappa \setminus A,$$

from which it follows that

$$|A| \leqslant \Big|\bigcup_p([\kappa\text{–}\mathbb{T}_2] \setminus D_p)\Big|.$$

Next, notice that for each p, $x \in [\kappa\text{–}\mathbb{T}_2] \setminus D_p$ if and only if $p \sqsubseteq x$ and, for all u with $p \sqsubseteq u$ and $o(u) > o(p)$, we have $u^\frown\tau(u) \not\sqsubseteq x$. Consequently, and using the fact that binary ordinal sequences are involved, $\big|[\kappa\text{–}\mathbb{T}_2] \setminus D_p\big| \leqslant 1$ for each p; hence, we get that $|A| \leqslant \kappa$.

For (ii), and for the forward direction, suppose that σ is a winning strategy for Player I. Then, we may recursively build a perfect subset of A by appealing to σ: we start with $s_0 = \langle\rangle$ and build a (binary) splitting subtree T of $\kappa\text{–}\mathbb{T}_2$ by considering, at each stage n, both extensions $s_n^\frown\langle 0\rangle$ and $s_n^\frown\langle 1\rangle$ of the current s_n. Then, $c^*_\kappa([T]) \subseteq A$ is perfect.

Conversely, suppose that $T \subseteq \kappa\text{–}\mathbb{T}_2$ is a splitting tree with $c^*_\kappa([T]) \subseteq A$. The winning strategy for Player I can be described as follows:

start by playing some $s_0 \in T$ such that both $s_0{}^\frown\langle 0\rangle$ and $s_0{}^\frown\langle 1\rangle$ are in T. For each n, and given Player II's response k_n, play a further s_{n+1} extending (the concatenation of) the current play such that both $s_{n+1}{}^\frown\langle 0\rangle$ and $s_{n+1}{}^\frown\langle 1\rangle$ are in T. The only subtle point is that our chosen s_n's should have strictly increasing orders $o(s_n)$'s, so that the final sequence converges. It is easy to see that the concatenation x of any complete play will be a member of $[T]$ and therefore $c^*_\kappa(x) \in A$. $\square$

From the previous theorem, we immediately obtain:

Corollary 7.6. *Suppose $\kappa > \omega$ and let $A \subseteq (0,1)_\kappa$. If $G(A)$ is determined then A has the κ–perfect set property.*

Recall that, for any given topological space X, a Borel subset of X is a member of the σ–algebra $\mathcal{B} \subseteq \mathcal{P}(X)$ generated by the open subsets of X.

Corollary 7.7. *Every Borel subset of κ–$\mathbb{R}$ has the κ–perfect set property.*

Proof. We just need to argue for the case $\kappa > \omega$. Let A be a Borel subset of κ–$\mathbb{R}$. We want to prove that A has the κ–perfect set property, but for this it suffices to assume that $A \subseteq (0,1)_\kappa$, since every translate of a closed set is closed. Now, as is well-known, the usual proof of Borel determinacy for subsets of the Baire space (Martin [28]) extends naturally to the general context of Borel subsets of the product space ${}^\omega X$, where X is any set endowed with the discrete topology. This establishes the desired result, since the payoff set of $G(A)$ is Borel in such a space ${}^\omega X$, for a suitable choice of X. $\square$

Note that the proofs of Theorem 7.5 and of the subsequent Corollary 7.7 used classical descriptive set-theoretic techniques applied to the case of the κ–reals. In this context, it seems that there is a natural way for generalizing traditional descriptive set theory of Polish spaces to that of "κ–Polish spaces" (that is, "κ–metric spaces" of density κ). See also Cohen and Goffman [6], Sikorski [34] and Stevenson and Thron [35] for related results on such metric spaces.

8. Open questions and final thoughts

Given that the subject of long reals seems to be quite broad, enjoying several connections with well-established fields of study, it is only inevitable that we are not able to cover the full depth and breadth of the emerging issues. We do hope, however, that our exposition is a coherent presentation of this intriguing topic.

At any rate, let us conclude with a (very non-exhaustive) list of issues and open questions which have arisen along the way.

Towards the end of Section 4 we mentioned that the Heine–Borel theorem fails for κ–$\mathbb{R}$ when $\kappa > \omega$. On the other hand, it is well-known that this and many other basic theorems from analysis are in fact equivalent (to each other and) to the axiom of (order) completeness of ω–$\mathbb{R}$. Hence, we may ask:

Question 8.1. Do any of these equivalences generalize for $\kappa > \omega$? If not, can we give a complete description of the implications between the corresponding statements? (see also Schmerl [32]).

Moreover, one can study the differential calculus of the κ–reals further; for instance, the theorems of Bolzano, Rolle, Fermat, Intermediate Value, etc. In this direction, one can perhaps find families of functions that satisfy them, and perhaps characterize them. A related issue is that of defining the notion of definite integral and then studying the corresponding integral calculus of the κ–reals.

Regarding our brief account in Section 5, we can ask:

Question 8.2. When $\mathrm{cf}(\kappa) = \omega$, are the "local" bijections (see Example 5.1) the only functions for which theorems such as Bolzano's hold?

In a somewhat more general flavor, it might be interesting to even look at "thicker" versions of κ–$\mathbb{R}$, for $\mathrm{cf}(\kappa) = \omega$: instead of rational coefficients $q_i \in \omega$–$\mathbb{Q}$ in the representation given in Theorem 4.7, one can take standard reals $r_i \in \omega$–$\mathbb{R}$ as coefficients, and study the resulting structure.

Turning to more set-theoretic issues, one basic question that has remained unanswered is:

Question 8.3. Is there a non-trivial and "interesting" (for example, one that preserves ω_1, or even all cardinals) quotient forcing algebra arising from κ–$\mathbb{R}$?

Furthermore, in the descriptive set-theoretic context:

Question 8.4. Is there any natural correspondence between levels in the projective hierarchy of κ–$\mathbb{R}$ and complexity classes in terms of definability over relevant structures (for instance, $\langle H_{\kappa^+}, \in, \ldots \rangle$)?

Question 8.5. What are the right analogues, for κ–$\mathbb{R}$, of the Baire property or of the notion of Lebesgue measurability?

For example, we may define appropriately and study κ–meager vs. λ–meager sets, giving rise to corresponding notions of Baire property. In a similar spirit:

Question 8.6. What about hierarchies of the perfect set property (PSP), by appropriately defining the κ–PSP_λ?

Regarding sizes of perfect sets, we may complement Corollary 7.2 by asking:

Question 8.7. Suppose that $\kappa > \omega$ and $\mathrm{cf}(\kappa) = \omega$. Let μ be a cardinal with $2^{\aleph_0} < \mu < \kappa^{\aleph_0}$ and $\kappa < \mu$. Does there exist a perfect set $P \subseteq \kappa$–$\mathbb{R}$ such that $|P| = \mu$?

As far as other basic results of descriptive set theory are concerned (such as Suslin's theorem, separation theorems, etc.), it remains to see if they can be adapted to the context of κ–reals, for some κ of countable cofinality. In addition, and as already suggested in the discussion right after Corollary 7.7, one can deal with the study of long κ–Polish and κ–metric spaces.

Let us conclude with the following thoughts. As we have seen, ω plays a very special role in the theory we have explored; in particular, the constructions corresponding to ordinals κ of countable cofinality and those corresponding to ordinals of uncountable cofinality have completely different properties. In fact, we saw that the theory becomes interesting only in the case $\mathrm{cf}(\kappa) = \omega$. The reason for this boils down to the representation theorem (Theorem 3.4). As a matter of fact, it is not difficult to see that ω would play exactly the same distinguished role, in exactly the same way, if we were to develop the present theory starting from any reasonable pair $\boxplus$, $\boxtimes$ of operations on the ordinals (instead of the Hessenberg operations) such that every ordinal $\kappa > \omega$ closed under $\boxplus$ and $\boxtimes$ is a limit of ordinals closed under $\boxplus$. In any case, it might be interesting to explore any possibilities for the following.

Issue 8.8. Consider other pairs of operations $\boxplus$ and $\boxtimes$ on the ordinals, satisfying the minimal algebraic requirements (commutativity, distributivity of $\boxtimes$ with respect to $\boxplus$, and so on), extending the usual addition and multiplication on ω, and relative to which the standard constructions of κ–$\mathbb{R}$ yield a theory that becomes interesting when $\mathrm{cf}(\kappa) = \mu$, for some choice of $\mu \neq \omega$.[24]

At this point, the possibilities seem open-ended.

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