Guessing and non-guessing of canonical functions

David Asperó

Abstract

It is possible to control to a large extent, via semiproper forcing, the parameters (β_0, β_1) measuring the *guessing density* of the members of any given antichain of stationary subsets of ω_1 (assuming the existence of an inaccessible limit of measurable cardinals). Here, given a pair (β_0, β_1) of ordinals, we will say that a stationary set $S \subseteq \omega_1$ has guessing density (β_0, β_1) if $\beta_0 = \gamma(S)$ and $\beta_1 = sup\{\gamma(S^*) : S^* \subseteq S,\, S^* \text{ stationary}\}$, where $\gamma(S^*)$ is, for every stationary $S^* \subseteq \omega_1$, the infimum of the set of ordinals $\tau \leq \omega_1 + 1$ for which there a function $F : S^* \longrightarrow \mathcal{P}(\omega_1)$ with $ot(F(\nu)) < \tau$ for all $\nu \in S^*$ and with $\{\nu \in S^* : g(\nu) \in F(\nu)\}$ stationary for every $\alpha < \omega_2$ and every canonical function g for α. This work involves an analysis of iterations of models of set theory relative to sequences of measures on possibly distinct measurable cardinals.

As an application of these techniques I show how to force, from the existence of a supercompact cardinal, a model of PFA^{++} in which there is a well–order of $H(\omega_2)$ definable, over $\langle H(\omega_2), \in\rangle$, by a formula without parameters.

1 Guessing densities of stationary sets

The present paper deals mostly with the manipulation, by forcing, of one particular guessing property for stationary subsets of ω_1 with respect to canonical functions for ordinals less than ω_2. As an application of the main forcing construction presented here I will show how to force, over a model with a supercompact cardinal, in order to obtain a model of the Proper Forcing Axiom, and in fact of its stronger form PFA^{++}, in which there is a well–order of $H(\omega_2)$ definable, over $\langle H(\omega_2), \in\rangle$, by a formula without parameters. But more of this later.

Recall that, for a given ordinal $\alpha < \omega_2$, a canonical function for α is a function g from ω_1 into ω_1 with the property that $\mathcal{P}(\omega_1)/NS_{\omega_1}$ – where NS_{ω_1} denotes the nonstationary ideal on ω_1 – forces that, letting $\langle M, E\rangle$ be the generic ultrapower of the ground model derived from the generic object, the set of M–ordinals below the class of g in M is well–ordered under E in length α. An equivalent presentation of canonical functions for α is the following:

Given a surjection $\pi : \omega_1 \longrightarrow \alpha$, the map $g : \omega_1 \longrightarrow \omega_1$ given by $g(\nu) = ot(\pi``\nu)$ is a canonical function for α (where, for a set X of ordinals, $ot(X)$ stands for the order type of X) and, furthermore, every canonical function for α coincides with g on a club (which justifies the use of the term 'canonical').

Given a function $F : S^* \longrightarrow \mathcal{P}(\omega_1)$, with S^* a subset of ω_1, we will say that F *guesses all canonical functions* if $\{\nu \in S^* : g(\nu) \in F(\nu)\}$ is stationary for every $\alpha < \omega_2$ and every (equivalently, some) canonical function g for α.[1] The combinatorial property for stationary subsets of ω_1 that I shall be looking at is the following.

Definition 1.1 *Let S be a stationary subset of ω_1. The guessing density of S is the unique pair (β_0, β_1) of ordinals given by*

$$\beta_0 = \gamma(S)$$

and

$$\beta_1 = sup\{\gamma(S^*) \, : \, S^* \subseteq S, \, S^* \text{ stationary}\},$$

where, given any stationary $S^* \subseteq \omega_1$, $\gamma(S^*)$ is the infimum of the set of ordinals τ with the property that there is a function $F : S^* \longrightarrow \mathcal{P}(\omega_1)$ with $ot(F(\nu)) < \tau$ for all $\nu \in S^*$ and such that F guesses all canonical functions.

Note that the guessing density (β_0, β_1) of every stationary subset of ω_1 satisfies $2 \leq \beta_0 \leq \beta_1 \leq \omega_1 + 1$. The above definition of guessing density has been proposed by an anonymous referee. In a previous version of the paper I was using the following different notion: A stationary $S \subseteq \omega_1$ was said to have guessing density I (for some $I \subseteq \omega_1$) in case for every stationary $S^* \subseteq S$

(a) there is a function $F : S^* \longrightarrow \mathcal{P}(\omega_1)$ guessing all canonical functions and with $ot(F(\nu)) \in I$ for all $\nu \in S^*$, and

[1] By 'canonical function' I will always mean 'canonical function for some ordinal less than ω_2'.

(b) no function $F : S^* \longrightarrow \mathcal{P}(\omega_1)$ such that $ot(F(\nu)) < min(I)$ for all $\nu \in S^*$ guesses all canonical functions.

One drawback of the original notion of 'guessing density' is that, as I observed in the older version of this paper and unlike the notion introduced in Definition 1.1, the guessing density – in the original sense – of a given stationary subset of ω_1 is not uniquely defined: If $S \subseteq \omega_1$ is a stationary set, I_0, $I_1 \subseteq \omega_1$ are such that $min(I_1) \leq min(I_0)$ and $sup(I_0) \leq sup(I_1)$, and S has guessing density I_0 in the old sense, then S also has guessing density I_1, again in the old sense.

The version of guessing density contained in Definition 1.1 compares to the old one in the following way: Given ordinals $\alpha_0 < \alpha_1$, a stationary set $S \subseteq \omega_1$ has guessing density, in the old sense, equal to the interval of ordinals $[\alpha_0, \alpha_1)$ if and only if S has guessing density (β_0, β_1), in the sense of Definition 1.1, and $\alpha_0 < \beta_0$ and $\beta_1 \leq \alpha_1$. It follows that all relevant statements involving the old notion of guessing density translate easily into statements involving the new notion. For these two reasons, in the current version of the paper I shall adopt Definition 1.1 as the official definition of guessing density.

It is easy to see that, for every stationary $S \subseteq \omega_1$, the assumption that $\diamondsuit(S^*)$ holds[2] for every stationary $S^* \subseteq S$ implies that S has guessing density $(2, 2)$.[3] On the other hand, Bounded Martin's Maximum[4] (BMM) is easily seen to imply that the guessing density (β_0, β_1) of every stationary subset of ω_1 satisfies $\omega_1 \leq \beta_0$:

Fact 1.1 *Let $S \subseteq \omega_1$ be stationary. If BMM holds and S has guessing density (β_0, β_1), then $\omega_1 \leq \beta_0$.*

Proof: Suppose BMM holds, let $S \subseteq \omega_1$ be stationary, let $\beta_0 < \omega_1$ be an ordinal, and let $F : S \longrightarrow \mathcal{P}(\omega_1)$ be a function with $ot(F(\nu)) < \beta_0$ for all $\nu \in S$. It suffices to see that there is an ordinal $\gamma < \omega_2$, a bijection $\pi : \omega_1 \longrightarrow \gamma$ and a club $C \subseteq \omega_1$ such that $ot(\pi``\nu) \notin F(\nu)$ for every

[2] $\diamondsuit(S^*)$ is the statement that there is a sequence $\langle X_\alpha : \alpha \in S^* \rangle$ with $X_\alpha \subseteq \alpha$ for all α and $\{\alpha \in S^* : X \cap \alpha = X_\alpha\}$ stationary for each $X \subseteq \omega_1$.

[3] In other words, for every stationary $S^* \subseteq S$ there is a function $F : S^* \longrightarrow \omega_1$ such that $\{\nu \in S^* : g(\nu) = F(\nu)\}$ is stationary for every $\alpha < \omega_2$ and every canonical function g for α.

[4] Bounded Martin's Maximum is (equivalent to) the statement that $H(\omega_2)$ is a Σ_1–elementary substructure of $H(\omega_2)^{V^{\mathcal{P}}}$ for every partial order $\mathcal{P}$ preserving stationary subsets of ω_1 (see [B]). Here and throughout the paper, given an infinite cardinal θ, $H(\theta)$ is the set of all those sets whose transitive closure has size less than θ.

$\nu \in C \cap S$. Let $\mathcal{P}$ be the natural forcing for adding, by initial segments, a $\subseteq$–increasing and $\subseteq$–continuous sequence $\langle X_\nu : \nu < \omega_1 \rangle$ of countable subsets of ω_2^V such that $\omega_2^V = \bigcup_{\nu<\omega_1} X_\nu$ and such that $ot(X_\nu) \notin F(X_\nu \cap \omega_1)$ whenever $X_\nu \cap \omega_1 \in S$ preserves stationary subsets of ω_1.[5] Now, given any function $H : [H(\omega_2)]^{<\omega} \longrightarrow H(\omega_2)$ and any stationary $T \subseteq \omega_1$, let $D = \{\delta < \omega_2 : H``[\delta]^{<\omega} \subseteq \delta\}$. Let N be a countable elementary substructure of $H((2^{\aleph_1})^+)$ containing β_0 and H and with $N \cap \omega_1 \in T$. Notice that $ot(D \cap N) > \beta_0$. Hence, there is some $\delta_0 \in D \cap N$ such that $ot(N \cap \delta_0) \notin F(N \cap \omega_1)$ in case $N \cap \omega_1 \in S$.[6] And, on the other hand, $H``[N \cap \delta_0]^{<\omega} \subseteq N \cap \delta_0$. By standard arguments this is enough to show that $\mathcal{P}$ preserves stationary subsets of ω_1 and that it adds a club of $[\omega_2^V]^{\aleph_0}$ with the required property. Finally, the desired conclusion follows from an application of BMM to $\mathcal{P}$. $\square$

Also, it is easy to see that if every function $f : \omega_1 \longrightarrow \omega_1$ is dominated on a club by some canonical function, then every stationary subset of ω_1 has guessing density $(\omega_1 + 1, \omega_1 + 1)$.[7]

The main purpose of this paper is to give a forcing construction, assuming the existence of an inaccessible limit of measurable cardinals, which preserves stationary subsets of ω_1 – and in fact is semiproper – and which, to each member S_i of any given antichain $\langle S_i : i < \Lambda \rangle$ of stationary subsets of ω_1 in the ground model, assigns guessing density (β_i^0, β_i^1), for ordinals $\beta_i^0 \leq \beta_i^1 < \omega_1$, where the sequence of pairs (β_i^0, β_i^1) can be controlled in advance with a good deal of accuracy and arbitrariness.

Here are some general pieces of notation related to forcing that will appear in the paper: Given a partial order $\langle \mathcal{P}, \leq \rangle$, a $\mathcal{P}$–name τ and a $\mathcal{P}$–generic filter G, τ_G – or $(\tau)_G$ – denotes the interpretation of τ by G. Also, if $\dot{x}$ is a $\mathcal{P}$–name and G is a $\mathcal{P}$–generic filter, then x may be used to denote $\dot{x}_G$. $\dot{G}$ will be a name such that $(\dot{G})_G = G$ for every $\mathcal{P}$–generic filter G. Given any $\mathcal{P}$–generic filter G and any set N, $N[G]$ denotes $\{\tau_G : \tau \in \dot{N}, \tau \text{ a } \mathcal{P}\text{–name}\}$. Given two $\mathcal{P}$–conditions p, q, $q \leq p$ will mean that q is stronger (carries more information) than p. Given a condition $p \in \mathcal{P}$, $\mathcal{P} \restriction p = \{q \in \mathcal{P} : q \leq p\}$.

[5] $\mathcal{P}$ is the set, ordered by $\supseteq$, of $\subseteq$–increasing and $\subseteq$–continuous sequences of the form $(X_\nu)_{\nu \leq \nu_0}$, for some countable ordinal ν_0, with $X_\nu \in [\omega_2]^{\aleph_0}$ and, furthermore, with $ot(X_\nu) \notin F(X_\nu \cap \omega_1)$ in case $X_\nu \cap \omega_1 \in S$.

[6] As, in that case, $F(N \cap \omega_1)$ has order type less than β_0.

[7] In other words, $\gamma(S) = \omega_1 + 1$ for every stationary $S \subseteq \omega_1$. It is not known whether BMM by itself implies this bounding principle.

For a set x in the ground model, $\check{x}$ will occasionally stand for the canonical $\mathcal{P}$–name whose interpretation is always x. If $\mathcal{Q}$ is another partial order, $\mathcal{P}$ is a complete suborder of $\mathcal{Q}$ (equivalently, $\mathcal{Q}$ completely extends $\mathcal{P}$) if $\mathcal{P}$ is a suborder of $\mathcal{Q}$ and if every maximal antichain of $\mathcal{P}$ is also a maximal antichain of $\mathcal{Q}$. In a context referring to a forcing iteration $\langle \mathcal{P}_\alpha : \alpha \leq \delta \rangle$ (based on a sequence of names $\langle \dot{\mathcal{Q}}_\alpha : \alpha < \delta \rangle$), we will assume that $\alpha \leq \beta$ implies $\mathcal{P}_\alpha \subseteq \mathcal{P}_\beta$. $\dot{G}_\alpha$ will denote a $\mathcal{P}_\alpha$–name for the generic filter added by $\mathcal{P}_\alpha$[8] (for $\alpha \leq \delta$). If G is a $\mathcal{P}_\delta$–generic filter and $\alpha < \delta$, then $\{p \restriction \alpha : p \in G\}$ may be denoted by $G \restriction \alpha$. Also, if $\alpha < \delta$, then $\mathcal{P}_\delta / \dot{G}_\alpha$ is a $\mathcal{P}_\alpha$–name for $\{p \in \mathcal{P}_\delta : p \restriction \alpha \in \dot{G}_\alpha\}$ with the ordering inherited from $\mathcal{P}_\delta$. For any given $\alpha \leq \delta$, $\leq_\alpha$ denotes the ordering of $\mathcal{P}_\alpha$ and $\Vdash_\alpha$ denotes the forcing relation for $\mathcal{P}_\alpha$. Finally, if p is a condition in $\mathcal{P}_\delta$, the support of p, $supp(p)$, is the set of $\alpha < \delta$ such that $p \restriction \alpha$ does not force that $p(\alpha)$ is the weakest condition in $\dot{\mathcal{Q}}_\alpha$ (if $\alpha \in dom(p)$).

We will use a notion of $(\tilde{N}, \mathcal{P})$–generic condition, where $\mathcal{P}$ is a forcing notion and, for some cardinal θ, $\tilde{N}$ is, not an elementary substructure of $H(\theta)$ in the ground model, but a $\mathcal{Q}$–name for an elementary substructure of $H(\theta)^V$ in the extension (by $\mathcal{Q}$), where $\mathcal{Q}$ is a partial order completely extending $\mathcal{P}$.

Definition 1.2 *Let $\mathcal{P}$ be and $\mathcal{Q}$ be partial orders such that $\mathcal{P}$ is a complete suborder of $\mathcal{Q}$, let θ be an infinite cardinal and let $\tilde{N}$ be a $\mathcal{Q}$–name for an elementary substructure of the structure $H(\theta)$ as computed in the ground model.*[9] *A $\mathcal{P}$–condition p is $(\tilde{N}, \mathcal{P})$–generic if p forces (in $\mathcal{Q}$) that $\tilde{N} \cap D \cap \dot{G}$ is nonempty for every dense subset D of $\mathcal{P}$ belonging to $\tilde{N}$.*

Thus, for $\mathcal{P}$ and θ as in the above definition and for $N \preccurlyeq H(\theta)$ in the ground model, a condition is $(N, \mathcal{P})$–generic in the usual sense if and only if it is $(\check{N}, \mathcal{P})$–generic (for the canonical $\mathcal{P}$–name $\check{N}$ for N).

We shall also make use of the following extended notion of properness.

Definition 1.3 *Let $\mathcal{P}$ be a forcing notion, let $\theta > |TC(\mathcal{P})|$ be a cardinal and let F be a function from $[H(\theta)]^{\aleph_0}$ into $\mathcal{P}([H(\theta)]^{\aleph_0})$. We say that $\mathcal{P}$ is F–proper in case for every countable $N \preccurlyeq H(\theta)$ containing $\mathcal{P}$ and every $p \in N \cap \mathcal{P}$ there is a condition q extending p and some $M \in F(N)$ such that q is $(M, \mathcal{P})$–generic.*

[8] That is, for $\{p \restriction \alpha : p \in G\}$, where G is the generic filter added by $\mathcal{P}_\delta$.

[9] Even though $\tilde{N}$ is required by $\mathcal{Q}$ to be included in the ground model V, it is not required to be a member of V.

Thus, a partial order $\mathcal{P}$ is proper if and only if for some (equivalently for any) regular cardinal $\theta > |TC(\mathcal{P})|$, $\mathcal{P}$ is F–proper for the function F sending a countable $N \preccurlyeq H(\theta)$ to $\{N\}$.

Recall that a partial order $\mathcal{P}$ is semiproper in case for every regular cardinal $\theta > |TC(\mathcal{P})|$, every countable $N \preccurlyeq H(\theta)$ containing $\mathcal{P}$ and every $p \in \mathcal{P} \cap N$ there is some condition q extending p and forcing $\tau \in \check{N}$ for every $\mathcal{P}$–name τ for a countable ordinal, $\tau \in N$.

The following notion of iteration will occur very often in this paper.

Definition 1.4 *Let γ be a measurable cardinal, let U be a normal measure on γ, let $\theta \geq (2^\gamma)^+$ be a cardinal and let N be an elementary substructure of $H(\theta)$ of size less than γ and containing U.*

The one–step extension of N relative to U is defined as

$$\{f(\eta) \,:\, f \in N,\ f \text{ a function with domain } \gamma\},$$

for $\eta = min(\bigcap(U \cap N))$.

The iteration of N relative to U is the unique $\subseteq$–increasing and $\subseteq$–continuous sequence $\langle N_\xi \,:\, \xi \leq \gamma\rangle$ such that $N_0 = N$ and such that, for every $\xi < \gamma$, $N_{\xi+1}$ is the one–step extension of N_ξ relative to U.

$\langle \eta_\xi \,:\, \xi < \gamma\rangle$, where $\eta_\xi = min(\bigcap(U \cap N_\xi))$ for all $\xi < \gamma$, is called the critical sequence of the iteration of N relative to U.

It is a standard fact that, in the above definition, $N_\xi \subseteq N_{\xi+1} \preccurlyeq H(\theta)$ and $N_\xi \cap \gamma$ is a proper initial segment of $N_{\xi+1} \cap \gamma$ for every $\xi < \gamma$.

Finally, at some point it will be convenient to use the *natural (or Hessenberg) product of ordinals.* Given two ordinals α and β, the natural product of α and β, to be denoted by $\alpha \otimes \beta$, is computed as follows: Let us first define the natural sum of α and β, $\alpha \oplus \beta$, as the result of adding the Cantor normal forms of α and β, where adding two finite sums, σ_0 and σ_1, each one of the form

$$\omega^{e_m} \cdot k_m + \omega^{e_{m-1}} \cdot k_{m-1} + \ldots + \omega^{e_0} \cdot k_0$$

with e_l an ordinal and $m_l < \omega$ for all $l \leq n$, means the result of treating the symbol 'ω' as an indeterminate and adding σ_0 and σ_1 as if they were polynomials in the indeterminate ω.[10] Now, the natural product of two

[10] Recall that every ordinal α has a unique Cantor normal form; that is, α can be written as a unique polynomial expression $\omega^{e_m} \cdot k_m + \omega^{e_{m-1}} \cdot k_{m-1} + \ldots + \omega^{e_0} \cdot k_0$ (for some $m < \omega$) with $(e_l)_{l \leq m}$ a strictly increasing sequence of ordinals and with $m_l < \omega$ for all l.

ordinals can be computed by multiplying their Cantor normal forms, again as if they were polynomials in the indeterminate ω, using the natural sum $\oplus$ (instead of the ordinary addition) when adding exponents.[11] Obviously, the natural product of ordinals is an associative operation and, unlike the ordinary ordinal product, it is also commutative. The following result is proved in [Bl-Gu].

Fact 1.2 *Let $\alpha_0, \dots \alpha_{n-1}$ (for some $n < \omega$) be ordinals. The set of ordinals τ such that τ is the order type of some well–order R of the Cartesian product $\alpha_0 \times \dots \times \alpha_{n-1}$ extending the product order*[12] *has a maximum, and this maximum is $\alpha_0 \otimes \dots \otimes \alpha_{n-1}$.*

The rest of the paper is structured as follows: Section 2 contains the main result (Theorem 2.1) and starts its proof. Section 3, the longest section in the paper, is a detour through some lemmas on certain types of (generalized) iterations of models of set theory, transitive or otherwise, relative to sequences of measures on cardinals. Using the general theory of Section 3, Section 4 concludes the proof of Theorem 2.1. Finally, in Section 5 I build, starting from the existence of a supercompact cardinal, a model of PFA^{++} with a well–order of $H(\omega_2)$ definable, over $\langle H(\omega_2), \in\rangle$, by a formula without parameters. The model is built by forcing with a suitable instance of the forcing construction for proving Theorem 2.1.

2 The main result

Theorem 2.1 is the main result in this paper.

Theorem 2.1 *Suppose κ is an inaccessible cardinal, $C \subseteq \kappa$ is a club with $\kappa \setminus C$ unbounded in κ, and $F : C \longrightarrow V_\kappa$ is a function. Suppose $\mathcal{M} \subseteq \kappa \setminus C$ is a set of measurable cardinals with $\kappa = sup(\mathcal{M})$ and such that, for all $\gamma \in \mathcal{M}$, $F \restriction (C \cap \gamma) \in V_\gamma$.*[13] *Let $\langle S_i : i < \Lambda\rangle$ be, for some $\Lambda \leq 2^{\aleph_1}$, a sequence of stationary subsets of ω_1 with $S_i \cap S_{i'}$ nonstationary for all $i < i' < \Lambda$, and let $\langle \alpha_i : i < \Lambda\rangle$ be a sequence of nonzero countable ordinals.*

Then there is a countable support iteration $\langle \mathcal{P}_\xi : \xi \leq \kappa\rangle$ based on a sequence of names $\langle \dot{\mathcal{Q}}_\xi : \xi < \kappa\rangle$ such that

[11] This definition of $\otimes$ appears in [H], pp. 68–70.

[12] That is, such that $\langle \eta^0_0, \dots \eta^0_{n-1}\rangle R \langle \eta^1_0, \dots \eta^1_{n-1}\rangle$ whenever $\eta^0_l \leq \eta^1_l \in \alpha_l$ for all $l < n$.

[13] This implies, in particular, that $\kappa \setminus (C \cup \mathcal{M})$ is unbounded in κ.

(1) for every $\xi < \kappa$, $\Vdash_\xi \dot{\mathcal{Q}}_\xi$ is a semiproper forcing of size less than κ,

(2) given any $\xi \in C$, if $F(\xi)$ is a $\mathcal{P}_\xi$–name for a proper poset, then $\mathcal{P}_\xi$ forces $\dot{\mathcal{Q}}_\xi = F(\xi)$,

(3) for every $\xi < \kappa$, in $V^{\mathcal{P}_\xi}$ it holds that $\mathcal{P}_\kappa/\dot{G}_\xi$ is semiproper and forces $\kappa = \omega_2$, and

(4) $\mathcal{P}_\kappa$ forces that, for every $i < \Lambda$, S_i has guessing density equal to a pair (β_i^0, β_i^1) with

($\circ$) $\alpha_i + 1 \leq \beta_i^0$ and $\beta_i^1 \leq \omega^{\alpha_i \cdot \omega}$ if $\alpha_i > 1$,[14] and

($\circ$) $\beta_i^0 = \beta_i^1 = 2$ if $\alpha_i = 1$.

The proof of Theorem 2.1 will stretch over Sections 2, 3 and 4 of this paper. In Section 5 we will see that an appropriate instance of Theorem 2.1 yields, when κ is in fact supercompact, a poset forcing both that PFA^{++} holds and that there is a well–order of $H(\omega_2)$ definable over $\langle H(\omega_2), \in\rangle$ by a parameter–free formula.

By first collapsing Λ to ω_1, we may assume that Λ is at most ω_1. Hence, by shrinking the S_i's slightly if necessary, we may also assume that S_i and $S_{i'}$ are disjoint for all $i < i' < \Lambda$.

Let $A \subseteq \kappa \setminus (C \cup \mathcal{M})$ be unbounded and let $\langle A_\zeta : \zeta \in A\rangle$ be a sequence of pairwise disjoint unbounded (in κ) subsets of $\mathcal{M}$ such that $(A \cup \bigcup_{\zeta \in A} A_\zeta) \cap \gamma$ is bounded in γ for every $\gamma \in \bigcup_{\zeta \in A} A_\zeta$, and such that $A \cap A_\zeta = \zeta \cap A_\zeta = \emptyset$ for every $\zeta \in A$.

Since the iteration is built with countable supports and since κ is inaccessible, by (1) $\mathcal{P}_\kappa$ will have the κ–chain condition, so we will be able to fix a function $\varphi : \kappa \longrightarrow V_\kappa$ such that each $\varphi(\xi)$ is a $\mathcal{P}_\xi$–name, with the property that for every $\mathcal{P}_\kappa$–name τ for a member of $H(\kappa)$ there is some $\zeta \in A$ such that $\Vdash_\zeta \tau = \varphi(\zeta)$, and with the property that for every $\zeta \in A$ and every $\mathcal{P}_\kappa$–name τ for a member of $H(\kappa)$ there is some $\gamma \in A_\zeta$ such that $\Vdash_\gamma \tau = \varphi(\gamma)$.

Our iteration $\langle \mathcal{P}_\xi : \xi \leq \kappa\rangle$ will be built with countable supports and will be based on a sequence $\langle \dot{\mathcal{Q}}_\xi : \xi < \kappa\rangle$ of names. Each $\dot{\mathcal{Q}}_\xi$ will be chosen to be a name for a semiproper forcing of size less than the least member of $\bigcup_{\zeta \in A} A_\zeta \setminus (\xi + 1)$, so that (1) will hold. Moreover, $\dot{\mathcal{Q}}_\xi$ will be forced, in $V^{\mathcal{P}_\xi}$, to be the trivial forcing $\{\emptyset\}$ if $\xi \in \kappa \setminus (C \cup A \cup \bigcup_{\zeta \in A} A_\zeta)$, and to be a proper

[14] $\alpha \cdot \beta$ and α^β denote, respectively, ordinal multiplication and ordinal exponentiation.

forcing of size $F(\xi)$ if $\xi \in C$. Furthermore, if $\xi \in C$ and $F(\xi)$ is a $\mathcal{P}_\xi$–name for a proper poset, then $\dot{\mathcal{Q}}_\xi$ will be forced to be $F(\xi)$. We also build our iteration in such a way that, for a fixed bookkeeping function φ as in the above paragraph, if $\zeta \in A$, then

(i) $\mathcal{P}_\zeta$ forces that $\dot{\mathcal{Q}}_\zeta$ is $\{\emptyset\}$ unless $\varphi(\zeta)$ is a stationary subset of ω_1 for which there is a (unique) $i < \Lambda$ such that $\varphi(\zeta) \subseteq S_i$, in which case $\dot{\mathcal{Q}}_\zeta$ is the forcing for adding, with countable conditions, a function $F : \varphi(\zeta) \longrightarrow [\omega_1]^{\leq\aleph_0}$ such that

   (i1) $F(\nu)$ is a subset of ω_1 of order type strictly less than $\omega^{\alpha_i\cdot\omega}$ for all $\nu \in dom(F)$ in case $\alpha_i > 1$ and such that

   (i2) $F(\nu)$ is a singleton for all $\nu \in dom(F)$ in case $\alpha_i = 1$,[15]

   and furthermore,

(ii) for each $\gamma \in A_\zeta$, $\mathcal{P}_\gamma$ forces that $\dot{\mathcal{Q}}_\gamma$ is $\{\emptyset\}$ unless $\varphi(\zeta)$ is a stationary subset of ω_1 for which there is a unique $i < \Lambda$ such that $\varphi(\zeta) \subseteq S_i$ and $\varphi(\gamma)$ is a function into $\mathcal{P}(\omega_1)$ with domain $\varphi(\zeta)$ such that $ot(\varphi(\gamma)(\nu)) < \alpha_i$ for all $\nu \in \varphi(\zeta)$, in which case $\dot{\mathcal{Q}}_\gamma$ is the natural forcing for adding, with countable conditions, a canonical function for γ avoiding $\varphi(\gamma)$; that is, $\dot{\mathcal{Q}}_\gamma$ is the forcing, ordered by extension, of $\subseteq$–increasing and $\subseteq$–continuous functions $p : \nu_0 + 1 \longrightarrow [\gamma]^{\aleph_0}$ (for some $\nu_0 < \omega_1$) such that $ot(p(\nu)) \notin \varphi(\gamma)(\nu)$ if $p(\nu) \cap \omega_1 \in \varphi(\zeta)$.

This finishes the description of the iteration. By a standard argument using the fact that this iteration has been built with countable supports, it can be seen that all ordinals less than κ are collapsed into ω_1 by $\mathcal{P}_\kappa$. Thus, since $\mathcal{P}_\kappa$ has the κ–chain condition, it forces $\kappa = \omega_2$.

Let U_γ be, for each $\gamma \in \bigcup_{\zeta\in A} A_\zeta$, a normal measure on γ. Given any γ as above and any $\xi \leq \gamma$, let $\tilde{U}^\gamma_\xi$ be a $\mathcal{P}_\xi$–name for the ultrafilter on γ generated by U_γ, that is, for $\{X \subseteq \gamma : (\exists Y \in U_\gamma)(Y \subseteq X)\}$. Since, by our construction, $\mathcal{P}_\gamma$ is of size less than γ for every $\gamma \in \bigcup_{\zeta\in A} A_\zeta$, each $\tilde{U}^\gamma_\xi$ (for $\gamma \in \bigcup_{\zeta\in A} A_\zeta$, $\xi \leq \gamma$) is forced by $\mathcal{P}_\xi$ to be a normal measure on γ.

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[15] It would cause the same effect to just specify that if $\zeta \in A$, then $\mathcal{P}_\zeta$ forces that $\dot{\mathcal{Q}}_\zeta$ is the poset for adding a function from ω_1 into $\mathcal{P}(\omega)$ by initial segments (such poset is certainly forcing–equivalent to the above description for $\dot{\mathcal{Q}}_\zeta$). The present description has been chosen in order to make the proof clearer.

It will be convenient to fix some notation for (names for) iterations of (names for) countable structures relative to some of the U_γ (or some of the $\tilde{U}^\gamma_\xi$): Given $\gamma \in \bigcup_{\zeta\in A} A_\zeta$, a cardinal $\theta > 2^\gamma$ and a countable $N \preccurlyeq H(\theta)$ containing U_γ, let $\langle (N)^\gamma_\alpha \,:\, \alpha \leq \gamma\rangle$ be the iteration of N relative to U_γ. Also, for γ and θ as above, given $\xi \leq \gamma$ and a $\mathcal{P}_\xi$–name $\dot{N}$ for a countable elementary substructure of $H(\theta)^{V[\dot{G}_\xi]}$ containing $\tilde{U}^\gamma_\xi$, let $\langle (\dot{N})^\gamma_{\xi,\alpha} \,:\, \alpha \leq \gamma\rangle$ be a sequence of $\mathcal{P}_\xi$–names such that $\mathcal{P}_\xi$ forces, for each α, that $(\dot{N})^\gamma_{\xi,\alpha}$ is the α-th member of the iteration of $\dot{N}$ relative to $\tilde{U}^\gamma_\xi$. Finally, let $\mathcal{U}^{\overline{\beta},\beta}_\xi$ be, for every three ordinals $\beta \geq \overline{\beta} \geq \xi$, a $\mathcal{P}_\xi$–name for $\langle \tilde{U}^\gamma_\xi \,:\, \overline{\beta} \leq \gamma < \beta,\, \gamma \in \bigcup_{\zeta\in A} A_\zeta\rangle$.

Conclusion (4) for $\mathcal{P}_\kappa$ is equivalent to the statement that, in $V^{\mathcal{P}_\kappa}$, every S_i has guessing density, in the old sense, equal to $[\alpha_i,\, \omega^{\alpha_i\cdot\omega})$ if $\alpha_i > 1$, and equal to $\{1\}$ if $\alpha_i = 1$. In other words, in order to prove (4) for $\mathcal{P}_\kappa$ it suffices to show, in $V^{\mathcal{P}_\kappa}$, that for every i and every stationary $S^* \subseteq S_i$,

(a) there is a function $F : S^* \longrightarrow \mathcal{P}(\omega_1)$ such that $ot(F(\nu)) \in [\alpha_i,\, \omega^{\alpha_i\cdot\omega})$ (if $\alpha_i > 1$) and $|F(\nu)| = 1$ (if $\alpha_i = 1$) and such that F guesses all canonical functions, and

(b) no function $F : S^* \longrightarrow \mathcal{P}(\omega_1)$ with $ot(F(\nu)) < \alpha_i$ for all $\nu \in S^*$ guesses all canonical functions.

Let us assume for a while that $\mathcal{P}_\kappa$ preserves the stationarity of all S_i[16] and let us verify one half of (4) for $\mathcal{P}_\kappa$ (in the above form). More specifically, let us prove, for a $\mathcal{P}_\kappa$–generic filter G, that if $i_0 < \Lambda$ and $\zeta \in A$ are such that $S^* := \varphi(\zeta)_{G\restriction\zeta}$ is, in $V[G]$, a stationary subset of S_{i_0} and $\gamma \in A_\zeta$ is such that $\varphi(\gamma)_{G\restriction\gamma}$ is a function into $\mathcal{P}(\omega_1)$ defined on S^* such that $ot(\varphi(\gamma)_{G\restriction\gamma}(\nu)) < \alpha_{i_0}$ for all ν, then there is a canonical function g for γ such that $g(\nu) \notin \varphi(\gamma)_{G\restriction\gamma}(\nu)$ for club–many $\nu \in S^*$.[17]

In the present situation, by (ii) above, $\mathcal{Q}_\gamma$ is the forcing for adding, by initial segments, a canonical function for γ avoiding $\varphi(\gamma)$. Let us show, by induction on $k < \omega_1$, that for every $\delta < \gamma$,

$$D^\delta_k := \{p \in \mathcal{Q}_\gamma \,:\, k \in dom(p),\, \delta \in \cup\, range(p)\}$$

[16]Of course later we will prove that $\mathcal{P}_\kappa$ is in fact semiproper.

[17]Since φ is supposed to be a suitable bookkeeping function, this will be enough to show, in $V[G]$, that the statement in (b) holds for every i and every stationary $S^* \subseteq S_i$.

is dense in $V[G \restriction \gamma]$,[18] from which it will follow that $\mathcal{Q}_\gamma$ forces that there is a surjection $\pi : \omega_1 \longrightarrow \gamma$ and a club $C \subseteq \omega_1$ such that, for all $j \in C$,

$$ot(\pi``j) \notin \varphi(\gamma)_{G\restriction\gamma}(\omega_1 \cap \pi``j)$$

if $\varphi(\gamma)_{G\restriction\gamma}(\omega_1 \cap \pi``j)$ is defined: In $V[G \restriction \gamma]$ let p be a condition in $\mathcal{Q}_\gamma$, let $N \preccurlyeq H(\theta)$ be countable, for a large enough cardinal θ, and containing all relevant objects, and let $\langle N_\alpha : \alpha \leq \gamma\rangle$ be the iteration of N relative to $(\tilde{U}^\gamma_\gamma)_{G\restriction\gamma}$.

First of all note that if p' is a condition with $k \in dom(p')$, then p' can trivially be extended to a condition p^* in D^δ_k by setting $p^* = p' \cup \{\langle dom(p'), X\rangle\}$, where $p'(max(dom(p')) \subseteq X \subseteq \gamma$, $\delta \in X$, and $ot(X) \notin \varphi(\gamma)_{G\restriction\gamma}(dom(p'))$ in case $dom(p') \in S^*$. Hence, we may assume $dom(p) < k$. If $k \notin S^*$, then we can build, by applying the induction hypothesis to $k' < k$, a decreasing sequence $(p_n)_{n<\omega}$ of conditions extending p with $\delta \in \cup range(p_0)$ and with $k \leq sup_{n<\omega} dom(p_n)$. If there is some n with $k < dom(p_n)$, then $p_n \in D^\delta_k$. Otherwise, $\bigcup_n p_n \cup \{\langle k, \bigcup_n \cup range(p_n)\rangle\}$ is a condition in D^δ_k extending p. Now suppose $k \in S^*$. $\{ot(N_\alpha \cap \gamma) : \alpha < \alpha_{i_0}\}$, being a subset of ω_1 of order type exactly α_{i_0}, is not contained in $\varphi(\gamma)_{G\restriction\gamma}(k)$, that is, there is some $\overline{\alpha} < \alpha_{i_0}$ such that $ot(N_{\overline{\alpha}} \cap \gamma) \notin \varphi(\gamma)_{G\restriction\gamma}(k)$. Since, by induction hypothesis, $D^{\delta'}_{k'}$ is dense for all $k' < k$ and all $\delta' < \gamma$, we can build a decreasing sequence $(p_n)_{n<\omega}$ of conditions in $N_{\overline{\alpha}}$ with $p_0 = p$ and such that $N_{\overline{\alpha}} \cap \gamma = \bigcup_n \cup range(p_n)$. By the above considerations, we may assume k is a limit ordinal and $k = sup_n dom(p_n)$. It follows then from the definition of $\mathcal{Q}_\gamma$ that $\bigcup_n p_n \cup \{\langle k, N_{\overline{\alpha}} \cap \gamma\rangle\}$ is a condition extending each p_n. This proves that D^δ_k is dense for every $\delta < \gamma$.

The following extra information can now be derived by arguing as above, replacing k by $N \cap \omega_1$ and ensuring that $(p_n)_{n<\omega}$ is an $(N, \mathcal{Q}_\gamma)$–generic sequence.

Lemma 2.2 *Given a large enough cardinal θ, $\mathcal{Q}_\gamma$ is F_γ–proper for the function F_γ sending a countable $N \preccurlyeq H(\theta)$ containing $\varphi(\gamma)_{G\restriction\gamma}$ and $(\tilde{U}^\gamma_\gamma)_{G\restriction\gamma}$ to*

[18] This is proved by a standard argument exploiting the fact, for a given countable $N \preccurlyeq H(\theta)$ (for a large enough cardinal θ), that $ot(N_\alpha \cap \gamma) \notin \varphi(\gamma)_{G\restriction\gamma}(N \cap \omega_1)$ for some appropriate member N_α of the iteration of N of length ω_1 relative to $(\tilde{U}^\gamma_\gamma)_{G\restriction\gamma}$ (if $N \cap \omega_1 \in S^*$). However, in our situation it will be convenient to derive some extra information from the argument – namely the fact that α can always be suitably bounded – which will be of crucial importance for the proof of the rest of the theorem.

the set $\{(N)^{\gamma}_{\gamma,\nu} : \nu < \alpha_{i_0}\}$ *if* $i_0 < \Lambda$ *is such that* $N \cap \omega_1 \in S_{i_0}$ *and to* $\{N\}$ *if* $N \cap \omega_1 \notin \bigcup_{i<\Lambda} S_i$ *(so in particular* $\mathcal{Q}_\gamma$ *is semiproper).*

We proceed to ascend to the proof of the rest of the theorem. It will be useful to collect several general lemmas concerning iterations of models of set theory relative to measures on cardinals.

3 General results about iterations relative to measures on cardinals

This section introduces two types of iterations of models of (a fragment of) set theory. The analysis of these iterations is then carried out to the level needed for the proof of Theorem 2.1. The theory surely can be developed with more generality, which I have not bothered to do here.

Let ZFC^* denote the theory ZFC without the Power Set Axiom.[19] First I am going to consider iterations $\langle M_\alpha : \alpha < \overline{\alpha}\rangle$ of arbitrary length $\overline{\alpha}$ of a transitive model of ZFC^* and in which, at every stage α with $\alpha + 1 < \overline{\alpha}$, $M_{\alpha+1}$ is the ultrapower of M_α by some measure on some measurable cardinal in M_α. Then I will consider iterations $\langle N_\alpha : \alpha < \overline{\alpha}\rangle$ of a – typically non-transitive – elementary substructure N of $H(\theta)$, for some regular cardinal θ. This time, at every given stage α with $\alpha + 1 < \overline{\alpha}$, $N_{\alpha+1}$ is the one–step extension of N_α relative to some normal measure in N_α on a measurable cardinal. The measures and even the measurable cardinals used at different stages need *not* be the same.

I will start by introducing notation to handle these types of iterations (in Definitions 3.1 and 3.2). Then I will observe that an iteration of the second kind of some $N \preccurlyeq H(\theta)$ (for some θ) corresponds, via transitive collapses, to an iteration of the first kind of its transitive collapse. Most of the section will be devoted to proving general lemmas on these types of iterations, first in Subsection 3.1, dealing with iterations of the first kind, and then in Subsection 3.2, dealing with iterations of the second kind. These lemmas will be primarily focused on the (vaguely stated) question "given a model N and an ordinal ϵ, to what extent can we control $ot(N' \cap \epsilon)$ for models N' occurring in an iteration (of the second kind) of N or of an expansion $N[G]$ of N via a generic filter G?". We shall need some knowledge of this kind for the rest of the proof of Theorem 2.1 in Section 4.

[19]Note that $H(\theta)$ satisfies ZFC^* for every regular cardinal θ.

Given a transitive model M of ZFC^*, an ordinal $\overline{\alpha}$ and, in M, a non-principal γ–complete ultrafilter U on a measurable cardinal γ, let

$$\langle M^U_\xi, j^{M,U}_{\xi,\xi'} \,:\, \xi \leq \xi' < \overline{\alpha}\rangle$$

denote the iteration of $\langle M, U\rangle$ of length $\overline{\alpha}$, that is, the directed system of length $\overline{\alpha}$ with direct limits taken at limit stages, where the M^U_ξ's are transitive ZFC^*–models and each $j^{M,U}_{\xi,\xi'} : M^U_\xi \longrightarrow M^U_{\xi'}$ is an elementary embedding, and where $M^U_0 = M$ and, for each ξ with $\xi + 1 < \overline{\alpha}$, $M^U_{\xi+1}$ is the ultrapower $Ult(M^U_\xi,\, j^{M,U}_{0,\xi}(U))$ and $j^{M,U}_{\xi,\xi+1} : M^U_\xi \longrightarrow M^U_{\xi+1}$ is given by $j^{M,U}_{\xi,\xi+1}(a) = [c_a]_{j^{M,U}_{0,\xi}(U)}$, where c_a is the constant function on $j^{M,U}_{0,\xi}(\gamma)$ with value a.[20] Also, letting $\vartheta_\xi = j^{M,U}_{0,\xi}(\gamma)$ whenever $\xi + 1 < \overline{\alpha}$, $\langle \vartheta_\xi \,:\, \xi + 1 < \overline{\alpha}\rangle$ is the *critical sequence* of this iteration.

The first type of iterations we shall present are linear compositions of iterations as in the above paragraph. For a given model M_0, they are directed systems $\langle M_\alpha, j_{\alpha,\beta} \,:\, \alpha \leq \beta < \overline{\alpha}\rangle$, with direct limit taken at limit stages, where the M_α's are transitive models of ZFC^*, the $j_{\alpha,\beta}$'s are elementary embeddings, where $M_{\alpha+1}$ is an ultrapower of M_α by a normal measure W_α on a measurable cardinal γ_α in M_α (if $\alpha + 1 < \overline{\alpha}$), and where, for $\alpha < \beta < \beta + 1 < \tau$, W_β *need not be* $j_{\alpha,\beta}(W_\alpha)$.[21]

Note that every such iteration can be split naturally into a chain of iterations each of which involves ultrapowers of the corresponding images of the same measure. More precisely, every such iteration $\mathcal{I}$ can be split into a chain of iterations, $\langle \mathcal{I}_i \,:\, i < \tau\rangle$, such that, for every i, $\mathcal{I}_i$ is the iteration of length ν_i – for some ν_i – of $\langle M_{\alpha_0}, W_{\alpha_0}\rangle$, where α_0 is the stage of $\mathcal{I}$ at which $\mathcal{I}_i$ starts. Furthermore, these iterations can be taken to be maximal, that is, they can be taken so that if $i + 1 < \tau$ and α_0 and β_0 are the stage of $\mathcal{I}$ at which, respectively, $\mathcal{I}_i$ and $\mathcal{I}_{i+1}$ start, then $W_{\beta_0} \neq j_{\alpha_0,\beta_0}(W_{\alpha_0})$. Definition 3.1 incorporates this way of looking at this kind of iterations.

Definition 3.1 *Given a transitive model M of ZFC^*, an ordinal τ, and two sequences $\mathcal{W} := \langle W_i \,:\, i < \tau\rangle$ and $\vec{\nu} := \langle \nu_i \,:\, i < \tau\rangle$,*[22] *the $\vec{\nu}$–iteration of M*

[20] We may denote the embedding of M^U_ξ derived from $j^{M,U}_{0,\xi}(U)$ by $j^{M,U}_\xi$.

[21] And, in fact, γ_β need not be $j_{\alpha,\beta}(\gamma_\alpha)$.

[22] None of the sequences $\vec{\nu}$ or $\mathcal{W}$ are required to be in M (but, by (c3), $W_0 \in M$ if $\nu_0 > 0$).

relative to $\mathcal{W}$, if it exists, is the directed system

$$\langle M_\alpha, j_{\alpha,\beta} \,:\, \alpha \leq \beta < \Sigma_{i<\tau}\, \nu_i\rangle,^{23}$$

where

- *(a)* $M_0 = M$,
- *(b) all* $j_{\alpha,\beta}$*'s are elementary embeddings, all* M_α*'s are transitive, and direct limits are taken at limit stages, and*
- *(c) for each* $i < \tau$,
  - *(c1)* W_i *is a collection of sets of ordinals and each* ν_i *is a nonzero ordinal,*
  - *(c2) if* $\Sigma_{j<i}\,\nu_j = \overline{\nu} + 1$ *for some* $\overline{\nu}$*, then* $M_{\Sigma_{j<i}\nu_j} = M_{\overline{\nu}}$,
  - *(c3) if* $\nu_i > 1$*, then* $\gamma_i := \bigcup W_i$ *is a measurable cardinal in* $M_{\Sigma_{j<i}\nu_j}$, $W_i \in M_{\Sigma_{j<i}\nu_j}$ *is, in* $M_{\Sigma_{j<i}\nu_j}$*, a non-principal* γ_i*–complete ultrafilter on* γ_i*, and*

    $$\langle M_\alpha, j_{\alpha,\beta} \,:\, \Sigma_{j<i}\,\nu_j \leq \alpha \leq \beta < \Sigma_{j\leq i}\,\nu_j\rangle$$

    is the iteration of $\langle M_{\Sigma_{j<i}\nu_j}, W_i\rangle$ *of length* ν_i*, and*
  - *(c4) if* $i+1 < \tau$, $W_{i+1} \neq j_{\Sigma_{j<i}\nu_j,\,\Sigma_{j\leq i}\nu_j}(W_i)$.

In the above system we may denote M_α *by* $M_\alpha^{\mathcal{W},\vec{\nu}}$ *and* $j_{\alpha,\beta}$ *by* $j_{\alpha,\beta}^{M,\mathcal{W},\vec{\nu}}$*. If each* W_i *is, in addition, a normal ultrafilter on* γ_i *in* $M_{\Sigma_{j<i}\nu_j}$*, then we say that the iteration is normal.*

Also, if $\langle \vartheta_\alpha \,:\, \alpha+1 < \Sigma_{i<\tau}\,\nu_i\rangle$ *is such that, for all* $i < \tau$ *with* $\nu_i > 1$ *and all* $\xi < \nu_i$,

$$\vartheta_{(\Sigma_{j<i}\nu_j)+\xi} = j^{M,\mathcal{W},\vec{\nu}}_{\Sigma_{j<i}\nu_j,\,(\Sigma_{j<i}\nu_j)+\xi}(\gamma_i),$$

(and $\vartheta_{\Sigma_{j<i}\nu_j} = \emptyset$ *if* $\nu_i = 1$*) then* $\langle \vartheta_\alpha \,:\, \alpha+1 < \Sigma_{i<\tau}\,\nu_i\rangle$ *is called the critical sequence of the iteration.*

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[23] $\Sigma_{i<\tau}\,\nu_i$ denotes ordinal addition. Specifically, $\Sigma_{i<\tau}\,\nu_i$, and occasionally $\Sigma(\nu_i \,:\, i<\tau)$, will denote the sum of the sequence $\langle \nu_i \,:\, i<\tau\rangle$ of ordinals.

Note that, given an ordinal i, if ν_j (for $j < i$) are ordinals, then $\Sigma_{j<i}\,\nu_j$ is a successor ordinal if and only if i and ν_{i^*} are successor ordinals (where $i = i^* + 1$). In that case ((c2) of the definition), the last model of the construction below stage $\Sigma_{i<j}\,\nu_j$ will be repeated at stage $\Sigma_{i<j}\,\nu_j$. I shall use this indexing of the models for notational convenience.

The second type of iteration we will be considering is the following generalization of Definition 1.4.

Definition 3.2 *Let θ be a regular cardinal and let N be an elementary substructure of $H(\theta)$. Let τ be an ordinal and let $\mathcal{W} = \langle W_i \,:\, i < \tau\rangle$ and $\vec{\nu} = \langle \nu_i \,:\, i < \tau\rangle$ be two sequences.*[24] *The $\vec{\nu}$–iteration of N relative to $\mathcal{W}$, if it exists, is the $\subseteq$–increasing and $\subseteq$–continuous sequence of the form*

$$\langle N_\alpha \,:\, \alpha < \Sigma_{i<\tau}\,\nu_i\rangle$$

such that $N_0 = N$ and such that, for each $i < \tau$,

(a) W_i is a collection of sets of ordinals and ν_i is a nonzero ordinal,

(b) if $\Sigma_{j<i}\,\nu_j = \overline{\nu} + 1$ for some $\overline{\nu}$, then $N_{\Sigma_{j<i}\,\nu_j} = N_{\overline{\nu}}$,

(c) if $\nu_i > 1$, then $\gamma_i := \bigcup W_i \in N_{\Sigma_{j<i}\,\nu_j}$ is a measurable cardinal (in $H(\theta)$) and $W_i \in N_{\Sigma_{j<i}\,\nu_j}$ is (in $H(\theta)$) a normal measure on γ_i,

(d) $|N_{\Sigma_{j<i}\,\nu_j}| < \gamma_i$ and $\nu_i \leq \gamma_i + 1$,

(e) $\langle N_\alpha \,:\, \Sigma_{j<i}\,\nu_j \leq \alpha < \Sigma_{j\leq i}\,\nu_j\rangle$ is the initial segment of length ν_i of the iteration of $N_{\Sigma_{j<i}\,\nu_j}$ relative to W_i; in other words, for every ordinal α with $\Sigma_{j<i}\,\nu_j < \alpha + 1 < \Sigma_{j\leq i}\,\nu_j$,

$$N_{\alpha+1} = \{f(\eta_\alpha) \,:\, f \in N_\alpha,\ f \text{ a function with domain } \gamma_i\}$$

where, for every α with $\Sigma_{j<i}\,\nu_j \leq \alpha < \Sigma_{j\leq i}\,\nu_j$, $\eta_\alpha = min(\bigcap(W_i \cap N_\alpha))$, and

(f) if $i + 1 < \tau$, then $W_{i+1} \neq W_i$.

[24] Similarly as in Definition 3.1, none of these sequences is required to be in N.

We shall call the sequence $\langle \overline{\eta}_\alpha : \alpha + 1 < \Sigma_{i<\tau}\,\nu_i\rangle$ such that $\overline{\eta}_\alpha = \eta_\alpha$ whenever there is some $i < \tau$ with $\Sigma_{j<i}\,\nu_j < \alpha+1 < \Sigma_{j\leq i}\,\nu_j$ (and $\overline{\eta}_{\Sigma_{j<i}\,\nu_j} = \emptyset$ if $\nu_i = 1$) the critical sequence of the iteration.

Similarly as in the previous definition, we may refer to N_α (in the expression $\langle N_\alpha : \alpha < \Sigma_{i<\tau}\,\nu_i\rangle$) as $N_\alpha^{\mathcal{W},\vec{\nu}}$.

Given an ordinal λ, we will say that the iteration is bounded by λ if $\nu_i < \lambda$ for all i.

Also, we will say that the iteration is closed in case $\Sigma_{i<\tau}\,\nu_i$ is a successor ordinal.

Each of the sequences $\langle N_{(\Sigma_{j<i}\,\nu_j)+\xi} : \xi < \nu_i\rangle$ (for $i < \tau$) will be called component iterations of $\langle N_\alpha : \alpha < \Sigma_{i<\tau}\,\nu_i\rangle$.

Finally, a component iteration $\langle N_{(\Sigma_{j<i}\,\nu_j)+\xi} : \xi < \nu_i\rangle$ will said to be closed if ν_i is a successor ordinal.

In analogy with what happened with the iterations presented in Definition 3.1, the iterations defined above can be construed, in a unique way, as concatenations of maximal iterations $\mathcal{I}_i$ ($i < \tau$, for some τ) such that, for each i, $\mathcal{I}_i$ is an initial segment of the iteration, in the sense of Definition 1.4, of the first model N_{α_0} occurring in $\mathcal{I}_i$ relative to a fixed normal measure in N_{α_0}. The notation chosen is intended to make this decomposition visible.

The following observation, which is easy to verify, shows that an iteration, in the sense of Definition 3.2, of an elementary substructure N of some $H(\theta)$ corresponds, via transitive collapses, to an iteration of the transitive collapse of N in the sense of Definition 3.1.

Fact 3.1 *Suppose*

(a) $\langle \gamma_i : i < \tau\rangle$ is a sequence of measurable cardinals,

(b) $\mathcal{W} = \langle W_i : i < \tau\rangle$ is such that each W_i is a normal measure on γ_i,

(c) $\vec{\nu} = \langle \nu_i : i < \tau\rangle$ is a sequence of ordinals with $1 \leq \nu_i \leq \gamma_i + 1$ for all i,

(d) θ is a regular cardinal such that $\theta \geq (2^{\gamma_i})^+$ for all i,

(e) N is an elementary substructure of $H(\theta)$ of size less than γ_i for all $i < \tau$ and containing $\mathcal{W}$, and

(f) the $\vec{\nu}$–iteration of N relative to $\mathcal{W}$ exists.

Let us define

(g) $M_{(\Sigma_{j<i}\nu_j)+\xi}$ *to be the transitive collapse of* $N_{(\Sigma_{j<i}\nu_j)+\xi}$ *via the collapsing function* $\pi_{(\Sigma_{j<i}\nu_j)+\xi}$ *for every* $i < \tau$ *and every* $\xi < \nu_i$,

(h) $\langle M_\alpha, j_{\alpha,\beta} : \alpha \leq \beta < \Sigma_{i<\tau}\nu_i\rangle$ *to be the directed system defined by letting* $j_{\alpha,\beta} = \pi_\beta \circ (\pi_\alpha)^{-1}$, *and*

(i) *for every* $i < \tau$, $\overline{W}_i = \pi_{\Sigma_{j<i}\nu_j}(W_i)$ *if* $\nu_i > 1$.

Then $\langle M_\alpha, j_{\alpha,\beta} : \alpha \leq \beta < \Sigma_{i<\tau}\nu_i\rangle$ *is the* $\vec{\nu}$*–iteration of* M_0 *relative to* $\langle \overline{W}_i : i < \tau\rangle$.

Incidentally, note that, if $\nu_0 > 1$, any N_0 and N_1 coming from the statement of Fact 3.1 have the property that N_0 is strictly included in N_1 whereas the transitive collapse of N_1 is strictly included in the transitive collapse of N_0.

3.1 Iterations of transitive models (as in Definition 3.1).

The following general fact can be extracted from classical results of Kunen ([Ku1]) on iterated ultrapowers. This fact will be used in the proof of Lemma 3.4.

Fact 3.2 *There is a formula* $\Phi(x, y, z, w)$ *with the property that, for every transitive* ZFC^**–model* M, *every* $b \in M$, *every ordinal* $\overline{\xi} \in M$ *and every* $U \in M$ *which is, in* M, *a normal ultrafilter on a measurable cardinal* γ,

$$c = j^{M,U}_{0,\overline{\xi}}(b) \text{ if and only if } M \models \Phi(U, \overline{\xi}, b, c)$$

Furthermore, $\Phi(x, y, z, w)$ *can be chosen so that* $\Phi(U, \overline{\xi}, b, w)$ *defines* $j^{M,U}_{0,\overline{\xi}}(b)$ *correctly in* M^* *for every* $b \in M^*$ *whenever* M^* *is an inner model of* M *containing* U *and* $\overline{\xi}$ *which is sufficiently closed in* M *with respect to* U, *in the sense that for every* $X \in U$, *every* $n < \omega$ *and every function* $f : [X]^n \longrightarrow M^*$ *in* M *there is* $Y \subseteq X$, $Y \in U$, *such that* $f \restriction [Y]^n \in M^*$.

Proof: Given an ultrafilter U on a cardinal δ (in some given model N), let us define the sequence $(U^i)_{0<i<\omega}$ by $U^1 = \{[X]^1 : X \in U\}$ and by specifying, for every $i > 1$ and every $X \in \mathcal{P}([\delta]^i) \cap N$, that X is in U^i if and only if

$$\{s \in [\delta]^{i-1} : \{\beta < \delta : s \cup \{\beta\} \in X\} \in U\} \in U^{i-1}$$

Every U^i thus defined is clearly an ultrafilter, in N, over $[\delta]^i$. Note also that if δ is a measurable cardinal and U is, in N, a normal ultrafilter, then

$$U^i = \{X \in \mathcal{P}([\delta]^i) \cap N \,:\, (\exists X_0 \in U)\,([X_0]^i \subseteq X)\}$$

for every $i > 0$.

Now, let M, ξ, U and γ be as in the statement. Let $\langle \vartheta_\xi \,:\, \xi < \overline{\xi}\rangle$ be the critical sequence of the iteration of $\langle M, U\rangle$ of length $\overline{\xi}+1$. Every member of $M^U_{\overline{\xi}}$ is of the form $j^{M,U}_{0,\overline{\xi}}(f)(\{\vartheta_{\xi_0}, \ldots \vartheta_{\xi_{n-1}}\})$ for some $n < \omega$, some function $f : [\gamma]^n \longrightarrow M$ in M and some sequence $\xi_0 < \ldots < \xi_{n-1} < \overline{\xi}$. Also, note that for R being $=$ or $\in$, for every two functions f and g in M with domain $[\gamma]^n$, and for every $\xi_0 < \ldots < \xi_{n-1} < \overline{\xi}$,

$$j^{M,U}_{0,\overline{\xi}}(f)(\{\vartheta_{\xi_0}, \ldots \vartheta_{\xi_{n-1}}\})\, R\, j^{M,U}_{0,\overline{\xi}}(g)(\{\vartheta_{\xi_0}, \ldots \vartheta_{\xi_{n-1}}\})$$

if and only if

$$\{s \in [\gamma]^n \,:\, f(s)\, R\, g(s)\} \in U^n,$$

where $(U^i)_{0<i<\omega}$ is defined, in M, as above.

Furthermore, if M^* is an inner model of M containing U and $\overline{\xi}$ and sufficiently closed in M with respect to U, in the sense of the statement, it can be proved that for every $X \in U^n$ and every two functions f and g in M with domain $[X]^n$ there is some $Y \subseteq X$, $Y \in U^n$, such that $Y \in (U^n)^{M^*}$ and such that $f \restriction Y$ and $g \restriction Y$ are members of M^*.[25] Hence, if R is as above and $\{s \in [\gamma]^n \,:\, f(s)\, R\, g(s)\} \in U^n$, then there is some $Y \in U^n \cap M^*$ such that

$$\{s \in Y \,:\, (f \restriction Y)(s)\, R\, (g \restriction Y)(s)\} \in U^n$$

holds in M^*.

Finally, note that although the expressions $j^{M,U}_{0,\overline{\xi}}(f)(\{\vartheta_{\xi_0}, \ldots \vartheta_{\xi_{n-1}}\})$ representing different members of $M^U_{\overline{\xi}}$ will have different dimensions in general,[26] they can nevertheless be made comparable through composition with the inverses of appropriate projection functions.

Hence, $\Phi(x, y, z, w)$ can be taken to express the following property:

[25]Because, both in M and in M^*, U^n is the ultrafilter on $[\gamma]^n$ generated by the sets of the form $[X_0]^n$ for $X_0 \in U$.

[26]That is, different f's may have $[\gamma]^n$, for different n's, as domain.

x is an ultrafilter on a measurable cardinal γ, y is an ordinal, and w is the transitive collapse of $\langle \overline{z}, E\rangle$, where $\overline{z}$ is the set of pairs $\langle f, s\rangle$ such that s is a finite subset of y and such that f is a function from $[\gamma]^n$ (where n is the size of s) into z, and, for $\langle f_\epsilon, s_\epsilon\rangle$ in $\overline{z}$ (for $\epsilon = 0, 1$), $\langle f_0, s_0\rangle E \langle f_1, s_1\rangle$ if and only if, letting $s = s_0 \cup s_1 = \{y_0, \ldots y_{r-1}\}_<$, if (for $\epsilon = 0, 1$) $i_0^\epsilon < \ldots < i_{n_\epsilon - 1}^\epsilon$ are such that $s_\epsilon = \{y_{i_0^\epsilon}, \ldots y_{i_{n_\epsilon-1}^\epsilon}\}$ and $\overline{f}_\epsilon$ is the unique function into z with domain $[\gamma]^r$ such that $\overline{f}_\epsilon(\xi_0, \ldots \xi_{r-1}) = f_\epsilon(\xi_{i_0^\epsilon}, \ldots \xi_{i_{n_\epsilon-1}^\epsilon})$ for all $\xi_0 < \ldots < \xi_{r-1} < \gamma$, then $\{s \in [\gamma]^r \, : \, \overline{f}_0(s) < \overline{f}_1(s)\} \in x^r$. $\Box$

Lemma 3.3 is a useful result concerning the closure, in a given model M, of the models arising in iterations of M.

Lemma 3.3 *Let M be a transitive model of ZFC^* and, in M, let $\gamma_0 < \gamma_1$ be measurable cardinals. Let*

(a) U be a non-principal γ_0–complete ultrafilter on γ_0 in M,

(b) W be a normal measure on γ_1 in M and

(c) $\xi < \gamma_0$ be an ordinal.

Then, given any $X \in U$, any integer n and any $f : [X]^n \longrightarrow M_\xi^W$, $f \in M$, there is some $X_0 \subseteq X$, $X_0 \in U$, such that $f \restriction [X_0]^n \in M_\xi^W$.

Proof: By induction on ξ. For $\xi = 0$ the result holds vacuously. Suppose $\xi = \xi_0 + 1$ and assume the result holds for ξ_0. Given $X \in U$ and $n < \omega$ and a function f from $[X]^n$ into M_ξ^W $(= Ult(M_{\xi_0}^W, j_{0,\xi_0}^{M,W}(W)))$ in M, let $f_0 : [X]^n \longrightarrow M_{\xi_0}^W$ be given by $f_0(s) = h_s$, where h_s is, for each $s \in [X]^n$, a function in $M_{\xi_0}^W$ on $j_{0,\xi_0}^{M,W}(\gamma_1)$ such that $j_{\xi_0,\xi}^{M,W}(h_s)(j_{0,\xi_0}^{M,W}(\gamma_1)) = f(s)$. By induction hypothesis there is $X_0 \subseteq X$ in U such that $f_0 \restriction [X_0]^n \in M_{\xi_0}^W$. But then, since M_ξ^W is closed under γ_0–sequences in $M_{\xi_0}^W$ (in fact under sequences of length $j_{0,\xi_0}^{M,W}(\gamma_1) > \gamma_0$),

$$f \restriction [X_0]^n = \{\langle s, j_{\xi_0,\xi}^{M,W}(f_0(s))(j_{0,\xi_0}^{M,W}(\gamma_1))\rangle \, : \, s \in [X_0]^n\}$$

is in M_ξ^W.

Finally, suppose ξ is a limit ordinal and suppose the result holds for all $\xi' < \xi$. Again let X, n and f be as in the statement. We define a function $f_0 : [X]^n \longrightarrow \bigcup_{\xi' < \xi} M_{\xi'}^W$ by letting $f_0(s)$ be, for each $s \in [X]^n$, a set in $M_{\xi_s}^W$

(for some $\xi_s < \xi$) such that $j^{M,W}_{\xi_s,\xi}(f_0(s)) = f(s)$. Since $\xi < \gamma_0$, by normality of U in M and by Rowbottom's theorem ([R]) for partitions of $[X]^n$ into less than γ_0–many pieces there is some $\bar{\xi} < \xi$ and some $X_0 \subseteq X$ in U such that $\xi_s = \bar{\xi}$ for every $s \in [X_0]^n$. By induction hypothesis there is then some $X_1 \subseteq X_0$ in U such that $f_0 \restriction [X_1]^n \in M^W_{\bar{\xi}}$. But then, as the critical point of $j^{M,W}_{\bar{\xi},\xi}$ is above γ_0, $f \restriction [X_1]^n = j^{M,W}_{\bar{\xi},\xi} \circ (f_0 \restriction [X_1]^n) \in M^W_\xi$. $\Box$

As a consequence of Lemma 3.3 we obtain the following commutativity lemma.

Lemma 3.4 *(Commutativity Lemma) Let M be a transitive ZFC^*–model, let $\gamma_0 < \gamma_1$ be measurable cardinals in M, let W_0 and W_1 be, in M, a normal measure on γ_0 and a normal measure on γ_1, respectively, and let $\xi_1 < \gamma_0$ and $\xi_0 < \gamma_1$ be ordinals. Let $j_0 = j^{M,W_0}_{0,\xi_0}$ and $j_1 = j^{M,W_1}_{0,\xi_1}$. Then,*

$$j_0 \circ j_1 = j_1 \circ j_0 = j^{M^{W_1}_{\xi_1}, j_1(W_0)}_{0,\xi_0} \circ j_1 = j^{M^{W_0}_{\xi_0}, j_0(W_1)}_{0,\xi_1} \circ j_0$$

Proof: Let $a \in M$ and let $\Phi(x, y, z, w)$ be a formula as given by Fact 3.2. Since W_0 and ξ_0 are fixed by j_1, $j_1(j_0(a))$ is the unique set c such that

$$M^{W_1}_{\xi_1} \models \Phi(W_0, \xi_0, j_1(a), c)$$

On the other hand, since $M^{W_1}_{\xi_1}$ is sufficiently closed in M by Lemma 3.3, this set c is $j_0(j_1(a))$ by the choice of Φ. This establishes the first equality.

As to the other equalities notice, again by the choice of Φ, that

$$M^{W_1}_{\xi_1} \models \Phi(W_0, \xi_0, j_1(a), j_1(j_0(a)))$$

implies $j_1(j_0(a)) = j^{M^{W_1}_{\xi_1}, j_1(W_0)}_{0,\xi_0}(j_1(a))$ (as W_0 is fixed by j_1) and that

$$M \models \Phi(W_1, \xi_1, a, j_1(a))$$

implies

$$M^{W_0}_{\xi_0} \models \Phi(j_0(W_1), \xi_1, j_0(a), j_0(j_1(a)))$$

(as $\xi_1 < \gamma_0$), which in turn implies $j_0(j_1(a)) = j^{M^{W_0}_{\xi_0}, j_0(W_1)}_{0,\xi_1}(j_0(a))$. $\Box$

The following result is a corollary of Lemma 3.4. It will be used in the proof of Lemma 4.2 in Section 4.

Lemma 3.5 *Let M be a transitive ZFC^*–model, let $(\gamma_k)_{k<n}$ be, for some $n<\omega$, a sequence of measurable cardinals, and let W_k be, for every $k<n$, a normal measure on γ_k. Let $\vec{\xi}=(\xi_k)_{k<n}$ be a sequence of ordinals below $min_{k<n}\gamma_k$. Then,*

$$j^{M,W_{\sigma(n-1)}}_{0,\xi_{\sigma(n-1)}}\circ\ldots\circ j^{M,W_{\sigma(0)}}_{0,\xi_{\sigma(0)}}=j^{M,\mathcal{W},\vec{\xi}}_{0,\Sigma_{k<n}\xi_k}$$

for every permutation σ of n, where $\mathcal{W}=\langle W^*_0,\ldots W^*_{n-1}\rangle$ is the sequence defined by $W^*_0=W_0$ and by $W^*_k=j^{M,\langle W^*_0,\ldots W^*_{k-1}\rangle,\langle\xi_0,\ldots\xi_{k-1}\rangle}_{0,\Sigma_{l<k}\xi_l}(W_k)$ for every nonzero $k<n$.

Proof: Extending the notation in the statement of Lemma 3.4, let $j_k=j^{M,W_k}_{0,\xi_k}$ for every $k<n$. It follows immediately from Lemma 3.4 that $j_{n-1}\circ\cdots\circ j_0=j_{\sigma(n-1)}\circ\ldots\circ j_{\sigma(0)}$ for every permutation $\sigma:n\longrightarrow n$.

Hence, it will be enough to prove $j^{M,\mathcal{W},\vec{\xi}}_{0,\Sigma_{k<n}\xi_k}=j_{n-1}\circ\cdots\circ j_0$ by induction on n. The case $n\leq 2$ is handled by Lemma 3.4, so suppose $n>2$. Let $j^*=j^{M,\mathcal{W}\restriction(n-1),\vec{\xi}\restriction(n-1)}_{0,\Sigma_{l<n-1}\xi_l}$ and let Φ be a formula as in Fact 3.2. Let us also fix $a\in M$. Notice that, by the fact that $M\models\Phi(W_{n-1},\xi_{n-1},a,j_{n-1}(a))$, together with the elementarity of j^* and the fact that ξ_{n-1} is fixed by j^*, we have

$$M^{\mathcal{W}\restriction(n-1),\vec{\xi}\restriction(n-1)}_{\Sigma_{l<n-1}\xi_l}\models\Phi(j^*(W_{n-1}),\xi_{n-1},j^*(a),j^*(j_{n-1}(a))),$$

and hence, by induction hypothesis and by the commutativity of the j_k's,

$$M^{\mathcal{W}\restriction(n-1),\vec{\xi}\restriction(n-1)}_{\Sigma_{l<n-1}\xi_l}\models\Phi(j^*(W_{n-1}),\xi_{n-1},j^*(a),(j_{n-1}\circ\ldots\circ j_0)(a))$$

But by the choice of Φ this means

$$(j_{n-1}\circ\ldots\circ j_0)(a)=j^{M^{\mathcal{W}\restriction(n-1),\vec{\xi}\restriction(n-1)}_{\Sigma_{l<n-1}\xi_l},j^*(W_{n-1})}_{0,\xi_{n-1}}(j^*(a))=j^{M,\mathcal{W},\vec{\xi}}_{0,\Sigma_{l<n}\xi_l}(a)$$

□

The following lemma will be of crucial importance. It is a strengthening of a result, due to Kunen ([Ku2]), saying that for every ordinal ϵ, the class of measurable cardinals γ for which there is a normal measure U on γ such that $\epsilon<j(\epsilon)$, where $j:V\longrightarrow Ult(V,U)$ is the embedding derived from U, is finite.

Lemma 3.6 *Let M be a transitive model of ZFC^*. Given any ordinal $\epsilon \in M$, in M it holds that there are only finitely many measurable cardinals γ with the property that there are a normal measure W on γ and an ordinal ξ less than the first measurable cardinal such that $j_\xi^{V,W}(\epsilon) > \epsilon$.*[27]

Proof: We will use a somewhat refined version of the proof, due to Fleissner (as presented in [K], Lemma 19.17), of the above result of Kunen. Pick an ordinal ϵ in M and let $\mathcal{M}_\epsilon$ be the set of all γ with the property expressed in the statement. For every $\gamma \in \mathcal{M}_\epsilon$ fix a pair (W_γ, ξ_γ) witnessing this. Let $\gamma \in \mathcal{M}_\epsilon$ be given. Let ϵ_γ^0 be the supremum of the set of fixed points of $j_{0,\xi_\gamma}^{M,W_\gamma}$ below ϵ and let ϵ_γ^1 be the least fixed point of $j_{0,\xi_\gamma}^{M,W_\gamma}$ above ϵ. Then $[\epsilon_\gamma^0,\, \epsilon_\gamma^1)$ is a *moving interval for ϵ with respect to* (W_γ, ξ_γ), meaning that $j_{0,\xi_\gamma}^{M,W_\gamma}$ fixes cofinally many ordinals in ϵ_γ^0 as well as ϵ_γ^1, that $j_{0,\xi_\gamma}^{M,W_\gamma}(\epsilon') > \epsilon'$ for every ϵ', $\epsilon_\gamma^0 \leq \epsilon' < \epsilon_\gamma^1$, and finally that $\epsilon_\gamma^0 \leq \epsilon < \epsilon_\gamma^1$. Let j be $j_{0,\xi_\gamma}^{M,W_\gamma}$.

Claim 3.6.1 $cf^M(\epsilon_\gamma^0) = \gamma$

Proof: Let I be a cofinal subset of ϵ_γ^0 of order type $cf^M(\epsilon_\gamma^0)$ consisting of fixed points of j. If $cf^M(\epsilon_\gamma^0) < \gamma$, then

$$j(\epsilon_\gamma^0) = j(sup(I)) = sup(j``I) = sup(I) = \epsilon_\gamma^0$$

since the critical point of j is γ. Suppose $cf^M(\epsilon_\gamma^0) > \gamma$. Every ordinal in $j(\epsilon_\gamma^0)$ is of the form $j(f)(\delta_0, \ldots \delta_{n-1})$ for some $n < \omega$, some function $f : \gamma^n \longrightarrow \epsilon_\gamma^0$ in M and some ordinals $\vartheta_0 < \ldots < \vartheta_{n-1}$ belonging to the critical sequence of the iteration of $\langle M, W_\gamma \rangle$ of length $\xi_\gamma + 1$. As every such function is then bounded by some ordinal in ϵ_γ^0,

$$j(\epsilon_\gamma^0) = sup(j``\epsilon_\gamma^0) = sup(j``I) = sup(I) = \epsilon_\gamma^0$$

□

On the other hand, ϵ_γ^1 is clearly equal to $sup_{n<\omega} j^n(\epsilon_\gamma^0)$, where $j^0 = id \restriction M$ and $j^{n+1} = j \circ j^n$ for every n.

The rest of the proof is as in Fleissner's argument (using Lemma 3.4): Given any distinct γ_0, γ_1 in $\mathcal{M}_\epsilon$, if $\epsilon_{\gamma_0}^0 < \epsilon_{\gamma_1}^0$, then $\epsilon_{\gamma_1}^1 < \epsilon_{\gamma_0}^1$. To see this,

[27]Note that $j_\xi^{V,W}(\epsilon) > \epsilon$ is indeed first order expressible from W, ξ and ϵ. Also, note that if W is, in a transitive ZFC^*–model M, a normal measure on an M–measurable cardinal γ, and ξ, ϵ are ordinals in M, $j_\xi^{M,W}(\epsilon) > \epsilon$ if and only if $M \models j_\xi^{V,W}(\epsilon) > \epsilon$.

suppose $\epsilon^0_{\gamma_0} < \epsilon^0_{\gamma_1} \leq \epsilon < \epsilon^1_{\gamma_0}$ and fix some η such that $\epsilon^0_{\gamma_0} \leq \eta < \epsilon^0_{\gamma_1}$ and such that $j^{M,W_{\gamma_1}}_{0,\xi_{\gamma_1}}(\eta) = \eta$. There is then some integer n such that $\epsilon^0_{\gamma_1} < (j^{M,W_{\gamma_0}}_{0,\xi_{\gamma_0}})^n(\eta) < \epsilon^1_{\gamma_0}$. By n applications of the first equality in the conclusion of Lemma 3.4, $j^{M,W_{\gamma_1}}_{0,\xi_{\gamma_1}}((j^{M,W_{\gamma_0}}_{0,\xi_{\gamma_0}})^n(\eta))$ is $(j^{M,W_{\gamma_0}}_{0,\xi_{\gamma_0}})^n(j^{M,W_{\gamma_1}}_{0,\xi_{\gamma_1}}(\eta))$, which is equal to $(j^{M,W_{\gamma_0}}_{0,\xi_{\gamma_0}})^n(\eta)$. Hence, $\epsilon^1_{\gamma_1}$ is at most $(j^{M,W_{\gamma_0}}_{0,\xi_{\gamma_0}})^n(\eta)$, and so $\epsilon^1_{\gamma_1} < \epsilon^1_{\gamma_0}$. Finally, suppose towards a contradiction that there is an infinite strictly increasing sequence $(\gamma_i)_{i<\omega}$ of ordinals $\mathcal{M}_\epsilon$. Since $i \neq j$ implies $\epsilon^0_{\gamma_i} \neq \epsilon^0_{\gamma_j}$, by passing to a subsequence we may assume that $i < j < \omega$ implies $\epsilon^0_{\gamma_i} < \epsilon^0_{\gamma_j}$. It follows that $\epsilon^1_{\gamma_0} > \epsilon^1_{\gamma_1} > \ldots$ is an infinite descending chain of ordinals, which is impossible. □

Any last member, in the sense of Definition 3.1, of an iteration of a model M bounded by an ordinal less than $|M|^+$ has the same size as M:

Fact 3.7 *Suppose τ is an ordinal, M is a transitive model M of ZFC^* and $\mathcal{W} := \langle W_i : i < \tau\rangle$ and $\vec{\nu} := \langle \nu_i : i < \tau\rangle$ are two sequences such that*

(a) $\mathcal{W} \in M$,

(a) the $\vec{\nu}$–iteration of M relative to $\mathcal{W}$ exists and is normal, and

(f) $\vec{\nu}$ is bounded by some ordinal $\lambda < |M|^+$.

Then $|M^{\mathcal{W},\vec{\nu}}_\alpha| = |M|$ for all $\alpha < \Sigma_{i<\tau}\,\nu_i$.

Proof: Given $i < \tau$, let $\gamma_i \in M$ be such that $\bigcup W_i = j^{M,\mathcal{W}}_{0,\Sigma_{j<i}\nu_j}(\gamma_i)$ if such an ordinal exists. We temporarily define, for every $n < \omega$, the collection $\mathcal{F}_n$ of Γ*–maps of depth at least* n by letting $\mathcal{F}_0$ be the collection of all functions with domain ${}^k\gamma_i$ (for $i < \tau$ and $k < \omega$) and by letting $\mathcal{F}_n$ be, for each nonzero $n < \omega$, the collection of all functions f in $\mathcal{F}_{n-1}$ such that

$$f(\alpha^n_0, \ldots \alpha^n_{k_n-1})(\alpha^{n-1}_0, \ldots \alpha^{n-1}_{k_{n-1}-1}) \ldots (\alpha^1_0, \ldots \alpha^1_{k_1-1}),$$

whenever this expression is defined, is a function with domain ${}^k\gamma_i$ for some $i < \tau$ and $k < \omega$.[28]

[28]The indices i and k may certainly depend on the values of $(\alpha^j_0, \ldots \alpha^j_{k_j-1})_{1\leq j\leq n}$.

Fix a nonzero $\alpha < \Sigma_{i<\tau}\nu_i$. Note that, if $\langle \vartheta_\beta \, : \, \beta + 1 < \Sigma_{i<\tau}\nu_i\rangle$ is the critical sequence of $\langle M_\alpha^{\mathcal{W},\vec{\nu}} \, : \, \alpha < \Sigma_{i<\tau}\nu_i\rangle$, then every set in $M_\alpha^{\mathcal{W},\vec{\nu}}$ is of the form

$$j_{0,\alpha}^{M,\mathcal{W},\vec{\nu}}(f)(\vartheta_{\beta_0^l}, \ldots \vartheta_{\beta_{k_l}^l})(\vartheta_{\beta_0^{l-1}}, \ldots \vartheta_{\beta_{k_{l-1}}^{l-1}}) \ldots (\vartheta_{\beta_0^0}, \ldots \vartheta_{\beta_{k_0}^0})$$

for some $l < \omega$, for indices

$$\beta_0^l < \ldots < \beta_{k_l}^l < \beta_0^{l-1} < \ldots < \beta_{k_{l-1}}^{l-1} < \ldots < \beta_0^0 < \ldots < \beta_{k_0}^0 < \alpha$$

and for a suitable $f \in M$, f a Γ–map of depth at least l. Hence, as each of the tuples $(\beta_0^j, \ldots \beta_{k_j}^j)$ depends only on $\langle(\beta_0^m, \ldots \beta_{k_m}^m)\rangle_{l\leq m<j}$ and on a unique choice of $k_j + 1$ ordinals less than λ– and thus, ultimately, on a unique choice of finitely many ordinals less than λ –, $|M_\alpha^{\mathcal{W},\vec{\nu}}| = |M \times \lambda^{<\omega}| = |M|$. $\square$

The following lemma and its proof are due to the referee. They replace previous incorrect versions.

Lemma 3.8 *Let M be a transitive model of ZFC^* and let $\epsilon_0 \in M$ be an ordinal. Let τ be a limit ordinal and let $\mathcal{W} = \langle W_i \, : \, i < \tau + 1\rangle$ and $\vec{\nu} = \langle \nu_i \, : \, i < \tau + 1\rangle$ be such that*

(a) the $\vec{\nu}$–iteration of M relative to $\mathcal{W}$ exists and is normal,

(b) for all $\beta < \Sigma_{i<\tau}\nu_i$, $j_{0,\beta}^{M,\mathcal{W},\vec{\nu}}(\epsilon_0) = \epsilon_0$, and

(c) for all $j < k < \tau$, $j_{\Sigma_{i<j}\nu_i, \Sigma_{i<k}\nu_i}^{M,\mathcal{W},\vec{\nu}}(\bigcup W_j) \neq \bigcup W_k$.

Then $j_{0,\Sigma_{i<\tau}\nu_i}^{M,\mathcal{W},\vec{\nu}}(\epsilon_0) = \epsilon_0$.

Proof: By induction on ϵ_0. The result clearly holds for $\epsilon_0 = 0$ and the successor stage of the induction is trivial, so assume $\epsilon_0 > 0$ is a limit ordinal. Fix M, $\mathcal{W}$ and $\vec{\nu}$ as in the statement of the lemma. Since

$$j_{0\,\Sigma_{i<\tau}\nu_i}^{M,\mathcal{W},\vec{\nu}}(\epsilon_0) = sup\{j_{\beta,\,\Sigma_{i<\tau}\nu_i}^{M,\mathcal{W},\vec{\nu}}(\epsilon) \, : \, \beta < \Sigma_{i<\tau}\nu_i, \, \epsilon < \epsilon_0\},$$

it suffices to show that for all $\beta < \Sigma_{i<\tau}\nu_i$ and all $\epsilon < \epsilon_0$, $j_{\beta,\,\Sigma_{i<\tau}\nu_i}^{M,\mathcal{W},\vec{\nu}}(\epsilon) < \epsilon_0$. Fix such β and ϵ. By Lemma 3.6 there is a finite set $E \in M_\beta$ such that in M_β it holds that no cardinal outside E carries a normal measure whose corresponding ultrapower embedding moves ϵ. Since τ is a limit ordinal, it follows from condition (c) that $\bigcup W_k \notin j^{M,\mathcal{W},\vec{\nu}}_{\beta,\Sigma_{i<k}\nu_i}(E)$ for a tail of $k<\tau$. Fix k in this tail, and let $\epsilon^* = j^{M,\mathcal{W},\vec{\nu}}_{\beta,\Sigma_{i<k}\nu_i}(\epsilon)$. Then

$$\epsilon^* \leq j^{M,\mathcal{W},\vec{\nu}}_{0,\Sigma_{i<k}\nu_i}(\epsilon) < j^{M,\mathcal{W},\vec{\nu}}_{0,\Sigma_{i<k}\nu_i}(\epsilon_0) = \epsilon_0$$

and the successor steps of the $\langle \nu_i : k \leq i < \tau\rangle$–iteration of $M_{\Sigma_{i<k}\nu_i}$ relative to $\langle W_i : k \leq i < \tau\rangle$ all fix the corresponding image of ϵ^*. Since the lemma holds for ϵ^*, it follows by induction hypothesis that the limit stages of this iteration also fix the image of ϵ^*, which means that $j^{M,\mathcal{W},\vec{\nu}}_{\Sigma_{i<k}\nu_i,\Sigma_{i<\tau}\nu_i}(\epsilon^*) = \epsilon^*$. But, since $j^{M,\mathcal{W},\vec{\nu}}_{\Sigma_{i<k}\nu_i,\Sigma_{i<\tau}\nu_i}(\epsilon^*) = j^{M,\mathcal{W},\vec{\nu}}_{\beta,\Sigma_{i<\tau}\nu_i}(\epsilon)$, this means that $j^{M,\mathcal{W},\vec{\nu}}_{\beta,\Sigma_{i<\tau}\nu_i}(\epsilon) < \epsilon_0$. $\square$

The proof of Lemma 3.11 below involves the following generalization of the ultrafilters $(U)^i$ defined in the proof of Fact 3.2 (for U an ultrafilter on a cardinal and i a nonzero integer).

Definition 3.3 *Given $m<\omega$, a tuple $(U_0, \ldots U_m)$ such that each U_l is an ultrafilter on a cardinal δ_l, and a tuple $\langle n_0 \ldots n_m\rangle$ of nonzero integers, let us define the sequence $(U_0, \ldots U_m)^{\langle n_0, \ldots n_m\rangle}$ of subsets of $[\delta_0]^{\nu_0} \times \ldots \times [\delta_m]^{n_m}$ by*

(a) letting $(U_0, \ldots U_m)^{\langle n_0, \ldots n_{m-1}, 1\rangle}$ be the set of all sets of the form $X_0 \times \ldots \times X_{m-1} \times X$, where $X_l \subseteq [\delta_l]^{n_l}$ for all $l<m$ and $X \subseteq [\delta_m]^1$, such that the set of

$$\langle s_0, \ldots s_{m-1}\rangle \in [\delta_0]^{n_0} \times \ldots \times [\delta_{m-1}]^{n_{m-1}}$$

with

$$\{\beta < \delta_m : \langle s_0, \ldots s_{m-1}\rangle \frown \langle \{\beta\}\rangle \in X_0 \times \ldots \times X_{m-1} \times X\} \in U_m$$

is in $(U_0, \ldots, U_{m-1})^{\langle n_0, \ldots n_{m-1}\rangle}$, and

(b) if $n_m > 1$, by specifying, for every sequence $X_l \subseteq [\delta_l]^{n_l}$ (for $l<m+1$) that $X_0 \times \ldots \times X_m$ is in $(U_0, \ldots, U_m)^{\langle n_0, \ldots n_m\rangle}$ if and only if the set of

$$\langle s_0, \ldots s_{m-1}\rangle \frown \langle s\rangle \in [\delta_0]^{n_0} \times \ldots \times [\delta_m]^{n_m - 1}$$

such that

$$\{\beta < \delta : \langle s_0, \ldots s_{m-1}\rangle \frown \langle s \cup \{\beta\}\rangle \in \langle X_0, \ldots X_m\rangle\} \in U_m$$

is in $(U_0, \ldots, U_m)^{\langle n_0, \ldots n_{m-1}, n_m - 1\rangle}$.

It is not difficult to check that, if each δ_l is a measurable cardinal and each U_l in the above definition is a normal ultrafilter on δ_l, then $(U_0, \ldots U_m)^{\langle n_0, \ldots n_m \rangle}$ is the set of $X_0 \times \ldots \times X_m$ such that, for all $l < m+1$, $X_l \subseteq [\delta_l]^{n_l}$ and there is $Y_l \in U_l$ with $[Y_l]^{n_l} \subseteq X_l$.[29]

Also, the following fact is an easy consequence of the definition of the $(U_0, \ldots, U_m)^{\langle n_0, \ldots n_m \rangle}$'s.

Fact 3.9 *Let M be a transitive ZFC^*–model. Let $\mathcal{W} = \langle W_i : i < \tau + 1 \rangle$ and $\vec{\nu} = \langle \nu_i : i < \tau + 1 \rangle$ be two sequences in M, let $m < \omega$, let $0 < n_l < \omega$ and $i_l < \tau$ (for $l < m$), let f and g in M, and let R such that*

(a) the $\vec{\nu}$–iteration $\mathcal{I}$ of M with respect to $\mathcal{W}$ exists and is normal,

(b) letting $\gamma_i = \bigcup W_i$ for all i, f and g are functions with domain $[\gamma_{i_0}]^{n_0} \times \ldots \times [\gamma_{i_{m-1}}]^{n_{m-1}}$, and

(c) $R \in \{=, \in\}$.

Let $\langle \vartheta_\xi : \xi < \Sigma_{i<\tau} \nu_i \rangle$ be the critical sequence of $\mathcal{I}$ and, for every $l < m$, let $W_l^* \in M$ be such that $j^{M,\mathcal{W},\vec{\nu}}_{0,\Sigma_{k<i_l} \nu_k}(W_l^*) = W_{i_l}$.

Then, given any $\xi_0^l < \ldots < \xi_{n_l-1}^l$ (for $l < m$) such that $\Sigma_{k<i_l} \nu_k \leq \xi_j^l < \Sigma_{k \leq i_l} \nu_k$ for all $j < n_l$, letting

$$t = \langle \{\vartheta_{\xi_0^0}, \ldots \vartheta_{\xi_{n_0-1}^0}\}, \ldots \{\vartheta_{\xi_0^{n_{m-1}}}, \ldots \vartheta_{\xi_{n_{m-1}-1}^{n_{m-1}}}\} \rangle,$$

the following conditions are equivalent:

(1) $j^{M,\mathcal{W},\vec{\nu}}_{0,\Sigma_{i<\tau} \nu_i}(f)(t) \, R \, j^{M,\mathcal{W},\vec{\nu}}_{0,\Sigma_{i<\tau} \nu_i}(g)(t)$.

(2) The set of $\langle s_0, \ldots s_{m-1} \rangle \in [\gamma_{i_0}]^{n_0} \times \ldots \times [\gamma_{i_{m-1}}]^{n_{m-1}}$ such that

$$f(\langle s_0, \ldots s_{m-1} \rangle) \, R \, g(\langle s_0, \ldots s_{m-1} \rangle)$$

is in $(W_0^*, \ldots W_{m-1}^*)^{\langle n_0, \ldots n_{m-1} \rangle}$

The next result follows now immediately from Fact 3.9 and the remark made after Definition 3.3.

[29]This fact, which is true in general, is easiest to verify in the case that $\delta_l \leq \delta_{l+1}$ for all $l < m$, which is the situation we shall be interested in.

Fact 3.10 *Let M be a transitive model of ZFC^* and let $\overline{M}$ be an inner model of M. Let $\mathcal{W} = \langle W_i : i < \tau + 1\rangle$ and $\vec{\nu} = \langle \nu_i : i < \tau + 1\rangle$ two sequences in M, let $\widetilde{\mathcal{W}} = \langle \tilde{W}_i : i < \tau + 1\rangle \in \overline{M}$, $m < \omega$, $0 < n_l < \omega$ and $i_l < \tau$ (for $l < m$), let f and g, and let R such that*

- *(a) both the $\vec{\nu}$–iteration $\mathcal{I}$ of M with respect to $\mathcal{W}$ and the $\vec{\nu}$–iteration $\widetilde{\mathcal{I}}$ of $\overline{M}$ with respect to $\widetilde{\mathcal{W}}$ exist and are normal,*
- *(b) $\tilde{W}_i = \overline{M} \cap W_i$ for all $i < \tau$,*
- *(c) letting $\gamma_i = \bigcup W_i$ for all i, f and g are functions in $\overline{M}$ with domain $[\gamma_{i_0}]^{n_0} \times \ldots \times [\gamma_{i_{m-1}}]^{n_{m-1}}$, and*
- *(d) $R \in \{=, \in\}$.*

Let $\langle \vartheta_\xi : \xi < \Sigma_{i<\tau}\nu_i\rangle$ and $\langle \tilde{\vartheta}_\xi : \xi < \Sigma_{i<\tau}\nu_i\rangle$ be, respectively, the critical sequence of $\mathcal{I}$ and the critical sequence of $\widetilde{\mathcal{I}}$.

Then, given any $\xi^l_0 < \ldots < \xi^l_{n_l-1}$ and $\zeta^l_0 < \ldots < \zeta^l_{n_l-1}$ (for $l < m$) such that $\Sigma_{k<i_l}\nu_k \leq \xi^l_j, \zeta^l_j < \Sigma_{k\leq i_l}\nu_k$ for all $j < n_l$, letting

$$t = \langle \{\vartheta_{\xi^0_0}, \ldots \vartheta_{\xi^0_{n_0-1}}\}, \ldots \{\vartheta_{\xi^{n_{m-1}}_0}, \ldots \vartheta_{\xi^{n_{m-1}}_{n_{m-1}-1}}\}\rangle$$

and

$$\tilde{t} = \langle \{\tilde{\vartheta}_{\zeta^0_0}, \ldots \tilde{\vartheta}_{\zeta^0_{n_0-1}}\}, \ldots \{\tilde{\vartheta}_{\zeta^{n_{m-1}}_0}, \ldots \tilde{\vartheta}_{\zeta^{n_{m-1}}_{n_{m-1}-1}}\}\rangle,$$

the following conditions are equivalent:

(1) $j^{M,\mathcal{W},\vec{\nu}}_{0,\Sigma_{i<\tau}\nu_i}(f)(t)\, R\, j^{M,\mathcal{W},\vec{\nu}}_{0,\Sigma_{i<\tau}\nu_i}(g)(t)$

(2) $j^{\overline{M},\widetilde{\mathcal{W}},\vec{\nu}}_{0,\Sigma_{i<\tau}\nu_i}(f)(\tilde{t})\, R\, j^{\overline{M},\widetilde{\mathcal{W}},\vec{\nu}}_{0,\Sigma_{i<\tau}\nu_i}(g)(\tilde{t})$

The next lemma shows, for a given ordinal ϵ in a model M, that if an iteration of M can be split nicely[30] into two parts such the first part does not move ϵ, then one can ignore the first part.

Lemma 3.11 *(One–step compression lemma) Let M be a transitive model of ZFC^*, let τ_0 and τ_1 be ordinals, let $\mathcal{W} := \langle W_i : i < \tau_0 + \tau_1 + 1\rangle$ and $\vec{\nu} := \langle \nu_i : i < \tau_0 + \tau_1 + 1\rangle$ be two sequences in M, and let ϵ be such that*

[30] 'Nicely' in the sense that conditions (b) and (c) apply.

(a) for each $i < \tau_0 + \tau_1$, $\gamma_i := \bigcup W_i$ is an ordinal,

(b) for all i' with $\tau_0 \leq i' < \tau_0 + \tau_1$, $\gamma_{i'} > \overline{\gamma} := sup_{i<\tau_0}\gamma_i$ and $\gamma_{i'} > sup_{i<\tau_0}\nu_i$,

(c) for all $i < \tau_0$ and all i' with $\tau_0 \leq i' < \tau_0 + \tau_1$, $\tau_1 < \gamma_i$ and $\nu_{i'} < \gamma_i$,

(d) ϵ is an ordinal in M,

(e) the $\vec{\nu}$–iteration of M relative to $\mathcal{W}$ exists and is normal, and

(f) $j^{M,\mathcal{W},\vec{\nu}}_{0,\Sigma_{i<\tau_0}\nu_i}(\epsilon) = \epsilon$.

Let $W^*_i \in M$, for $i < \tau_1 + 1$, be such that $j^{M,\mathcal{W},\vec{\nu}}_{0,\Sigma(\nu_k\,:\,k<\tau_0+i)}(W^*_i) = W_{\tau_0+i}$ for every $i < \tau_1$, and let $\vec{\nu}^* = \langle \nu^*_i : i < \tau_1 + 1\rangle$ be given by $\nu^*_i = \nu_{\tau_0+i}$ for all $i < \tau_1 + 1$.

Then, given any $i < \tau_1$,

$$j^{M,\mathcal{W},\vec{\nu}}_{0,\Sigma_{k<\tau_0}\nu_k}(W^*_i) = W^*_i \cap M^{\mathcal{W},\vec{\nu}}_{\Sigma_{k<\tau_0}\nu_k},$$

and thus

$$j^{M,\mathcal{W},\vec{\nu}}_{0,\Sigma(\nu_i\,:\,i<\tau_0+\tau_1)}(\epsilon) = j^{M,\mathcal{W}^*,\vec{\nu}^*}_{0,\tau_1}(\epsilon),$$

where $\mathcal{W}^* = \langle W^{**}_i : i < \tau_1 + 1\rangle \in M$ is defined recursively by

(i) $W^{**}_0 = W^*_0$ and by

(ii) $W^{**}_i = j^{M,\mathcal{W}^*\restriction i,\vec{\nu}^*\restriction i}_{0,\Sigma_{k<i}\nu^*_k}(W^*_i)$ for $i > 0$.

Proof: Let $\overline{M} = M^{\mathcal{W},\vec{\nu}}_{\Sigma_{i<\tau_0}\nu_i}$. To start with, note that, since $\vec{\nu}$ and $\mathcal{W}$ are both in M and M is transitive, $\overline{M}$ is definable in M. In particular, $\overline{M} \subseteq M$.

Let $j_0 = j^{M,\mathcal{W},\vec{\nu}}_{0,\Sigma_{k<\tau_0}\nu_k}$, let $\widetilde{\mathcal{W}} = \langle \tilde{W}_i : i < \tau_1 + 1\rangle$ be defined recursively by

(i) $\tilde{W}_0 = j_0(W^*_0)$ and by

(ii) $\tilde{W}_i = j^{M^{\mathcal{W},\vec{\nu}}_{\Sigma_{j<\tau_0}\nu_j},\widetilde{\mathcal{W}}\restriction i,\vec{\nu}^*\restriction i}_{\Sigma_{k<i}\nu^*_k}(j_0(W^*_i))$ if $i > 0$,

and let $j_1 = j^{\overline{M},\widetilde{\mathcal{W}},\vec{\nu}^*}_{0,\Sigma_{k<\tau_1}\nu^*_k}$. Fix $i < \tau_1$, and let us show $j_0(W^*_i) = W^*_i \cap \overline{M}$. Let $X_0 \in W^*_i$ be the set of M–inaccessible cardinals between $\overline{\gamma}$ and γ_{τ_0+i}. Note that every ordinal in X_0, as well as γ_{τ_0+i}, is a fixed point of j_0. Now, given any $X \in j_0(W^*_i)$, $X = j_0(f)(\{\eta_0, \ldots \eta_{n-1}\})$ for some n, some function

$f : [\overline{\gamma}]^n \longrightarrow W_i^*$ in M and some $\eta_0 < \ldots < \eta_{n-1}$ on the critical sequence of the $\vec{\nu} \restriction (\tau_0 + 1)$–iteration of M with respect to $\mathcal{W} \restriction (\tau_0 + 1)$. By γ_1–completeness of W_i^* in M, there is then some $Y \in W_i^*$ such that $Y \subseteq f(s)$ for all $s \in [\overline{\gamma}]^n$. Note that for every $\delta \in Y \cap X_0$,

$$\delta = j_0(\delta) \in j_0(f)(\{\eta_0, \ldots \eta_{n-1}\}) = X$$

But $Y \cap X_0$ is in W_i^*, and so $X \supseteq Y \cap X_0$ is in W_i^* as well. This shows $j_0(W_i^*) \subseteq W_i^* \cap \overline{M}$. For the reverse inclusion, note that $X \in (W_i^* \cap \overline{M}) \setminus j_0(W_i^*)$ implies $\gamma_{\tau_0+i} \setminus X \in j_0(W_i^*) \subseteq W_i^*$, which is a contradiction.

It remains to show

$$j_1(\epsilon) = j_1(j_0(\epsilon)) \geq j^{M, \mathcal{W}^*, \vec{\nu}^*}_{0, \Sigma(\nu_k^* : k < \tau_1)}(\epsilon) \geq j_1(\epsilon)$$

This will finish the proof since $j_1 \circ j_0 = j^{M, \mathcal{W}, \vec{\nu}}_{0, \Sigma(\nu_k : k < \tau_0 + \tau_1)}$.

The equality holds by hypothesis. As to the first inequality, let $\langle \vartheta_\xi : \xi < \tau_1 \rangle$ be the critical sequence of the $\vec{\nu}^*$–iteration of M with respect to $\mathcal{W}^*$. Note that, for any transitive ZFC^*–model N, any normal measure W in N and any ordinal $\overline{\nu} \in N$, the critical sequence of the iteration of $\langle N, W \rangle$ of length $\overline{\nu} + 1$ is definable, within N, from W and $\overline{\nu}$. Hence, since the critical point of j_0 is above ν_i^* for all $i < \tau_1$ and above τ_1, $\langle j_0(\vartheta_\xi) : \xi < \tau_1 \rangle$ is the critical sequence of the $\vec{\nu}^*$–iteration of $\overline{M}$ with respect to $\widetilde{\mathcal{W}}$. Also, let $m < \omega$, fix a sequence of nonzero $n_l < \omega$ and of $i_l < \tau_1$ $(l < m)$, let f and g be two functions in M from $[\gamma_{i_0}]^{n_0} \times \ldots \times [\gamma_{i_{m-1}}]^{n_{m-1}}$ into ϵ, and let $R \in \{=, \in\}$. Let $U = (W_{i_0}^*, \ldots W_{i_{m-1}}^*)^{\langle n_0, \ldots n_{m-1} \rangle}$ be as in Definition 3.3. Let also ξ_j^l (for $l < m$ and $j < n_l$) be such that $\Sigma_{k < i_l} \nu_k^* \leq \xi_0^l < \xi_1^l < \ldots < \Sigma_{k \leq i_l} \nu_k^*$ for all $l < m$, and let $t = \langle \{\vartheta_{\xi_0^0}, \ldots \vartheta_{\xi_{n_0-1}^0}\} \ldots \{\vartheta_{\xi_0^{m-1}} \ldots \vartheta_{\xi_{n_{m-1}-1}^{m-1}}\} \rangle$. Note that

$$j^{M, \mathcal{W}^*, \vec{\nu}^*}_{0, \Sigma(\nu_k^* : k < \tau_1)}(f)(t) \, R \, j^{M, \mathcal{W}^*, \vec{\nu}^*}_{0, \Sigma(\nu_k^* : k < \tau_1)}(f)(t)$$

if and only if

$$\{s \in [\gamma_{i_0}]^{n_0} \times \ldots \times [\gamma_{i_{m-1}}]^{n_{m-1}} \, : \, f(s) \, R \, g(s)\} \in U,$$

by Fact 3.9, if and only if

$$\{s \in [\gamma_{i_0}]^{n_0} \times \ldots \times [\gamma_{i_{m-1}}]^{n_{m-1}} \, : \, j_0(f)(s) \, R \, j_0(g)(s)\} \in j_0(U),$$

by elementarity of j_0, if and only if

$$j_1(j_0(f))(\overline{t}) \, R \, j_1(j_0(g))(\overline{t}),$$

for $\bar{t} = \langle\{j_0(\vartheta_{\xi^0_0}), \ldots j_0(\vartheta_{\xi^0_{n_0-1}})\} \ldots \{j_0(\vartheta_{\xi^{m-1}_0}) \ldots j_0(\vartheta_{\xi^{m-1}_{n_{m-1}-1}})\}\rangle$, again by Fact 3.9.

It follows that the mapping sending an ordinal which can be expressed as $j^{M,\mathcal{W}^*,\vec{\nu}^*}_{0,\Sigma(\nu^*_k\,:\,k<\tau_1)}(f)(t)$, for $f \in M$ and t as above, to $j_1(j_0(f))(\bar{t})$, where $\bar{t}$ is obtained from t as above, is a well-defined one–to–one order–preserving function from $j^{M,\mathcal{W}^*,\vec{\nu}^*}_{0,\Sigma(\nu_k\,:\,k<\tau_1)}(\epsilon)$ into $j_1(j_0(\epsilon))$.

The second inequality follows from the fact that $\overline{M} \subseteq M$ and that $W^*_i \cap \overline{M} = j_0(W^*_i)$ for all $i < \tau_1$. As before, let $\langle\vartheta_\xi : \xi < \tau_1\rangle$ be the critical sequence of the $\vec{\nu}^*$–iteration of M with respect to $\mathcal{W}^*$. Consider the mapping from $j_1(\epsilon)$ into $j^{M,\mathcal{W}^*,\vec{\nu}^*}_{0,\Sigma_{k<\tau_1}\nu^*_k}(\epsilon)$ sending an ordinal of the form $j_1(f)(\bar{t})$, where $\bar{t} = \langle\{j_0(\vartheta_{\xi^0_0}), \ldots j_0(\vartheta_{\xi^0_{n_0-1}})\} \ldots \{j_0(\vartheta_{\xi^{n_{m-1}}_0}), \ldots j_0(\vartheta_{\xi^{n_{m-1}}_{n_{m-1}-1}})\}\rangle$,[31] and where m, n_l, i_l (for $l < m$), ξ^l_j (for $l < m$ and $j < n_l$) and f are as before, except that f is also supposed to be in $\overline{M}$, to $j^{M,\mathcal{W}^*}_{0,\,\Sigma_{k<\tau_1}\nu^*_k}(f)(t)$, where $t = \langle\{\vartheta_{\xi^0_0}, \ldots \vartheta_{\xi^0_{n_0-1}}\} \ldots \{\vartheta_{\xi^{n_{m-1}}_0}, \ldots \vartheta_{\xi^{n_{m-1}}_{n_{m-1}-1}}\}\rangle$. This mapping is well–defined, since every function in $\overline{M}$ is also in M, and it is defined on all $j_1(\epsilon)$, one–to–one and order–preserving. This follows from considerations as in the above paragraph, using the fact that $j_0(W^*_i) = W^*_i \cap \overline{M}$ for all $i < \tau_1$, together with Fact 3.10. $\square$

Lemma 3.12 is a consequence of Lemmas 3.8 and 3.11.

Lemma 3.12 *(Compression lemma) Let M be a transitive set model of ZFC^*. Let ϵ, $\mathcal{W} = \langle W_i : i < \tau + 1\rangle$ and $\vec{\nu} = \langle\nu_i : i < \tau + 1\rangle$ be such that*

(a) ϵ is an ordinal in M,

(b) $\mathcal{W} = \langle W_i : i < \tau + 1\rangle$ and $\vec{\nu} = \langle\nu_i : i < \tau + 1\rangle$ are two sequences in M,

(c) for each $i < \tau$, ν_i is less than the first measurable cardinal in M,

(d) for all $j < k < \tau$, $j^{M,\mathcal{W},\vec{\nu}}_{\Sigma_{i<j}\nu_i,\Sigma_{i<k}\nu_i}(\bigcup W_j) < \bigcup W_k$, and

(e) the $\vec{\nu}$–iteration of M relative to $\mathcal{W}$ exists and is normal.

[31] Remember that $\langle j_0(\vartheta_\xi) : \xi < \Sigma_{k<\tau_1}\nu^*_k\rangle$ is the critical sequence of the $\vec{\nu}^*$–iteration of $\overline{M}$ with respect to $\widetilde{\mathcal{W}}$.

Let $\{i_0, \dots i_{n-1}\}_<$[32] *be the finite set of indices* $i < \tau$ *such that*

$$j^{M,\mathcal{W},\vec{\nu}}_{0,\Sigma_{j<i}\nu_j}(\epsilon) < j^{M^{\mathcal{W},\vec{\nu}}_{\Sigma_{j<i}\nu_j},W_i}_{0,\nu_i}(j^{M,\mathcal{W},\vec{\nu}}_{0,\Sigma_{j<i}\nu_j}(\epsilon))^{33}$$

and let $(W^*_k)_{k<n} \in M$ *be such that* $W_{i_k} = j^{M^{\mathcal{W},\vec{\nu}}}_{0,\Sigma(\nu_j\,:\,j<i_k)}(W^*_k)$ *for all* $k < n$.

Let $(M_k)_{k\leq n}$ *be the sequence of* ZFC^**–models given by* $M_0 = M$ *and by* $M_{k+1} = (M_k)^{j_k(W^*_k)}_{\nu_{i_k}}$ *whenever* $k < n$*, where* $(j_k)_{k<n}$ *is the sequence of mappings defined by setting*

(i) $j_0 = id_{M_0}$*, and*

(ii) $j_{k+1} = j^{M_k,\,j_k(W^*_k)}_{0,\nu_{i_k}} \circ j_k$ *for every* $k < n$.

Then,

$$j^{M,\mathcal{W},\vec{\nu}}_{0,\Sigma_{i<\tau}\nu_i}(\epsilon) = (j^{M_n,\,j_n(W^*_n)}_{0,\nu_{i_n}} \circ j^{M_{n-1},\,j_{n-1}(W^*_{n-1})}_{0,\nu_{i_{n-1}}} \circ \dots \circ j^{M_0,\,j_0(W^*_0)}_{0,\nu_{i_0}})(\epsilon)$$

Proof: Let us partition $\tau + 1$ into m blocks $(I_j)_{j<m}$, for some m with $1 \leq m \leq 2n+1$, in such a way that

(1) $i < i'$ whenever $i \in I_j$, $i' \in I_{j'}$ and $j < j'$,

(2) for all j with $j + 1 < m$, all $i \in I_j$ and all $i' \in I_{j+1}$,

$$j^{M,\mathcal{W},\vec{\nu}}_{0,\Sigma_{k<i}\nu_k}(\epsilon) < j^{M^{\mathcal{W},\vec{\nu}}_{\Sigma_{k<i}\nu_k},W_i}_{0,\nu_i}(j^{M,\mathcal{W},\vec{\nu}}_{0,\Sigma_{k<i}\nu_k}(\epsilon))$$

if and only if

$$j^{M,\mathcal{W},\vec{\nu}}_{0,\Sigma_{k<i'}\nu_k}(\epsilon) = j^{M^{\mathcal{W},\vec{\nu}}_{\Sigma_{k<i'}\nu_k},W_{i'}}_{0,\nu_{i'}}(j^{M,\mathcal{W},\vec{\nu}}_{0,\Sigma_{k<i'}\nu_k}(\epsilon))$$

This partition exists, by Lemma 3.8, and is clearly unique. Also, again by Lemma 3.8, every I_j has a maximum. Also, notice that if j is such that $j^{M,\mathcal{W},\vec{\nu}}_{0,\Sigma_{k<i}\nu_k}(\epsilon) < j^{M^{\mathcal{W},\vec{\nu}}_{\Sigma_{k<i}\nu_k},W_i}_{0,\nu_i}(j^{M,\mathcal{W},\vec{\nu}}_{0,\Sigma_{k<i}\nu_k}(\epsilon))$ for all $i \in I_j$, then I_j is finite. Let us temporarily call $(I_j)_{j<m}$ the *moving indices decomposition for* $(M, \mathcal{W}, \vec{\nu}, \epsilon)$.

[32] That is, $i_0 < \dots < i_{n-1}$.

[33] This set is finite by Lemma 3.6 and by elementarity of each $j^{M,\mathcal{W},\vec{\nu}}_{0,\Sigma_{j<i}\nu_j}$.

We will prove the lemma by induction on m, the size of the moving indices decomposition for $(M,\, \mathcal{W},\, \vec{\nu},\, \epsilon)$.

If $m = 1$, then either ϵ is fixed by $j^{M,\mathcal{W},\vec{\nu}}_{0,\,\Sigma_{i<\tau}\nu_i}$, and so the result holds, or every W_i moves the corresponding image of ϵ (in this case of course τ is finite), and so $j^{M,\mathcal{W},\vec{\nu}}_{0,\Sigma_{i<\tau}\nu_i}$ is already of the desired form.

If $m = 2$, the result follows trivially (by Lemma 3.8) if $j^{M,\mathcal{W},\vec{\nu}}_{0,\,\Sigma_{k<i}\nu_k}(\epsilon)$ is a fixed point of $j^{M^{\mathcal{W},\vec{\nu}}_{\Sigma_{k<i}\nu_k},W_i}_{0,\,\nu_i}$ for all $i \in I_1$, and from the one–step compression lemma (Lemma 3.11) if $j^{M,\mathcal{W},\vec{\nu}}_{0,\,\Sigma_{k<i}\nu_k}(\epsilon)$ is a fixed point of $j^{M^{\mathcal{W},\vec{\nu}}_{\Sigma_{k<i}\nu_k},W_i}_{0,\,\nu_i}$ for all $i \in I_0$.

Now suppose $m > 2$ and suppose that the result holds for $(M',\, \mathcal{W}',\, \vec{\nu}\,',\, \epsilon')$ whenever this tuple satisfies the hypothesis of the lemma and its moving indices decomposition is of size less than m.

Suppose $j^{M,\mathcal{W},\vec{\nu}}_{0,\,\Sigma(\nu_k\,:\,k<max(I_{m-2})+1)}(\epsilon)$ is a fixed point of $j^{M^{\mathcal{W},\vec{\nu}}_{\Sigma_{k<i}\nu_k},W_i}_{0,\,\nu_i}$ for every $i \in I_{m-1}$. Then, $j^{M,\mathcal{W},\vec{\nu}}_{0,\,\Sigma_{i<\tau}\nu_i}(\epsilon) = j^{M,\mathcal{W},\vec{\nu}}_{0,\,\Sigma(\nu_k\,:\,k<max(I_{m-2})+1)}(\epsilon)$, by Lemma 3.8, and by induction hypothesis the right-side term of this equation is equal to

$$(j^{M_n,\,j_n(W^*_n)}_{0,\nu_{i_n}} \circ j^{M_{n-1},\,j_{n-1}(W^*_{n-1})}_{0,\nu_{i_{n-1}}} \circ \ldots \circ j^{M_0,\,j_0(W^*_0)}_{0,\nu_{i_0}})(\epsilon)$$

for $(M_k)_{k\leq n}$ and $(j_k)_{k\leq n}$ obtained in the required way.[34]

The remaining case is when $j^{M^{\mathcal{W},\vec{\nu}}_{\Sigma_{k<i}\nu_k},W_i}_{0,\,\nu_i}(j^{M,\mathcal{W},\vec{\nu}}_{0,\,\Sigma_{k<i}\nu_k}(\epsilon)) > j^{M,\mathcal{W},\vec{\nu}}_{0,\,\Sigma_{k<i}\nu_k}(\epsilon)$ for every $i \in I_{m-1}$ (and thus $j^{M^{\mathcal{W},\vec{\nu}}_{\Sigma_{k<i}\nu_k},W_i}_{0,\,\nu_i}(j^{M,\mathcal{W},\vec{\nu}}_{0,\,\Sigma_{k<i}\nu_k}(\epsilon)) = j^{M,\mathcal{W},\vec{\nu}}_{0,\,\Sigma_{k<i}\nu_k}(\epsilon)$ for every $i \in I_{m-2}$).

By applying Lemma 3.11,[35] we first compress the tail of the iteration by eliminating the segment of it corresponding to I_{m-2}. Then we apply the induction hypothesis to the iteration thus obtained (whose moving indices decomposition has less than m elements).

More precisely, let $I_{m-1} \subseteq \{i_0, \ldots i_{n-1}\}$ be of the form $\{i_m, i_{m+1}, \ldots i_{n-1}\}$ for some $m < \omega$. Suppose $|I_{m-1}| = r$. Let

$$\mathcal{W}^* = \mathcal{W} \restriction (max(I_{m-3}) + 1) \frown \langle \widetilde{W}_0, \ldots \widetilde{W}_{r-1} \rangle$$

and

$$\vec{\nu}^* = \vec{\nu} \restriction (max(I_{m-3}) + 1) \frown \langle \tilde{\nu}_0, \ldots \tilde{\nu}_{r-1} \rangle,$$

[34]The induction hypothesis applies because the moving indices decomposition for $(M, \mathcal{W} \restriction (max(I_{m-2}) + 1), \vec{\nu} \restriction (max(I_{m-2}) + 1), \epsilon)$ is of size $m - 1$.

[35]To $M^{\mathcal{W},\vec{\nu}}_{\Sigma(\nu_k\,:\,k<max(I_{m-3})+1)}$ as M, $j^{M,\mathcal{W},\vec{\nu}}_{0,\Sigma(\nu_k\,:\,k<max(I_{m-3})+1)}(\epsilon)$ as ϵ, etc.

where $\tilde{\nu}_l = \nu_{i_{m+l}}$ for all $l < r$, and where $\langle \widetilde{W}_l : l < r \rangle$ is defined recursively by setting

(i) $\widetilde{W}_0 = j^{M,\mathcal{W},\vec{\nu}}_{0,\Sigma(\nu_k\,:\,k<max(I_{m-3})+1)}(W^*_{i_m})$, and

(ii) by setting, for every nonzero $l < r$, $\widetilde{W}_l = j^{M,\mathcal{W}^l,\vec{\nu}^l}_{0,\Sigma(\nu_k\,:\,k<max(I_{m-3})+1)}(W^*_{i_{m+l}})$, where $\mathcal{W}^l = \mathcal{W} \restriction (max(I_{m-3}) + 1) \frown \langle \widetilde{W}_0, \ldots, \widetilde{W}_{l-1} \rangle$ and $\vec{\nu}^l = \vec{\nu} \restriction (max(I_{m-3}) + 1) \frown \langle \nu_{i_m}, \ldots, \nu_{i_{m+l-1}} \rangle$.

Now, by Lemma 3.11,[36]

$$j^{M,\mathcal{W},\vec{\nu}}_{0,\Sigma_{i<\tau}\nu_i}(\epsilon) = j^{M,\mathcal{W}^*,\vec{\nu}^*}_{0,\Sigma(\nu_k\,:\,k<max(I_{m-3})+1)+\nu_{i_0}+\ldots+\nu_{i_{r-1}}}(\epsilon)$$

Finally, since the moving indices decomposition for $(M, \mathcal{W}^*, \vec{\nu}^*, \epsilon)$ is of size $m-2$, it can be seen by induction hypothesis that the right-side term of this equation is equal to

$$(j^{M_n,\,j_n(W^*_n)}_{0,\nu_{i_n}} \circ j^{M_{n-1},\,j_{n-1}(W^*_{n-1})}_{0,\nu_{i_{n-1}}} \circ \ldots \circ j^{M_0,\,j_0(W^*_0)}_{0,\nu_{i_0}})(\epsilon)$$

for $(M_k)_{k\leq n}$ and $(j_k)_{k\leq n}$ as required. $\square$

3.2 Iterations of non-transitive models (as in Definition 3.2).

Given two sets N^0, N^1 and a set $\mathcal{F}$ of functions, we will say that N^0 *and* N^1 *are dense in each other relative to* $\mathcal{F}$ if for every $\epsilon \in \{0,1\}$ and every $f \in \mathcal{F} \cap N^\epsilon$ there is a function $g \in \mathcal{F} \cap N^{1-\epsilon}$ with $dom(g) = dom(f)$ and $g(x) \subseteq f(x)$ for all $x \in dom(f)$.

The following lemma is quite standard.

Lemma 3.13 *Suppose*

(a) γ *is a measurable cardinal and* W *is a normal measure on* γ*,*

[36] With $M^{\mathcal{W},\vec{\nu}}_{\Sigma(\nu_k\,:\,k<max(I_{m-3})+1)}$ as M, $ot(I_{m-2})$ as τ_0, $ot(I_{m-1})$ as τ_1, $\mathcal{W} \restriction [min(I_{m-2}), \tau+1)$ as $\mathcal{W}$, $\vec{\nu} \restriction [min(I_{m-2}), \tau+1)$ as $\vec{\nu}$, and $j^{M,\mathcal{W},\vec{\nu}}_{0,\Sigma(\nu_k\,:\,k<max(I_{m-3})+1)}(\epsilon)$ as ϵ.

(b) $\theta > 2^\gamma$ *is a regular cardinal,*

(c) N^0 *and* N^1 *are elementary substructures of* $H(\theta)$ *containing* W*,*

(d) given $\epsilon \in \{0,1\}$, $\langle N^\epsilon_\alpha : \alpha \leq \gamma\rangle$ *and* $\langle \eta^\epsilon_\alpha : \alpha < \gamma\rangle$ *are, respectively, the iteration of* N^ϵ *relative to* W *and its critical sequence, and*

(e) N^0 *and* N^1 *are dense in each other relative to* $\bigcup_{n<\omega}({}^{(n\gamma)}W)$.

Then, for every $\alpha < \gamma$, N^0_α *and* N^1_α *are dense in each other relative to* $\bigcup_{n<\omega}({}^{(n\gamma)}W)$.

In particular, $\langle \eta^0_\alpha : \alpha < \gamma\rangle = \langle \eta^1_\alpha : \alpha < \gamma\rangle$.[37] *Hence, if, in addition,* $N_0 \subseteq N_1$, *then* $N^0_\alpha \subseteq N^1_\alpha$ *for all* $\alpha \leq \gamma$.

Lemma 3.13 can be easily proved by induction on $\alpha < \gamma$. The following fact will be used, together with Lemma 3.13, in the proof of Lemma 3.15.

Lemma 3.14 *Suppose*

(a) $\Gamma = \langle \gamma_i : i < \tau\rangle$ *is a sequence of measurable cardinals,*

(b) for each $i < \tau$, W_i *is a normal measure on* γ_i,

(c) γ *is a measurable cardinal,* $\gamma \notin range(\Gamma)$,

(d) W *is a non-principal* γ*–complete ultrafilter on* γ,

(e) $\theta > 2^{sup_{i<\tau}\gamma_i}$ *is a regular cardinal,*

(f) N *is an elementary substructure of* $H(\theta)$ *of size less than* γ_0 *and containing* $\mathcal{W} := \langle W_i : i < \tau\rangle$ *and* W,

(g) $\vec{\nu} = \langle \nu_i : i < \tau\rangle$ *is a sequence of nonzero ordinals bounded by* $\gamma_0 + 1$, *and*

(h) the $\vec{\nu}$*–iteration of* N *relative to* $\mathcal{W}$ *exists.*

Then, given any $\alpha < \tau$, N *and* $N^{\mathcal{W},\vec{\nu}}_\alpha$ *are dense in each other relative to* $\bigcup_{n<\omega}({}^{(n\gamma)}W)$.

[37] Because for each $\epsilon \in \{0,1\}$ and $\alpha < \gamma$ and each $X \in W \cap N^\epsilon_\alpha$ there is then some $Y \in W \cap N^{1-\epsilon}_\alpha$ with $Y \subseteq X$, so that in particular $min(\bigcap(W \cap N^\epsilon_\alpha)) = min(\bigcap(W \cap N^{1-\epsilon}_\alpha))$.

Proof: By induction on α. For $\alpha = 0$ there is nothing to prove and, for α a nonzero limit ordinal, the result follows immediately by induction hypothesis as then $N_\alpha^{\mathcal{W},\vec{\nu}} = \bigcup_{\beta<\alpha} N_\beta^{\mathcal{W},\vec{\nu}}$. For the successor case, suppose $N_\alpha^{\mathcal{W},\vec{\nu}}$ is the one–step extension of $N_{\alpha_0}^{\mathcal{W},\vec{\nu}}$ – so $\alpha = \alpha_0 + 1$ – relative to W_i for some $i < \tau$. Fix $n < \omega$ and a function f from ${}^n\gamma$ into W in $N_\alpha^{\mathcal{W},\vec{\nu}}$. We want to see that there is a function g from ${}^n\gamma$ into W in N such that $g(\beta_0, \dots \beta_{n-1}) \subseteq f(\beta_0, \dots \beta_{n-1})$ for all $\beta_0, \dots \beta_{n-1} \in \gamma$.

If $\gamma_i > \gamma$, then $2^\gamma < sup(\gamma_i \cap N_{\alpha_0}^{\mathcal{W},\vec{\nu}})$. In particular, $N_\alpha^{\mathcal{W},\vec{\nu}}$ and $N_{\alpha_0}^{\mathcal{W},\vec{\nu}}$ have the same intersection with $(\bigcup_{n<\omega}({}^{({}^n\gamma)}W))$, and the desired result follows from the induction hypothesis applied to α_0. Hence, we may assume $\gamma_i < \gamma$. $f = h(\eta)$, where $\eta = min(\bigcap(W_i \cap N_{\alpha_0}^{\mathcal{W},\vec{\nu}}))$, for a function h from γ_i into ${}^{({}^n\gamma)}W$ in $N_{\alpha_0}^{\mathcal{W},\vec{\nu}}$. Let $\overline{h} : {}^n\gamma \longrightarrow W$ be defined by

$$\overline{h}(\beta_0, \dots \beta_{n-1}) = \bigcap\{h(\beta)(\beta_0, \dots \beta_{n-1}) \; : \; \beta < \gamma_i\}$$

$\overline{h}$ is a function in $N_{\alpha_0}^{\mathcal{W},\vec{\nu}}$ and, as $\gamma_i < \gamma$ and W is γ–complete, its range is indeed included in W. By induction hypothesis there is a function g from ${}^n\gamma$ into W in N such that

$$g(\beta_0, \dots \beta_{n-1}) \subseteq \overline{h}(\beta_0, \dots \beta_{n-1}) \subseteq h(\eta)(\beta_0, \dots \beta_{n-1}) = f(\beta_0, \dots \beta_{n-1})$$

for all $\beta_0, \dots \beta_{n-1} \in \gamma$. $\square$

Lemma 3.15 shows that for every ordinal λ, every N and every sequence $\mathcal{W}$ of measures (on distinct cardinals) there is a unique maximal closed iteration $\mathcal{I}$ of N relative to $\mathcal{W}$ such that any member of any iteration, in any possible transitive outer model of set theory, of N relative to $\mathcal{W}$ and bounded by λ is included in the last model of $\mathcal{I}$. This last model of $\mathcal{I}$ can thus be thought of as the hull of N relative to all such possible iterations.

Lemma 3.15 *Suppose*

(a) $\Gamma = \langle \gamma_i \; : \; i < \tau \rangle$ *is a one–to–one sequence of measurable cardinals,*[38]

(b) for each $i < \tau$*,* W_i *is a normal measure on* γ_i*,*

(c) $\theta > 2^{sup_{i<\tau}\gamma_i}$ *is a regular cardinal,*

[38] Γ need not be increasing.

(d) *N is an elementary substructure of $H(\theta)$ of size less than γ_0 and containing $\mathcal{W} := \langle W_i : i < \tau\rangle$,*

(e) *λ is an ordinal and $\vec{\lambda}$ is the τ–sequence with constant value λ,*

(f) *$\overline{\tau} \leq \tau$ and $\sigma : \overline{\tau} \longrightarrow \tau$ is a strictly increasing function,*

(g) *$\vec{\nu} = \langle \nu_j : j < \overline{\tau}\rangle$ is a sequence of nonzero ordinals bounded by $\lambda + 1$, and*

(h) *both the $\vec{\lambda}$–iteration of N relative to $\mathcal{W}$ and the $\vec{\nu}$–iteration of N relative to $\overline{\mathcal{W}} := \langle W_{\sigma(j)} : j < \overline{\tau}\rangle$ exist.*[39]

Let $\langle \eta^*_\alpha : \alpha + 1 < \lambda \cdot \tau\rangle$ and $\langle \eta_\alpha : \alpha + 1 < \Sigma_{j<\overline{\tau}}\nu_j\rangle$ be, respectively, the critical sequence of the $\vec{\lambda}$–iteration of N relative to $\mathcal{W}$, and the critical sequence of the $\vec{\nu}$–iteration of N relative to $\overline{\mathcal{W}}$.

Then, for all $j < \overline{\tau}$, $\eta_{(\Sigma_{k<j}\,\nu_k)+\alpha} = \eta^*_{\lambda\cdot\sigma(j)+\alpha}$ for all α with $\alpha + 1 < \nu_j$ and

$$N^{\mathcal{W},\vec{\nu}}_{(\Sigma_{k<j}\,\nu_k)+\alpha} \subseteq N^{\mathcal{W},\vec{\lambda}}_{\lambda\cdot\sigma(j)+\alpha}$$

for all $\alpha < \nu_j$.

Proof: We prove the result by induction on $j < \overline{\tau}$.

$N^{\mathcal{W},\vec{\nu}}_{\lambda\cdot\sigma(j)}$ is the last member of an iteration of N relative to the sequence $\mathcal{W}' = \langle W_k : k < \sigma(j)\rangle$ of measures. Since each W_k is a measure on γ_k and $\gamma_{\sigma(j)} \notin \{\gamma_k : k < \sigma(j)\}$, it follows by Lemma 3.14 that N and $N^{\mathcal{W},\vec{\lambda}}_{\lambda\cdot\sigma(j)}$ are dense in each other relative to $\bigcup_{n<\omega}({}^{({}^n\gamma_{\sigma(j)})}W_{\sigma(j)})$. For the same reason, N and $N^{\mathcal{W},\vec{\nu}}_{\Sigma_{k<j}\,\nu_k}$ are also dense in each other relative to $\bigcup_{n<\omega}({}^{({}^n\gamma_{\sigma(j)})}W_{\sigma(j)})$. By the transitivity of the relation of being dense in each other relative to $\bigcup_{n<\omega}({}^{({}^n\gamma_{\sigma(j)})}W_{\sigma(j)})$, it follows that $N^{\mathcal{W},\vec{\nu}}_{\Sigma_{k<j}\,\nu_k}$ and $N^{\mathcal{W},\vec{\lambda}}_{\lambda\cdot\sigma(j)}$ are dense in each other relative to $\bigcup_{n<\omega}({}^{({}^n\gamma_{\sigma(j)})}W_{\sigma(j)})$.

By Lemma 3.13, for every $\alpha < \nu_j$, $N^{\mathcal{W},\vec{\nu}}_{(\Sigma_{k<j}\,\nu_k)+\alpha}$ and $N^{\mathcal{W},\vec{\lambda}}_{\lambda\cdot\sigma(j)+\alpha}$ are then dense in each other relative to $\bigcup_{n<\omega}({}^{({}^n\gamma_{\sigma(j)})}W_{\sigma(j)})$ and $\eta_{(\Sigma_{k<j}\,\nu_k)+\alpha} = \eta^*_{\lambda\cdot\sigma(j)+\alpha}$ for every α with $\alpha+1 < \nu_j$. Since $N^{\mathcal{W},\vec{\nu}}_{\Sigma_{k<j}\,\nu_k} \subseteq N^{\mathcal{W},\vec{\lambda}}_{\lambda\cdot\sigma(j)}$, this implies $N^{\mathcal{W},\vec{\nu}}_{(\Sigma_{k<j}\,\nu_k)+\alpha} \subseteq N^{\mathcal{W},\vec{\lambda}}_{\lambda\cdot\sigma(j)+\alpha}$ for all $\alpha < \nu_j$, again by Lemma 3.13. This finishes the proof. □

Lemma 3.16 follows from Lemma 3.12 via Fact 3.1 and Lemma 3.15.

[39]Note that this implies $\lambda \leq \gamma_i + 1$ for all $i < \tau$.

Lemma 3.16 *Suppose*

(a) $\langle \gamma_i \,:\, i < \tau \rangle$ *is a one–to–one sequence of measurable cardinals,*

(b) W_τ *is any set and* W_i *is a normal measure on* γ_i *for each* $i < \tau$,

(c) $\theta > 2^{sup_{i<\tau}\gamma_i}$ *is a regular cardinal,*

(d) $N \preccurlyeq H(\theta)$ *is countable and with* $\mathcal{W} := \langle W_i \,:\, i < \tau + 1 \rangle \in N$,

(e) $\epsilon \in N$ *is an ordinal,*

(f) $\vec{\nu} = \langle \nu_i \,:\, i < \tau + 1 \rangle$ *is, in some outer model of set theory, a sequence of ordinals with* $0 < \nu_i < N \cap \omega_1$ *and with* $\nu_\tau = 1$,

(g) $\{i_0, \ldots i_{n-1}\}_<$ *is the finite set of indices* $i < \tau$ *such that* $\epsilon < j_{0,\omega_1}^{H(\theta),\, W_i}(\epsilon)$,[40] *and*

(h) the $\vec{\nu}$*–iteration of* N *relative to* $\mathcal{W}$ *exists.*

Then,

$$ot(N^{\mathcal{W},\vec{\nu}}_{\Sigma_{i<\tau}\nu_i} \cap \epsilon) = ot(N^{\langle W_{i_0}, W_{i_1}, \ldots W_{i_{n-1}}\rangle,\, \langle \nu_{i_0}, \nu_{i_1}, \ldots \nu_{i_{n-1}}+1\rangle}_{\nu_{i_0}+\nu_{i_1}+\ldots+\nu_{i_{n-1}}} \cap \epsilon)$$

Proof: Let $\vec{\nu}^* = \langle \nu_i^* \,:\, i < \tau + 1 \rangle$ be the $\tau + 1$–sequence of ordinals defined by $\nu_\tau^* = 1$, by $\nu_i^* = \nu_i$ if $i \in \{i_0, \ldots i_{n-1}\}$, and by $\nu_i^* = \omega_1 \cap N$ otherwise. Clearly, the $\vec{\nu}^*$–iteration of N relative to $\mathcal{W}$ exists. Then,

$$ot(N^{\mathcal{W},\vec{\nu}}_{\Sigma_{i<\tau}\nu_i} \cap \epsilon) \leq ot(N^{\mathcal{W},\vec{\nu}^*}_{\Sigma_{i<\tau}\nu_i^*} \cap \epsilon) = ot(N^{\langle W_{i_0}, W_{i_1}, \ldots W_{i_{n-1}}\rangle,\, \langle \nu_{i_0}, \nu_{i_1}, \ldots \nu_{i_{n-1}}+1\rangle}_{\nu_{i_0}+\nu_{i_1}+\ldots+\nu_{i_{n-1}}} \cap \epsilon)$$

The inequality holds by Lemma 3.15. As to the equality, let $\langle N_\alpha \,:\, \alpha < \overline{\alpha} \rangle$ be the $\vec{\nu}^*$–iteration of N relative to $\mathcal{W}$ and let π_α be, for every $\alpha < \overline{\alpha}$, the collapsing function of N_α. Also, let $\overline{W}_i = \pi_{\Sigma_{j<i}\nu_j^*}(W_i)$ for every $i < \tau$, and let $\overline{W}_\tau$ be, for example, $\emptyset$. Note that, by Fact 3.1, $\langle \overline{W}_i \,:\, i < \tau + 1 \rangle$ can be computed, inside $M := \pi_0``N_0$, from $\pi_0(\mathcal{W})$,[41] and hence it is an element of M (and therefore $\vec{\nu}^*$ also belongs to M). Hence, by Lemma 3.12,

$$j^{M,\overline{\mathcal{W}},\vec{\nu}^*}_{0,\,\Sigma_{i<\tau}\nu_i^*}(\pi(\epsilon)) = (j^{M_{n-1},\, j_{n-1}(\overline{W}_{i_{n-1}})}_{0,\nu_{i_{n-1}}} \circ j^{M_{n-2},\, j_{n-2}(\overline{W}_{i_{n-2}})}_{0,\nu_{i_{n-2}}} \circ \ldots \circ j^{M_0,\, j_0(\overline{W}_{i_0})}_{0,\nu_{i_0}})(\pi(\epsilon))$$

[40] Note that this set is indeed finite, once again by Lemma 3.6, and definable over $H(\theta)$ from $\mathcal{W}$ and ϵ (and in particular it belongs to N).

[41] $\langle \overline{W}_i \,:\, i < \tau + 1 \rangle$ need not be $\pi_0(\mathcal{W})$ in general.

where $(M_k)_{k<n}$ and $(j_k)_{k<n}$ are defined by recursion as in its statement. By Fact 3.1, this implies

$$ot(N^{\mathcal{W},\vec{\nu}^*}_{\Sigma_{i<\tau}\nu^*_i} \cap \epsilon) = ot(N^{\langle W_{i_0}, W_{i_1}, \ldots W_{i_{n-1}}\rangle, \langle \nu_{i_0}, \nu_{i_1}, \ldots \nu_{i_{n-1}}+1\rangle}_{\nu_{i_0}+\nu_{i_1}+\ldots+\nu_{i_{n-1}}} \cap \epsilon)$$

Finally, $ot(N^{\langle W_{i_0}, W_{i_1}, \ldots W_{i_{n-1}}\rangle, \langle \nu_{i_0}, \nu_{i_1}, \ldots \nu_{i_{n-1}}+1\rangle}_{\nu_{i_0}+\nu_{i_1}+\ldots+\nu_{i_{n-1}}} \cap \epsilon) \leq ot(N^{\mathcal{W},\vec{\nu}}_{\Sigma_{i<\tau}\nu_i} \cap \epsilon)$ again by Lemma 3.15, and the desired result follows from putting all the inequalities (and the fifth equality) together. □

The final result in this section (Lemma 3.18) deals with iterations relative to measures in a forcing extension. It will be used in the proof of Lemma 4.1 in the next section. The proof of Lemma 3.18 relies on the following preliminary fact.

Lemma 3.17 *Let γ be a measurable cardinal, let W be a normal measure on γ, let $\theta > 2^\gamma$ be a regular cardinal, and let $\mathcal{P}$, $\mathcal{Q}$, $\tilde{W}$, G be such that*

- *(a) $\mathcal{P} \subseteq \mathcal{Q}$ are partial orders such that $\mathcal{P} \in H(\theta)$ is a complete suborder of $\mathcal{Q}$ of size less than γ,*
- *(b) $\tilde{W}$ is a $\mathcal{P}$–name for the ultrafilter on γ generated by W,*
- *(c) $\dot{N}$ is a $\mathcal{Q}$–name for a countable elementary substructure of $H(\theta)^V$ containing W and $\mathcal{P}$, and*
- *(d) G is a $\mathcal{Q}$–generic filter over V and, in $V[G]$, every countable set of ordinals is included in a countable set of ordinals in V.*

In $V[G]$,

- *(e) let N' be the one–step extension of $\dot{N}_G$ relative to W, and*
- *(f) let $N^\dagger$ be the one–step extension of $\dot{N}_G[G \cap \mathcal{P}]$ relative to $\tilde{W}_G$.[42]*

Then, $N^\dagger = N'[G \cap \mathcal{P}]$. If, in addition, $G \cap D \cap \dot{N}_G \neq \emptyset$ for every dense subset D of $\mathcal{P}$ in $\dot{N}_G$, then $G \cap D \cap N' \neq \emptyset$ for every dense $D \subseteq \mathcal{P}$ in N'.

[42]Note that, since every countable subset of V in $V[G]$ is covered by a countable set in V, $\bigcap(X \cap W)$ and $\bigcap(Y \cap \tilde{W}_G)$ are nonempty for all countable sets X, Y in $V[G]$ with $W \in X$ and $\tilde{W}_G \in Y$, so it makes sense to define, in the same way as in Definition 3.2, the one–step iterations of $\dot{N}_G$ and of $\dot{N}_G[G \cap \mathcal{P}]$ relative to, respectively, W and $\tilde{W}_G$.

Proof: Let $\eta = min(\bigcap(\dot{N}_G \cap W))$ and $\overline{\eta} = min(\bigcap(\dot{N}_G[G \cap \mathcal{P}] \cap \tilde{W}_G))$. $N'[G \cap \mathcal{P}]$ and $N^\dagger$ are, respectively, the set of all $(\tilde{f}_\eta)_G$, where $(\tilde{f}_\alpha)_{\alpha<\gamma}$ is a γ–sequence in $\dot{N}_G$ consisting of $\mathcal{P}$–names, and the set of $\tilde{f}_G(\overline{\eta})$, where $\tilde{f} \in \dot{N}_G$ is a $\mathcal{P}$–name for a function with domain γ. Therefore, in order to check the equality $N^\dagger = N'[G \cap \mathcal{P}]$, it suffices to prove that $\overline{\eta}$ is forced to be equal to η. In fact, we can prove $\bigcap(W \cap \dot{N}_G) = \bigcap(\tilde{W}_G \cap \dot{N}_G[G \cap \mathcal{P}])$. In order to verify this equality, it is enough to show that for every $X \in \tilde{W}_G \cap \dot{N}_G[G \cap \mathcal{P}]$ there is a $Y \subseteq X$ in $\dot{N}_G \cap W$.

Thus, let $\tilde{X}$ be a $\mathcal{P}$–name in $\dot{N}_G$ for X. The set D of conditions $p \in \mathcal{P}$ for which there is some $Y \in W$ with $p \Vdash_{\mathcal{P}} Y \subseteq \tilde{X}$ is dense in $\mathcal{P}$. For every $p \in D$ we fix some Y_p which works as Y for p in the above sentence. Furthermore, we may assume that $\mathcal{F} := \{(p, Y_p) : p \in D\}$ is in $\dot{N}_G$. By γ–completeness of W, $Y := \bigcap\{Y_p : p \in D\}$ is in W, and as $\mathcal{F}$ is in $\dot{N}_G$, Y is in $\dot{N}_G$ as well. Since D is dense, there is some $p \in D \cap G$, and thus $Y \subseteq Y_p \subseteq X$.

Finally, note that, since γ is inaccessible and $\gamma \cap \dot{N}_G$ is an initial segment of $\gamma \cap N'$, $\dot{N}_G$ and N' have the same subsets of $\mathcal{P}$. Hence, if $G \cap D \cap \dot{N}_G$ is nonempty for every dense subset D of $\mathcal{P}$ in $\dot{N}_G$, then it is also true that $G \cap D \cap N'$ is nonempty for every dense subset D of $\mathcal{P}$ in $\dot{N}'$. $\square$

Lemma 3.18 *Let $\langle \gamma_i : i < \tau \rangle$ be a sequence of measurable cardinals and let W_i be, for each i, a normal measure on γ_i. Let $\theta > 2^{sup_{i<\tau}\gamma_i}$ be a regular cardinal, and let $\mathcal{P}$, $\mathcal{Q}$, $\tilde{W}_i$ (for $i < \tau$), $\mathcal{W}$, $\mathcal{W}^*$, $\dot{N}$, G and $\vec{\nu}$ be such that*

(a) $\mathcal{P} \subseteq \mathcal{Q}$ are partial orders such that $\mathcal{P} \in H(\theta)$ is a complete suborder of $\mathcal{Q}$ of size less than γ_i for all $i < \tau$,

(b) for each $i < \tau$, $\tilde{W}_i$ is a $\mathcal{P}$–name for the ultrafilter on γ_i generated by W_i,

(c) $\mathcal{W} = \langle W_i : i < \tau \rangle$ and $\mathcal{W}^*$ is a $\mathcal{P}$–name for $\langle \tilde{W}_i : i < \tau \rangle$,

(d) $\dot{N}$ is a $\mathcal{Q}$–name for a countable elementary substructure of $H(\theta)^V$ containing $\mathcal{W}$ and $\mathcal{P}$,

(e) G is $\mathcal{Q}$–generic over V and every countable set of ordinals in $V[G]$ is included in a countable set in V,

(f) $\vec{\nu} \in V[G]$ is a τ–sequence of nonzero ordinals, and

(g) *the $\vec{\nu}$–iteration $\langle N_\alpha : \alpha < \overline{\alpha}\rangle$ of $\dot{N}_G$ relative to $\mathcal{W}$ and the $\vec{\nu}$–iteration $\langle N^*_\alpha : \alpha < \overline{\alpha}\rangle$ of $\dot{N}_G[G \cap \mathcal{P}]$ relative to $\mathcal{W}^*$ exist.*[43]

Then, for every $\alpha < \overline{\alpha}$, $N^*_\alpha = N_\alpha[G \cap \mathcal{P}]$. Furthermore, if $\dot{N}_G \cap D \cap G \neq \emptyset$ for every dense subset D of $\mathcal{P}$ in $\dot{N}_G$, then $N_\alpha \cap D \cap G \neq \emptyset$ for every $\alpha < \overline{\alpha}$ and every dense $D \subseteq \mathcal{P}$ in N_α.

Proof: Since $|\mathcal{P}| < \gamma_i$ for all i and since all γ_i are inaccessible, all N_α's have the same subsets of $\mathcal{P}$. Hence, as in Lemma 3.17, if $\dot{N}_G \cap D \cap G \neq \emptyset$ for every dense subset D of $\mathcal{P}$ in $\dot{N}_G$, then also $N_\alpha \cap D \cap G \neq \emptyset$ for every dense $D \subseteq \mathcal{P}$ in N_α and every $\alpha < \overline{\alpha}$.

Next we prove, by induction on $i < \tau$, that for every $\xi < \nu_i$, $N^*_{(\Sigma_{j<i}\nu_j)+\xi} = N_{(\Sigma_{j<i}\nu_j)+\xi}[G \cap \mathcal{P}]$.

Since $N^*_{\Sigma_{j<i}\nu_j} = \bigcup_{\alpha<\Sigma_{j<i}\nu_j} N^*_\alpha$ if $\Sigma_{j<i}\nu_j$ is a limit ordinal and $N^*_{\Sigma_{j<i}\nu_j} = N^*_{\overline{\xi}}$ if $\Sigma_{j<i}\nu_j = \overline{\xi} + 1$, by induction hypothesis applied to $j < i$, $N^*_{\Sigma_{j<i}\nu_j} = N_{\Sigma_{j<i}\nu_j}[G \cap \mathcal{P}]$.

Now we can obtain the conclusion for $\xi < \nu_i$ by induction on ξ. The desired conclusion for the case $\xi = 0$ has already been proved and, as $N_{(\Sigma_{j<i}\nu_j)+\xi} = \bigcup_{\xi'<\xi} N_{(\Sigma_{j<i}\nu_j)+\xi'}$ and $N^*_{(\Sigma_{j<i}\nu_j)+\xi} = \bigcup_{\xi'<\xi} N^*_{(\Sigma_{j<i}\nu_j)+\xi'}$ for ξ limit, the limit case of the induction is trivial.

Now suppose $\xi = \overline{\xi}+1$ and assume $N^*_{(\Sigma_{j<i}\nu_j)+\overline{\xi}} = N_{(\Sigma_{j<i}\nu_j)+\overline{\xi}}[G\cap\mathcal{P}]$. Then, since $N_{(\Sigma_{j<i}\nu_j)+\xi}$ and $N^*_{(\Sigma_{j<i}\nu_j)+\xi}$ are, respectively, the one–step extension of $N_{(\Sigma_{j<i}\nu_j)+\overline{\xi}}$ relative to $\dot{W}_i$ and the one–step extension of $N^*_{(\Sigma_{j<i}\nu_j)+\overline{\xi}}$ relative to $(\tilde{W}_i)_G$, by Lemma 3.17 it follows that $N^*_{(\Sigma_{j<i}\nu_j)+\xi} = N_{(\Sigma_{j<i}\nu_j)+\xi}[G \cap \mathcal{P}]$. $\square$

4 Resuming the proof of Theorem 2.1.

Now we are back in our forcing construction for proving Theorem 2.1 and with enough notions and tools for stating and proving the following forcing iteration lemma.

Lemma 4.1 *Let $\alpha \leq \kappa$ be an ordinal. Then, any countable set of ordinals in any forcing extension via $\mathcal{P}_\alpha$ is included in a countable set in the ground model.*

[43] By the same remark as in the footnote to Lemma 3.17, it makes sense to define, in $V[G]$, the iterations $\langle N_\alpha : \alpha < \overline{\alpha}\rangle$ and $\langle N^*_\alpha : \alpha < \overline{\alpha}\rangle$.

Furthermore, let $\xi_0 < \alpha$ be given and let q_0 be a condition in $\mathcal{P}_{\xi_0}$. Then, the following holds in $V^{\mathcal{P}_{\xi_0}\restriction q_0}$:

Let $i_0 < \Lambda$ be an ordinal, let $\theta > 2^\kappa$ be a regular cardinal, and let $\overline{\alpha}$, $\dot{N}$, N, q and $\tilde{p}$ be such that

(a) $\xi_0 \le \overline{\alpha} \le \alpha$, $\overline{\alpha} < \kappa$,

(b) *$\dot{N}$ is a $\mathcal{P}_{\overline{\alpha}}/\dot{G}_{\xi_0}$–name for a countable elementary substructure of $H(\theta)^{V[\dot{G}_{\xi_0}]}$ with $\dot{N} \cap \omega_1^V \in S_{i_0}$,*

(c) *N is a countable elementary substructure of $H(\theta)^{V[\dot{G}_{\xi_0}]}$ containing $\mathcal{U}_{\xi_0}^{\overline{\alpha}}$, $\dot{G}_{\xi_0}$, $\langle \mathcal{P}_\xi : \xi \le \alpha \rangle$ and $\overline{\alpha}$,*

(d) *$\dot{N}$ is forced to be either N or else the last model of an iteration of N relative to $\mathcal{U}_{\xi_0}^{\xi_0,\overline{\alpha}}$, with closed component iterations, and bounded by sup$\{\nu + 2 : \nu < \alpha_{i_0}\}$,*[44]

(e) *q is a $(\dot{N}, \mathcal{P}_{\overline{\alpha}}/\dot{G}_{\xi_0})$–generic condition in $\mathcal{P}_{\overline{\alpha}}/\dot{G}_{\xi_0}$ with $q \restriction \xi_0 = q_0$,*

(f) *$\tilde{p}$ is a $\mathcal{P}_{\overline{\alpha}}/\dot{G}_{\xi_0}$–name for a condition in $\dot{N} \cap \mathcal{P}_\alpha/\dot{G}_{\xi_0}$, and*

(g) $q \Vdash_{\mathcal{P}_{\overline{\alpha}}/\dot{G}_{\xi_0}} (\tilde{p})_{\dot{G}_{\overline{\alpha}}} \restriction \overline{\alpha} \in \dot{G}_{\overline{\alpha}}$.

Then there are a condition q^+ in $\mathcal{P}_\alpha/\dot{G}_{\xi_0}$ and a $\mathcal{P}_\alpha/\dot{G}_{\xi_0}$–name $\tilde{N}$ such that

(i) *$\tilde{N}$ is forced to be either $\dot{N}$ or the last model of an iteration of $\dot{N}$ relative to $\mathcal{U}_{\xi_0}^{\overline{\alpha},\alpha}$ with closed component iterations and bounded by sup$\{\nu + 2 : \nu < \alpha_{i_0}\}$,*

(ii) $q^+ \restriction \overline{\alpha} = q$,

(iii) $\Vdash_{\mathcal{P}_\alpha/\dot{G}_{\xi_0}} \mathit{supp}(q^+) \setminus \overline{\alpha} \subseteq \tilde{N}$,

(iv) *$q^+ \Vdash_{\mathcal{P}_\alpha/\dot{G}_{\xi_0}} (\tilde{p})_{\dot{G}_\alpha} \in \dot{G}_\alpha$, and*

(v) *q^+ is $(\tilde{N}, \mathcal{P}_\alpha/\dot{G}_{\xi_0})$–generic.*

[44] In other words, the iteration is bounded by α_{i_0} if α_{i_0} is a limit ordinal, and by $\alpha_{i_0} + 1$ if α_{i_0} is a successor ordinal. Equivalently, this means that every component iteration of the iteration in question has length $\nu + 1$ for some $\nu < \alpha_{i_0}$.

Proof: We can prove the lemma by induction on α. Fix $\alpha \leq \kappa$ and suppose the result holds for all $\alpha_0 < \alpha$. Note that, by Fact 3.7 together with Fact 3.1 and with Lemma 3.15, the second conclusion of the lemma for α (with $\xi_0 = \overline{\alpha} = 0$) implies the first conclusion, namely that every countable set of ordinals in any extension by $\mathcal{P}_\alpha$ is included in some countable set in the ground model. Hence, in order to prove the lemma for α, it suffices to prove the second conclusion (for α).

Fix $\xi_0 < \alpha$ and $q_0 \in \mathcal{P}_{\xi_0}$ and work in $V^{\mathcal{P}_{\xi_0}\restriction q_0}$. Fix i_0, θ, $\overline{\alpha}$, $\dot{N}$, q and $\tilde{p}$ as in the hypothesis. Suppose $q \Vdash_{\mathcal{P}_{\overline{\alpha}}/\dot{G}_{\xi_0}} (\tilde{p})_{\dot{G}_{\overline{\alpha}}} \restriction \overline{\alpha} \in \dot{G}_{\overline{\alpha}}$. For $\alpha = \overline{\alpha}$ the desired conclusion holds trivially.

Suppose $\alpha = \alpha_0 + 1$ for $\alpha_0 \geq \overline{\alpha}$. By induction hypothesis there are a condition q^* in $\mathcal{P}_{\alpha_0}/\dot{G}_{\xi_0}$ and a $\mathcal{P}_{\alpha_0}/\dot{G}_{\xi_0}$–name $\tilde{N}^*$ for $\dot{N}$ or for the last model of an iteration of $\dot{N}$ relative to $\mathcal{U}^{\overline{\alpha},\alpha_0}_{\xi_0}$ such that (i)–(iv) from the conclusion hold with α_0, q^*, (a name for) $\tilde{p} \restriction \alpha_0$ and $\tilde{N}^*$ replacing α, q^+, $\tilde{p}$ and $\tilde{N}$. Suppose $\alpha_0 \in \bigcup_{\zeta\in A} A_\zeta$. Since, by Lemma 2.2, $\dot{\mathcal{Q}}_{\alpha_0}$ is forced by $\mathcal{P}_{\alpha_0}$ to be F_{α_0}–proper for the function F_{α_0} sending a countable $N' \preccurlyeq H(\theta)$ containing everything relevant to $\{(N')^{\alpha_0}_{\alpha_0,\nu} : \nu < \alpha_{i_0}\}$, we may fix a name $\tilde{\nu}_{\alpha_0}$ for an ordinal less than α_{i_0}, together with a $\mathcal{P}_{\alpha_0}/\dot{G}_{\xi_0}$–name $\dot{q}$ such that q^* forces that $\dot{q}$ is a $((\tilde{N}^*[\dot{G}_{\alpha_0}])^{\alpha_0}_{\alpha_0,\tilde{\nu}_{\alpha_0}}, \dot{\mathcal{Q}}_{\alpha_0})$–generic condition in $\dot{\mathcal{Q}}_{\alpha_0}$ extending $(\tilde{p})_{\dot{G}_{\overline{\alpha}}}(\alpha_0)$.

Let q^+ be q^* followed by $\dot{q}$. Then q^+ is a condition forcing $(\tilde{p})_{\dot{G}_\alpha} \in \dot{G}_\alpha$, and $supp(q^+) \setminus \overline{\alpha} \subseteq (supp(q^*) \setminus \overline{\alpha}) \cup \{\alpha_0\}$, so $supp(q^+)$ is certainly forced to be included in $(\tilde{N}^*)^{\alpha_0}_{\xi_0,\tilde{\nu}_{\alpha_0}}$. Note that, by Lemma 3.18, q^* is $((\tilde{N}^*)^{\alpha_0}_{\xi_0,\tilde{\nu}_{\alpha_0}}, \mathcal{P}_{\alpha_0}/\dot{G}_{\xi_0})$–generic and $q^* \Vdash_{\mathcal{P}_{\alpha_0}/\dot{G}_{\xi_0}} (\tilde{N}^*[\dot{G}_{\alpha_0}])^{\alpha_0}_{\alpha_0,\tilde{\nu}_{\alpha_0}} = (\tilde{N}^*)^{\alpha_0}_{\xi_0,\tilde{\nu}_{\alpha_0}}[\dot{G}_{\alpha_0}]$[45] (so that q^* forces that $\dot{q}$ is $((\tilde{N}^*)^{\alpha_0}_{\xi_0,\tilde{\nu}_{\alpha_0}}[\dot{G}_{\alpha_0}], \dot{\mathcal{Q}}_{\alpha_0})$–generic). It follows that q^+ is an $((\tilde{N}^*)^{\alpha_0}_{\xi_0,\tilde{\nu}_{\alpha_0}}, \mathcal{P}_\alpha/\dot{G}_{\xi_0})$–generic condition (and $(\tilde{N}^*)^{\alpha_0}_{\xi_0,\tilde{\nu}_{\alpha_0}}$ is clearly forced to be the last member of an iteration of $\dot{N}$ relative to $\mathcal{U}^{\overline{\alpha},\alpha_0+1}_{\xi_0}$ with closed component iterations and bounded by $sup\{\nu + 1 : \nu < \alpha_0\}$). The proof when α_0 is not in $\bigcup_{\zeta\in A} A_\zeta$ is a simpler version of the above argument. The point is that $\dot{\mathcal{Q}}_{\alpha_0}$ is then forced to be a proper poset. This finishes the proof in this case.

It remains to consider the case when $\alpha > \overline{\alpha}$ and α is a limit ordinal.

Suppose $cf(\alpha) = \omega$. Let us move over to $V^{\mathcal{P}_{\overline{\alpha}}\restriction q}$. Let $(\alpha_n)_{n<\omega}$ be a strictly increasing sequence of ordinals in $\dot{N} \cap \alpha$ converging to α and such that $\alpha_0 = \overline{\alpha}$. We build four sequences $(q_n)_{n<\omega}$, $(\tilde{p}_n)_{n<\omega}$, $(\tilde{N}_n)_{n<\omega}$ and $(\tilde{\mathcal{D}}_n)_{n<\omega}$ such that

[45] Because q^* is $(\tilde{N}^*, \mathcal{P}_{\alpha_0}/\dot{G}_{\xi_0})$–generic and, if $\alpha_0 \in \bigcup_{\zeta\in A} A_\zeta$, $|\mathcal{P}_{\alpha_0}| < \alpha_0$ in V (and so in particular $|\mathcal{P}_{\alpha_0}/\dot{G}_{\xi_0}| < \alpha_0$ in $V^{\mathcal{P}_{\xi_0}}$).

$q_0 = q$, $\tilde{p}_0 = \tilde{p}$ and $\tilde{N}_0 = \dot{N}[\dot{G}_{\overline{\alpha}}]$ and such that, for all $n < \omega$,

(a) $\tilde{N}_{n+1}$ is a $\mathcal{P}_{\alpha_{n+1}}/\dot{G}_{\overline{\alpha}}$–name forced to be either $\tilde{N}_n$ or else the last model of a closed iteration of $\tilde{N}_n$ relative to $\mathcal{U}_{\overline{\alpha}}^{\alpha_n,\alpha_{n+1}}$ with closed component iterations and bounded by $sup\{\nu + 2 \,:\, \nu < \alpha_{i_0}\}$,

(b) q_n is a condition in $\mathcal{P}_{\alpha_n}/\dot{G}_{\overline{\alpha}}$ which is forced to be $(\tilde{N}_n,\, \mathcal{P}_{\alpha_n}/\dot{G}_{\overline{\alpha}})$–generic, $q_{n+1} \restriction \alpha_n = q_n$ and $\Vdash_{\mathcal{P}_{\alpha_{n+1}}/\dot{G}_{\overline{\alpha}}} supp(q_{n+1} \setminus \alpha_n) \subseteq \tilde{N}_{n+1}$,

(c) $\tilde{p}_n$ is a $\mathcal{P}_{\alpha_n}/\dot{G}_{\overline{\alpha}}$–name for a condition in $\tilde{N}_n \cap (\mathcal{P}_\alpha/\dot{G}_{\overline{\alpha}})$ and $\Vdash_{\mathcal{P}_{\alpha_{n+1}}/\dot{G}_{\overline{\alpha}}}$ $\tilde{p}_{n+1} \leq_\alpha \tilde{p}_n$, and

(d) $\tilde{\mathcal{D}}_n$ is a $\mathcal{P}_{\alpha_{n+1}}/\dot{G}_{\overline{\alpha}}$–name forced to be an enumeration in length ω of all dense and open subsets of $\mathcal{P}_\alpha/\dot{G}_{\overline{\alpha}}$ in $\tilde{N}_{n+1}$.

Moreover, the sequences will be chosen in such a way that, for every n,

(e) $\mathcal{P}_{\alpha_{n+1}}/\dot{G}_{\overline{\alpha}}$ forces that $\tilde{p}_{n+1}$ is in $\bigcap_{k,\, l \leq n} \tilde{\mathcal{D}}_k(l)$ and that $\tilde{p}_{n+1} \restriction \alpha_{n+1}$ is in $\dot{G}_{\alpha_{n+1}}$ in case $\tilde{p}_n \restriction \alpha_{n+1}$ is in $\dot{G}_{\alpha_{n+1}}$, and finally

(f) $q_n \Vdash_{\alpha_n} (\tilde{p}_n)_{\dot{G}_{\alpha_n}} \restriction \alpha_n \in \dot{G}_{\alpha_n}$.

Suppose q_n, $\tilde{p}_n$ and $\tilde{N}_n$ have been defined. Then $\tilde{N}_{n+1}$ and q_{n+1} can be found by an application of the induction hypothesis applied to α_{n+1}. Furthermore, in $V^{\mathcal{P}_{\alpha_{n+1}}/\dot{G}_{\overline{\alpha}}}$, $\tilde{p}_{n+1}$ can be found in $\tilde{N}_{n+1}$ by an application, within $\tilde{N}_{n+1}$, of the general fact (see Lemma 3.17 in [G]) that if $\langle \mathbb{P}_\xi \,:\, \xi \leq \delta \rangle$ is a forcing iteration, $\beta < \delta$, D is a dense subset of $\mathbb{P}_\delta$ and $\tilde{p}_1$ is a $\mathbb{P}_\beta$–name for a condition in $\mathbb{P}_\delta$, then $\mathbb{P}_\beta$ forces that there is a condition $\tilde{p}_2$ in D extending $\tilde{p}_1$ such that $\tilde{p}_2 \restriction \beta \in \dot{G}_\beta$ in case $\tilde{p}_1 \restriction \beta \in \dot{G}_\beta$.

Let $\tilde{N}$ be a $\mathcal{P}_\alpha/\dot{G}_{\overline{\alpha}}$–name for $\bigcup_n \tilde{N}_n$. Since the iteration has been built with countable supports and since, by induction hypothesis applied to $\overline{\alpha}$, every countable set of ordinals in $V^{\mathcal{P}_{\overline{\alpha}}}$ is covered by a countable set in V, there is a condition $q^* \in \mathcal{P}_\alpha$ such that $q^* \restriction \beta \Vdash_\beta q^*(\beta) = \bigcup_n q_n(\beta)$ for all $\beta < \alpha$. Also, both $q^* \restriction \overline{\alpha} = q$ and $\Vdash_{\mathcal{P}_\alpha/\dot{G}_{\overline{\alpha}}} supp(q^*) \setminus \overline{\alpha} \subseteq \tilde{N}$ hold by construction. Since q^* extends all q_n and since every q_n forces that $(\tilde{p})_{\dot{G}_{\overline{\alpha}}} \restriction \alpha_n$ is extended by $(\tilde{p}_n)_{\dot{G}_{\alpha_n}} \restriction \alpha_n \in \dot{G}_{\alpha_n}$, q^* forces $(\tilde{p})_{\dot{G}_{\overline{\alpha}}} \restriction \alpha_n \in \dot{G}_{\alpha_n}$ for all n, and therefore it forces $(\tilde{p})_{\dot{G}_{\overline{\alpha}}} \in \dot{G}_\alpha$. In order to see that q^* is $(\tilde{N}, \mathcal{P}_\alpha/\dot{G}_{\overline{\alpha}})$–generic, let l, $k < \omega$ be given. Let n be an integer above l and k. $\tilde{p}_{n+1}$ is forced by $\mathcal{P}_{\alpha_{n+1}}/\dot{G}_{\overline{\alpha}}$ to be in $\tilde{N}_{n+1}\cap\tilde{\mathcal{D}}_l(k)$. Since, by an argument as above, $q^*\Vdash_{\mathcal{P}_\alpha/\dot{G}_{\overline{\alpha}}}(\tilde{p}_{n+1})_{\dot{G}_{\alpha_{n+1}}}\in\dot{G}_\alpha$, $q^*\Vdash_{\mathcal{P}_\alpha/\dot{G}_{\overline{\alpha}}}\dot{G}_\alpha\cap\tilde{\mathcal{D}}_l(k)\cap\tilde{N}\neq\emptyset$.

Let $\dot{q}^*$ be a $\mathcal{P}_{\overline{\alpha}}/\dot{G}_{\xi_0}$–name for q^*. By Lemma 3.15, q forces $supp(\dot{q}^*)\setminus\overline{\alpha}\subseteq\overline{N}$, where $\overline{N}$ is the last model of the $\vec{\alpha_{i_0}}$–iteration of $\dot{N}[G_{\overline{\alpha}}]$ relative to $\mathcal{U}^{\overline{\alpha},\alpha}_{\overline{\alpha}}\frown\langle\emptyset\rangle$ (where $\vec{\alpha}_{i_0}$ is $\langle\alpha_{i_0}\,:\,\gamma\in\bigcup_{\zeta\in A}A_\zeta\cap[\overline{\alpha},\,\alpha)\rangle\frown\langle 1\rangle$). Since q is $(\dot{N},\mathcal{P}_{\overline{\alpha}}/\dot{G}_{\xi_0})$–generic and $\mathcal{P}_{\overline{\alpha}}/\dot{G}_{\xi_0}$ has size less than $min(\bigcup_{\zeta\in A}A_\zeta\setminus\overline{\alpha})$, by Lemma 3.18 q is also $(\overline{N}^*,\mathcal{P}_{\overline{\alpha}}/\dot{G}_{\xi_0})$–generic, where $\overline{N}^*$ is now the last member of the $\vec{\alpha_{i_0}}$–iteration of $\dot{N}$ relative to $\mathcal{U}^{\overline{\alpha},\alpha}_{\xi_0}\frown\langle 1\rangle$, and furthermore it forces $\overline{N}=\overline{N}^*[\dot{G}_{\overline{\alpha}}]$. Thus, by Facts 3.7 and 3.1, there is, in $V^{\mathcal{P}_{\xi_0}\restriction q_0}$, a countable set $X\subseteq\alpha$ – namely $\overline{N}^*\cap\alpha$ – such that $q\Vdash_{\overline{\alpha}}supp(\dot{q}^*)\subseteq\check{X}$. Now, since every countable set of ordinals in $V^{\mathcal{P}_{\xi_0}\restriction q_0}$ is covered by a countable set in V, it follows that there is a real $\mathcal{P}_\alpha$–condition q^+ with $q^+\restriction\overline{\alpha}=q$ and $q\Vdash_{\overline{\alpha}}\dot{q}^*=q^+$. Again by Lemma 3.18, in $V^{\mathcal{P}_{\overline{\alpha}}\restriction q}$ there is a closed iteration of $\dot{N}$ relative to $\mathcal{U}^{\overline{\alpha},\alpha}_{\xi_0}\frown\langle\emptyset\rangle$ with closed component iterations and bounded by $sup\{\nu+1\,:\,\nu<\alpha_{i_0}\}$ such that q is $(\tilde{N}^*,\mathcal{P}_{\overline{\alpha}}/\dot{G}_{\xi_0})$–generic for its last member $\tilde{N}^*$ and such that $\tilde{N}$ is $\tilde{N}^*[\dot{G}_{\overline{\alpha}}]$. It follows, since q^+ is $(\tilde{N},\mathcal{P}_\alpha/\dot{G}_{\overline{\alpha}})$–generic, that q^+ is $(\tilde{N}^*,\mathcal{P}_\alpha/\dot{G}_{\xi_0})$–generic as well. Hence, q^+ and (any name for) $\tilde{N}^*$ satisfy the desired conclusion for α in this case.

Finally, suppose α has uncountable cofinality. It is conceivable, in $V^{\mathcal{P}_\alpha/\dot{G}_{\xi_0}}$, that $sup(\dot{N}\cap\alpha)<sup(N'\cap\alpha)$ holds for some N' occurring in some of the relevant iterations of $\dot{N}$. However, by Fact 3.1 and by Lemmas 3.6 and 3.8, there is some $\hat{\alpha}<\alpha$ such that this does not happen if the measures used in the iteration are on cardinals above $\hat{\alpha}$. Furthermore, this $\hat{\alpha}$ has the same property, in $V^{\mathcal{P}_\alpha/\dot{G}_{\xi_0}}$, with respect to any $N\preccurlyeq H(\theta)^{V[\dot{G}_{\xi_0}]}$, not just $\dot{N}$. Hence, in $V^{\mathcal{P}_\alpha/\dot{G}_{\xi_0}}$ it may be that there is a relevant iteration of $\dot{N}$ which makes $sup(N\cap\alpha)$ grow, but nevertheless $sup(N\cap\alpha)$ will be fixed by the tail of the iteration given by some stage ξ if it is the case that we only apply measures on cardinals above $\hat{\alpha}$ from stage ξ on. To be more precise, there is an ordinal $\hat{\alpha}<\alpha$ above $\overline{\alpha}$ such that, in $V^{\mathcal{P}_\alpha/\dot{G}_{\xi_0}}$, it holds that there is no countable $N\preccurlyeq H(\theta)^{V[\dot{G}_{\xi_0}]}$ for which there is a $\vec{\nu}$–iteration of N, for any sequence $\vec{\nu}$ bounded by $sup\{\nu+2\,:\,\nu<\alpha_{i_0}\}$, relative to $\mathcal{U}:=\mathcal{U}^{\hat{\alpha},\alpha}_{\xi_0}$, and such that $sup(N\cap\alpha)<sup(N^{\mathcal{U},\vec{\nu}}_\beta\cap\alpha)$ for any β. We may assume that $\hat{\alpha}$ is definable in $H(\theta)^{V^{\mathcal{P}_{\xi_0}}}$ from $\mathcal{U}^{\overline{\alpha},\alpha}_{\xi_0}$ and α, and therefore that it is forced to be in $\dot{N}$. Further, by Lemma 3.8 we may assume as well that $\hat{\alpha}$ has the same property as above in every outer model of set theory.

By induction hypothesis applied to $\hat{\alpha}$ we may fix a condition $q^\dagger \in \mathcal{P}_{\hat{\alpha}}/\dot{G}_{\xi_0}$ and a $\mathcal{P}_{\hat{\alpha}}/\dot{G}_{\xi_0}$–name $\tilde{N}^\dagger$ such that (i)–(iv) from the conclusion hold for $\hat{\alpha}$, $q^\dagger$, (a $\mathcal{P}_{\overline{\alpha}}/\dot{G}_{\xi_0}$–name for) $\tilde{p} \restriction \hat{\alpha}$ and $\tilde{N}^\dagger$ in place of α, q^+, $\tilde{p}$ and $\tilde{N}$.

Let us move over to $V^{\mathcal{P}_{\hat{\alpha}} \restriction q^\dagger}$. Let $\alpha^* = sup(\tilde{N}^\dagger \cap \alpha)$. Fix a strictly increasing sequence $(\alpha_n)_{n<\omega}$ of ordinals in $\tilde{N}^\dagger$ converging to α^* and with $\alpha_0 = \hat{\alpha}$. We build sequences $(q_n)_{n<\omega}$, $(\tilde{p}_n)_{n<\omega}$, $(\tilde{N}_n)_{n<\omega}$ and $(\tilde{\mathcal{D}}_n)_{n<\omega}$ such that $q_0 = q^\dagger$, $\tilde{p}_0 = \tilde{p}$ and $\tilde{N}_0 = \tilde{N}^\dagger[\dot{G}_{\hat{\alpha}}]$ and such that, for all $n < \omega$, (a)–(f) as in the $cf(\alpha) = \omega$–case construction hold for them and for all $n < \omega$. Let q' and $\tilde{N}'$ be, respectively, $\bigcup_n q_n$ and $\bigcup_n \tilde{N}_n$.

Using the induction hypothesis on $\hat{\alpha}$ and arguing as in the previous case we can verify that q' is a condition in $\mathcal{P}_\alpha/\dot{G}_{\xi_0}$. By construction, $q' \restriction \hat{\alpha} = q^\dagger$ and $q' \Vdash_{\mathcal{P}_\alpha/\dot{G}_{\xi_0}} supp(q') \setminus \overline{\alpha} \subseteq \tilde{N}'$. Arguing again as before we obtain that q' forces $\tilde{p}_n \restriction \alpha^* \in \dot{G}_{\alpha^*}$ for all n. Now fix any n and note that there is a dense set of conditions in $\tilde{N}_n \cap (\mathcal{P}_{\alpha_n}/\dot{G}_{\xi_0})$ deciding the value of the supremum of $supp(\tilde{p}_n)$. Hence, since q' is $(\tilde{N}_n, \mathcal{P}_{\alpha_n}/\dot{G}_{\xi_0})$–generic, it forces that there is a condition in $\dot{G}_{\alpha_n} \subseteq \dot{G}_\alpha$ belonging to $\tilde{N}_n$ and deciding the value of the supremum of $supp(\tilde{p}_n)$ (and therefore deciding that this supremum is some ordinal less than $sup(\tilde{N}_n \cap \alpha)$). But, by the choice of $\hat{\alpha}$, $sup(\tilde{N}_n \cap \alpha) = \alpha^*$. This is because, by Lemma 3.18, q' forces $\tilde{N}_n \cap Ord = \overline{N} \cap Ord$, where $\overline{N}$ is either $\tilde{N}^\dagger$ or the last member of a closed iteration of $\tilde{N}^\dagger$ relative to $\mathcal{U}_{\xi_0}^{\hat{\alpha},\alpha}$ and bounded by $sup\{\nu + 2 : \nu < \alpha_{i_0}\}$. Thus, q' forces, for each n, that the support of $\tilde{p}_n$ is bounded by α^*, and therefore that $\tilde{p}_n$ is in $\dot{G}_\alpha$. Hence, q' is an $(\tilde{N}', \mathcal{P}_\alpha/\dot{G}_{\xi_0})$–generic condition forcing $\tilde{p} \in \dot{G}_\alpha$.

Now we move back to $V^{\mathcal{P}_{\xi_0} \restriction q_0}$. Let $\dot{q}'$ be a name for q' in the above paragraph. It remains to see that there is a condition $q^+ \in \mathcal{P}_\alpha$ such that, for all $\beta < \alpha$, $q^+ \restriction \beta$ forces $q^+(\beta) = \dot{q}'(\beta)$. This can be checked by arguing in the same fashion as we did for the $cf(\alpha) = \omega$–case (using the fact that q' forces $supp(q') \setminus \overline{\alpha} \subseteq \tilde{N}'$). Now, q^+ and a name for $\tilde{N}'$ (as in the above paragraph) satisfy the desired conclusion for α . $\square$

One immediate corollary of the proof of Lemma 4.1 is that $\mathcal{P}_\kappa/\dot{G}_{\xi_0}$ is semiproper, in $V^{\mathcal{P}_{\xi_0}}$, for every $\xi_0 < \kappa$, which was the part of (3) in Theorem 2.1 that remained to be proved.

The proof of Theorem 2.1 will be complete once we show, in $V^{\mathcal{P}_\kappa}$, that for every $i_0 < \Lambda$ and every stationary $S^* \subseteq S_{i_0}$ there is a function $F : S^* \longrightarrow \mathcal{P}(\omega_1)$ with $\alpha_{i_0} \leq ot(F(\nu)) < \omega^{\alpha_{i_0} \cdot \omega}$ for all $\nu \in S^*$ if $\alpha_{i_0} > 1$ and $F(\nu)$ a singleton for all $\nu \in S^*$ if $\alpha_{i_0} = 1$, and such that $\{\nu \in S^* : g(\nu) \in F(\nu)\}$ is stationary for every ordinal $\epsilon < \omega_2$ and every canonical function g for ϵ. For this it suffices to prove[46] in $V^{\mathcal{P}_{\zeta_0}}$, for any given $\zeta_0 \in A$, that the following holds:

Let $\epsilon_0 < \kappa$, $i_0 < \Lambda$ be given, let N_0 be a countable elementary substructure of $H(\theta)$ (for some large enough regular cardinal θ) containing $\mathcal{P}_\kappa$, $\dot{G}_{\zeta_0}$, ϵ_0, α_{i_0} and $\mathcal{U}_{\zeta_0}^{\zeta_0,\kappa}$ and with $N_0 \cap \omega_1 \in \varphi(\zeta_0)$ and let $p \in N_0 \cap (\mathcal{P}_\kappa/\dot{G}_{\zeta_0})$. Suppose $\varphi(\zeta_0)$ is a stationary subset of S_{i_0}.[47] Then there is a condition $q \in \mathcal{P}_\kappa$ extending p and with $N_0 \cap \omega_1 \in dom(q(\zeta_0))$ and there is a $\mathcal{P}_\kappa/\dot{G}_{\zeta_0}$–name $\tilde{N}$ for a countable elementary substructure of $H(\theta)^{V[\dot{G}_{\zeta_0}]}$ such that

(i) $q \Vdash_\kappa N_0 \subseteq \tilde{N} \wedge N_0 \cap \omega_1 = \tilde{N} \cap \omega_1$,

(ii) q is an $(\tilde{N}, \mathcal{P}_\kappa/\dot{G}_{\zeta_0})$–generic condition, and

(iii) $q \Vdash_\kappa ot(\tilde{N} \cap \epsilon_0) \in q(\zeta_0)(N_0 \cap \omega_1)$.

Let us work in $V^{\mathcal{P}_{\zeta_0}}$. Let $\overline{\alpha} := \alpha_{i_0}$. By Lemma 3.6, the set

$$\mathcal{K} := \{\gamma \in \bigcup_{\zeta \in A} A_\zeta \,:\, \gamma > \zeta_0 \wedge (\exists \nu < \alpha)(\epsilon_0 < j_\nu^{H(\theta), \tilde{U}_{\zeta_0}^{\gamma}}(\epsilon_0))\}$$

is finite.

It will be useful to fix some more notation:

Let $\mathcal{K}_{\epsilon_0} = \{\gamma_0, \ldots \gamma_{n-1}\}_<$ for some $n < \omega$. Note that γ_{n-1}, if $n > 0$, is at most ϵ_0 and that $\tilde{U}_{\zeta_0}^{\gamma_j}$ is in N_0 for all $j < n$. We define a sequence $\langle Y_k \,:\, k < n\rangle$ by setting $Y_0 = \{(N_0)_{\zeta_0,\nu}^{\gamma_0} \,:\, \nu < \overline{\alpha}\}$ (if $n > 0$) and by specifying that, given any nonzero $k < n$, if Y_{k-1} has been defined, then $Y_k = \{(N)_{\zeta_0,\nu}^{\gamma_k} \,:\, N \in Y_{k-1},\, \nu < \overline{\alpha}\}$. Of course, if $\overline{\alpha} = 1$, then $Y_k = \{N_0\}$ for every $k < n$. In general we have the following.

Lemma 4.2 *For every $k < n$, $\{ot(N \cap \epsilon_0) \,:\, N \in Y_k\}$ has order type at most equal to the $k+1$–fold $\otimes$–product of $\overline{\alpha}$ with itself, and hence at most*

[46]Since φ is a suitable bookkeeping function and since every $\mathcal{P}_\kappa/\dot{G}_{\zeta_0}$ preserves stationary subsets of ω_1 in $V^{\mathcal{P}_{\zeta_0}}$.

[47]So that $\dot{\mathcal{Q}}_{\zeta_0}$ is the forcing for adding, with countable conditions, a function $F : \varphi(\zeta_0) \longrightarrow \mathcal{P}(\omega_1)$ with $ot(F(\nu)) < \omega^{\alpha_{i_0} \cdot \omega}$ for all ν if $\alpha_{i_0} > 1$ and $|F(\nu)| = 1$ for all ν if $\alpha_{i_0} = 1$.

$\omega^{\overline{\alpha}\cdot(k+1)}$.[48]

Proof: To start with, notice that Fact 1.2, together with $\overline{\alpha} \leq \omega^{\overline{\alpha}}$, implies that the $k+1$–fold $\otimes$–product of $\overline{\alpha}$ with itself – let us call it α^* – is at most $\omega^{\overline{\alpha}\cdot(k+1)}$.

We are going to prove $ot(\{ot(N \cap \epsilon_0) : N \in Y_k\}) \leq \alpha^*$. Let M be the transitive collapse of N_0. Let $\overline{\epsilon_0} = \pi(\epsilon_0)$ and let $\overline{U_l} = \pi(\tilde{U}^{\gamma_l}_{\zeta_0})$ for every $l \leq k$, where π is the collapsing function for N_0. Let $j^l_{e^l}$ denote $j^{M,\overline{U_l}}_{0,e^l}$ for every $l \leq k$.

Notice, in the first place, that by Fact 3.1 and by Lemma 3.5, the order type of $\{ot(N \cap \epsilon_0) : N \in Y_k\}$ is equal to the order type of the set of all ordinals of the form

$$(j^k_{\nu_k} \circ j^{k-1}_{\nu_{k-1}} \circ \ldots \circ j^0_{\nu_0})(\overline{\epsilon}_0),$$

where $\nu_l < \overline{\alpha}$ for all $l \leq k$.

Secondly, notice that, for every $t \in^{k+1} \overline{\alpha}$ and every permutation σ of $k+1$, both

$$(j^k_{t(k)} \circ j^{k-1}_{t(k-1)} \circ \ldots \circ j^0_{t(0)})(\overline{\epsilon}_0) = (j^{\sigma(k)}_{t(\sigma(k))} \circ j^{\sigma(k-1)}_{t(\sigma(k-1))} \circ \ldots \circ j^{\sigma(0)}_{t(\sigma(0))})(\overline{\epsilon}_0)$$

and

$$(j^{\sigma(k)}_{t(\sigma(k))} \circ j^{\sigma(k-1)}_{t(\sigma(k-1))} \circ \ldots \circ j^{\sigma(0)}_{t(\sigma(0))})(\overline{\epsilon}_0) < (j^{\sigma(k)}_{t(\sigma(k))+1} \circ j^{\sigma(k-1)}_{t(\sigma(k-1))} \circ \ldots \circ j^{\sigma(0)}_{t(\sigma(0))})(\overline{\epsilon}_0)$$

hold. This follows easily from Lemma 3.5 and from the fact that $\overline{\epsilon}_0$ is moved by $j^{M,\overline{U_{\sigma(k)}}}_{0,1}$. Hence, if $\langle t_i : i < \tau\rangle$ is an enumeration of ${}^{k+1}\overline{\alpha}$ such that $i < i' < \tau$ implies

$$(j^k_{t_i(k)} \circ j^{k-1}_{t_i(k-1)} \circ \ldots \circ j^0_{t_i(0)})(\overline{\epsilon}_0) \leq (j^k_{t_{i'}(k)} \circ j^{k-1}_{t_{i'}(k-1)} \circ \ldots \circ j^0_{t_{i'}(0)})(\overline{\epsilon}_0),$$

then τ is at most α^*. This is because

$$(j^k_{t(k)} \circ j^{k-1}_{t(k-1)} \circ \ldots \circ j^0_{t(0)})(\overline{\epsilon}_0) < (j^k_{t'(k)} \circ j^{k-1}_{t'(k-1)} \circ \ldots \circ j^0_{t'(0)})(\overline{\epsilon}_0)$$

holds whenever t, t' are distinct tuples in ${}^{k+1}\overline{\alpha}$ such that $t \leq t'$ (in the product order on ${}^{k+1}\overline{\alpha}$), which follows from the above observation.

[48]It is conceivable that a better bound for $\{ot(N \cap \epsilon_0) : N \in Y_{n-1}\}$ can be found, and thus that a finer form of Theorem 2.1 is true, in the sense that the distance between β^0_i and β^1_i, in (4) of the statement of Theorem 2.1, can be made smaller (and similarly with Theorem 5.4 in Section 5). Nevertheless, the bound proved here suffices for the present purposes.

These two facts together imply the desired conclusion. $\square$

Let now $(f_i)_{i<\omega}$ be an $(N_0, \dot{\mathcal{Q}}_{\zeta_0})$–generic sequence extending $p(\zeta_0)$ and let $\overline{f}$ be $\bigcup_i f_i \cup \{\langle N_0 \cap \omega_1, \{ot(N \cap \epsilon_0) \,:\, N \in Y\}\rangle\}$ for $Y = Y_{n-1}$ if $n > 0$ and for $Y = \{N_0\}$ if $n = 0$.

Since, by Lemma 4.2, $\{ot(N \cap \epsilon_0) \,:\, N \in Y\}$ has order type less than $\omega^{\overline{\alpha}\cdot\omega}$ if $n > 0$, we have that $\overline{f}$ is a condition in $\dot{\mathcal{Q}}_{\zeta_0}$ extending $p(\zeta_0)$. Thus we will be done if we can show that there is a $\mathcal{P}_\kappa$–name $\tilde{N}$ for a countable elementary substructure of $H(\theta)^{V[\dot{G}_{\zeta_0}]}$ extending N_0 and with the same intersection with ω_1 as N_0 and there is a $(\tilde{N}, \mathcal{P}_\kappa/\dot{G}_{\zeta_0})$–generic condition extending p^* and forcing $ot(\tilde{N} \cap \epsilon_0) \in \{ot(N \cap \epsilon_0) \,:\, N \in Y\}$ for Y as above and for the $\mathcal{P}_\kappa$–condition p^* such that $p^* \restriction \zeta_0 = p \restriction \zeta_0$ and $p^* \restriction [\zeta_0 + 1, \kappa) = p \restriction [\zeta_0 + 1, \kappa)$ and such that $p^*(\zeta_0)$ is (a $\mathcal{P}_{\zeta_0}$–name for) $\overline{f}$.

We apply Lemma 4.1 with $\xi_0 = \zeta_0$, $\alpha = \kappa$, $\overline{\alpha} = \zeta_0 + 1$, $\dot{N} = \check{N}_0$, $q = p^* \restriction \zeta_0 + 1$, and with $\tilde{p} = \check{p}$. By this instance of the lemma we know that there is a $\mathcal{P}_\kappa/\dot{G}_{\xi_0}$–name $\tilde{N}$ for the last model of an iteration of N_0 relative to $\mathcal{U}^{\zeta_0,\kappa}_{\dot{\zeta}_0}$ with closed component iterations and bounded by $sup\{\nu + 2 \,:\, \nu < \alpha\}$ and that there is a condition $q^+ \in \mathcal{P}_\kappa/\dot{G}_{\zeta_0}$ extending p^* such that $q^+ \restriction \zeta_0 + 1 = p^* \restriction \zeta_0 + 1$ and such that q^+ is $(\tilde{N}, \mathcal{P}_\kappa)$–generic. Thus, we just need to show that q^+ forces $ot(\tilde{N} \cap \epsilon_0) \in \{ot(N \cap \epsilon_0) \,:\, N \in Y\}$. But, by Lemma 3.16, $\{ot(N \cap \epsilon_0) \,:\, N \in Y\}$ is precisely the set of all ordinals of the form $ot(N \cap \epsilon_0)$, where N is, in $V^{\mathcal{P}_\kappa}$, the last model of a closed iteration of N_0 bounded by $sup\{\nu + 2 \,:\, \nu < \alpha\}$. This finishes the proof of Theorem 2.1.

5 A model of PFA^{++} with a definable (without parameters) well–order of $H(\omega_2)$.

This final section presents an application of Theorem 2.1 to the problem of forcing a well–order of $H(\omega_2)$ definable, over the structure $\langle H(\omega_2), \in\rangle$, by a formula without parameters.[49] Each of the papers [A1], [A2] and [A3] present forcing iterations for constructing models in which there are well–orders of $H(\omega_2)$ as above. All of those constructions are flexible enough to allocate arbitrary posets with the countable chain condition. As a consequence, all models constructed can be taken to be models of MA_{ω_1}.

On the other hand, we run into difficulties if we want to modify any

[49] The reader is referred to [A1] for background on this problem.

of the constructions in [A1], [A2] or [A3] so as to produce a model of a stronger forcing axiom, for example of $BPFA$.[50] The reason is that all of those constructions involve some form or another of guessing of clubs by club–sequences. By a club–sequence here I mean a sequence of the form $\overline{\alpha} = \langle \alpha_\delta : \delta \in S\rangle$, for S a subset of countable limit ordinals, such that each α_δ is a club of δ. All of the models built in the above papers exhibit some nontrivial pattern of club–guessing properties (of some kind or another) for club–sequences. In fact, for the relevant codings to work, it is essential, in all those models, that there be club–sequences $\overline{\alpha} = \langle \alpha_\delta : \delta \in S\rangle$ with stationary domain and with the property that for every club $C \subseteq \omega_1$ there are stationarily many $\delta \in S$ such that $sup(\alpha_\delta \cap C) = \delta$. However, it is not difficult to see that if $\overline{\alpha}$ is as above, then the standard poset for adding a club avoiding $\overline{\alpha}$[51] is proper. Hence, none of the codings considered in any of those papers can work in the presence of $BPFA$.

Recall the following strengthening of the Proper Forcing Axiom.

Definition 5.1 *PFA^{++} is the statement that for every proper poset $\mathcal{P}$, every sequence $\langle D_i : i < \omega_1\rangle$ of dense subsets of $\mathcal{P}$ and every sequence $\langle \tau_i : i < \omega_1\rangle$ of $\mathcal{P}$–names for stationary subsets of ω_1 there is a filter $G \subseteq \mathcal{P}$ such that, for every $i < \omega_1$, $G \cap D_i \neq \emptyset$ and $\{\nu < \omega_1 : (\exists p \in G)(p \Vdash_{\mathcal{P}} \nu \in \tau_i)\}$ is a stationary subset of ω_1.*

The Levy hierarchy $\bigcup_{n<\omega}(\Sigma_n \cup \Pi_n)$ of formulas in the language for the structure $\langle H(\omega_2), \in, NS_{\omega_1}\rangle$ can be defined, in the natural way, as for the language of set theory. The main results in this section are the following.

Theorem 5.1 *Suppose κ is a supercompact cardinal and A is a subset of ω_1. Then there is a semiproper poset $\mathcal{P} \subseteq V_\kappa$ such that*

(1) $\mathcal{P}$ forces PFA^{++}, and

(2) $\mathcal{P}$ forces that A is definable, over the structure $\langle H(\omega_2), \in, NS_{\omega_1}\rangle$, by a Σ_4 formula without parameters. In particular, $\mathcal{P}$ forces that A is definable, over the structure $\langle H(\omega_2), \in\rangle$, by a formula without parameters.

[50] The Bounded Proper Forcing Axiom, $BPFA$, is the weak form of the Proper Forcing Axiom asserting that, for every proper poset $\mathcal{P}$, $\langle H(\omega_2), \in\rangle$ is a Σ_1–elementary substructure of $\langle H(\omega_2), \in\rangle^{V^{\mathcal{P}}}$.

[51] That is, for adding a club $C \subseteq \omega_1$ such that $\alpha_\delta \cap C$ is bounded in δ for every $\delta \in dom(\overline{\alpha}) \cap C$.

Theorem 5.2 *Suppose κ is a supercompact cardinal. Then there is a semi-proper poset $\mathcal{P} \subseteq V_\kappa$ such that*

(1) $\mathcal{P}$ forces PFA^{++}, and

(2) $\mathcal{P}$ forces the existence of a well–order of $H(\omega_2)$ definable, over the structure $\langle H(\omega_2), \in, NS_{\omega_1}\rangle$, by a Σ_4 formula without parameters. In particular, $\mathcal{P}$ forces the existence of a well–order of $H(\omega_2)$ definable, over $\langle H(\omega_2), \in\rangle$, by a formula without parameters.

Note that the last sentence in (2) of the statement of each of these theorems follows from the fact that NS_{ω_1} is a definable predicate over $\langle H(\omega_2), \in\rangle$.

It is worth recalling at this point that forcing axioms, even in their bounded forms, are known (see for example [M] and [C-V]) to imply the existence of well–orders of $H(\omega_2)$ definable over $\langle H(\omega_2), \in\rangle$, but that these definitions always depend on some parameter. In fact it is not known whether any strong enough forcing axiom implies the existence of a well–order of $H(\omega_2)$ definable over $\langle H(\omega_2), \in\rangle$ by a parameter–free formula (see the questions at the end of the paper).

Lemma 5.3 *Let $\langle(\alpha^0_\xi, \alpha^1_\xi) : \xi < \omega_1\rangle$ be a sequence of pairs of ordinals with $2 \leq \alpha^0_\xi < \alpha^1_\xi < \alpha^0_{\xi'} < \omega_1$ for all $\xi < \xi' < \omega_1$. Suppose $\langle S_i : i < \Lambda\rangle$ is a partition of ω_1 into stationary sets such that $S_i \cap (i+1) = \emptyset$ for every $i > 0$, and suppose $(\xi_i)_{i<\Lambda}$ is such that each S_i has guessing density equal to a pair (β_0, β_1) with $\alpha^0_{\xi_i} \leq \beta_0$ and $\beta_1 \leq \alpha^1_{\xi_i}$.*

Then $\{\xi_i\}_{i<\Lambda}$ is equal to the set of indices ξ such that there is a stationary subset of ω_1 with guessing density equal to a pair (β_0, β_1) such that $\alpha^0_\xi \leq \beta_0$ and $\beta_1 \leq \alpha^1_\xi$.

Proof: Let $S \subseteq \omega_1$ be stationary and suppose $\xi < \omega_1$ is such that S has guessing density equal to a pair (β_0, β_1) with $\alpha^0_\xi \leq \beta_0$ and $\beta_1 \leq \alpha^1_\xi$. The function sending every ordinal $\nu < \omega_1$ to the unique $i < \Lambda$ such that $\nu \in S_i$ is regressive. Thus, there is some i with $S \cap S_i$ stationary. Suppose $\xi > \xi_i$. Then, $\gamma(S \cap S_i) \geq \alpha^0_\xi > \alpha^1_{\xi_i}$,[52] which is impossible since $\alpha^1_{\xi_i} \geq sup\{\gamma(S^*) : S^* \subseteq S_i,\ S^* \text{ stationary}\}$. Now suppose $\xi_i > \xi$. Then, $\gamma(S \cap S_i) \leq \alpha^1_\xi < \alpha^0_{\xi_i}$, which again is a contradiction since $\alpha^0_{\xi_i} \leq \gamma(S_i) \leq \gamma(S \cap S_i)$. Hence, $\xi = \xi_i$. □

Theorem 5.1 is a consequence of the following result.

[52] Where $\gamma(\cdot)$ is as in Definition 1.1.

Theorem 5.4 *Suppose κ is a supercompact cardinal, A is a nonempty subset of ω_1 and $\langle S_i : i < ot(A)\rangle$ is a partition of ω_1 into stationary sets such that $S_i \cap (i+1) = \emptyset$ for every $i > 0$. Let $(\eta_\xi)_{\xi<\omega_1}$ be the strictly increasing enumeration of the club C_0 of nonzero $\eta < \omega_1$ such that $\omega^{\epsilon\cdot\omega} < \eta$ for all $\epsilon < \eta$. If $\langle \xi_i : i < ot(A)\rangle$ is an enumeration of A, then there is a semiproper poset $\mathcal{P} \subseteq V_\kappa$ such that*

(1) $\mathcal{P}$ forces PFA^{++}, and

(2) $\mathcal{P}$ forces, for every $i < ot(A)$, that S_i has guessing density equal to a pair (β_0, β_1) with $\eta_{\xi_i} + 1 \leq \beta_0$ and $\beta_1 \leq \omega^{\eta_{\xi_i}\cdot\omega}$. Hence, $\mathcal{P}$ forces that A is the set of ordinals ξ such that there is a stationary subset of ω_1 with guessing density equal to some pair (β_0, β_1) such that $\eta_\xi + 1 \leq \beta_0$ and $\beta_1 \leq \omega^{\eta_\xi\cdot\omega}$.[53]

Consider the following property $P(x)$:
"x is a countable ordinal and there is a stationary subset of ω_1 with guessing density equal to an ordered pair (β_0, β_1) with $\eta_x + 1 \leq \beta_0$ and $\beta_1 \leq \omega^{\eta_x\cdot\omega}$". This property can be expressed by writing

$$x \in \omega_1 \wedge (\exists S \subseteq \omega_1)(S \notin NS_{\omega_1} \wedge (\forall S^* \subseteq S)(S^* \notin NS_{\omega_1} \rightarrow Q(S^*, x)))$$

where $Q(S^*, x)$ is the conjunction of $Q_0(S^*, x)$ and $Q_1(S^*, x)$, with $Q_0(S^*, x)$ and $Q_1(S^*, x)$ being, respectively, the properties "there is a function $F : S^* \longrightarrow \mathcal{P}(\omega_1)$ with $ot(F(\nu)) \in [\eta_x, \omega^{\eta_x\cdot\omega})$ for all $\nu \in S^*$ and such that F guesses all canonical functions" and "for every function $F : S^* \longrightarrow \mathcal{P}(\omega_1)$ such that $ot(F(\nu)) < \eta_x$ for every $\nu \in S^*$ there is some ordinal $\alpha < \omega_2$ and some surjection $\pi : \omega_1 \longrightarrow \alpha$ such that $\{\nu \in S^* : ot(\pi``\nu) \in F(\nu)\} \in NS_{\omega_1}$". "$F$ guesses all canonical functions" can be expressed by saying that for every ordinal $\alpha < \omega_2$ and every surjection $\pi : \omega_1 \longrightarrow \alpha$, $\{\nu \in S^* : ot(\pi``\nu) \in F(\nu)\} \notin NS_{\omega_1}$. Thus, since the club C_0 is Δ_1 definable, $Q_0(S^*, x)$ and $Q_1(S^*, x)$ can be expressed over $\langle H(\omega_2), \in, NS_{\omega_1}\rangle$ by formulas with S^* and x as parameters of complexity, respectively, Σ_2 and Π_2. It follows that $Q(S^*, x)$ can be expressed over $\langle H(\omega_2), \in, NS_{\omega_1}\rangle$ as a Π_3 formula with S^* and x as parameters and, finally, that $P(x)$ can be expressed as a Σ_4 formula without parameters.

[53] This is by Lemma 5.3 with $\langle(\alpha^0_\xi, \alpha^1_\xi) : \xi < \omega_1\rangle$ being $\langle(\eta_\xi + 1, \omega^{\eta_\xi\cdot\omega}) : \xi < \omega_1\rangle$.

Theorem 5.1 follows from Theorem 5.4, since the property $P(x)$ from the above paragraph is expressible over $\langle H(\omega_2), \in, NS_{\omega_1}\rangle$ by a Σ_4 formula without parameters.

The proof of Theorem 5.2 uses Theorem 5.4, together with the following formulation of a result of Moore.

Theorem 5.5 *([M]) There is a Σ_2 formula $\Theta(x, y, z)$ in the language of set theory such that $BPFA$ implies that*

$$\{(x, y) \in H(\omega_2) \times H(\omega_2) \, : \, \langle H(\omega_2), \in\rangle \models \Theta(x, y, p)\}$$

is a well–order of $H(\omega_2)$ of order type ω_2 whenever $\overline{\epsilon}$ is a ladder system[54] *with stationary domain, $(U_i)_{i<\omega_1}$ is a sequence of pairwise disjoint stationary subsets of ω_1 and $p = \langle\overline{\epsilon}, (U_i)_{i<\omega_1}\rangle$.*

Let us fix a formula $\Theta(x, y, z)$ as given by Theorem 5.5. In order to prove Theorem 5.2, we apply Theorem 5.4 to a subset A of ω_1 coding, in some canonical Δ_1 way, a parameter of the form $p = \langle\overline{\epsilon}, (U_i)_i\rangle$, with $\overline{\epsilon}$ and $(U_i)_i$ as in Theorem 5.5. The definition of the well–order is then given by a formula $\Phi(x, y)$ expressing the following property:

There is a sequence $\langle S_i \, : \, i < \omega_1\rangle$ of stationary subsets of ω_1 such that $S_i \cap (i+1) = \emptyset$ for all $i < \omega_1$ and there is a sequence $\langle \xi_i \, : \, i < \omega_1\rangle$ of nonzero countable ordinals such that each S_i has guessing density a pair (β_0, β_1) with $\eta_{\xi_i} + 1 \leq \beta_0$ and $\beta_1 \leq \omega^{\eta_{\xi_i} \cdot \omega}$ (where $(\eta_\xi)_{\xi<\omega_1}$ is the strictly increasing enumeration of the club C_0 from Theorem 5.4), such that $\{\xi_i \, : \, i < \omega_1\}$ codes (in some fixed Δ_1 way) a parameter p as in Theorem 5.5, and $\Theta(x, y, p)$.

Thus, it remains to prove Theorem 5.4. Recall the result of Laver ([Lav]) saying that for every supercompact cardinal κ there is a function $F : \kappa \longrightarrow V_\kappa$ with the property that for every set x and every ordinal λ there is a transitive class M closed under sequences of length λ and there is an elementary embedding $j : V \longrightarrow M$ with critical point κ and $j(\kappa) > \lambda$ and such that $j(F)(\kappa) = x$. We fix such a function F. We also fix a set $\mathcal{M}$ of measurable cardinals with $sup(\mathcal{M}) = \kappa$, together with a club $C \subseteq \kappa$, such that $\kappa \setminus C$ is unbounded in κ, $\mathcal{M} \subseteq \kappa \setminus C$, and such that $F \restriction (C \cap \gamma) \in V_\gamma$ for every $\gamma \in \mathcal{M}$. The desired conclusion follows now from Theorem 2.1 with F, $\mathcal{M}$, C

[54] That is, $\overline{\epsilon}$ is a club–sequence with $ot(\overline{\epsilon}(\delta)) = \omega$ for all δ in its domain.

and $\Lambda = ot(A)$, with $\langle \alpha_i : i < \Lambda\rangle = \langle \xi_i : i < ot(A)\rangle$ (where $\langle \xi_i : i < ot(A)\rangle$ is as in the hypothesis of Theorem 5.4), and with $\langle S_i : i < ot(A)\rangle$ again as in the hypothesis of Theorem 5.4.

Let $\mathcal{P}$ be the poset $\mathcal{P}_\kappa$, where $\langle \mathcal{P}_\xi : \xi \leq \kappa\rangle$ is the forcing iteration coming from this instance of Theorem 2.1. By the classical Baumgartner's construction of a model of PFA^{++}, this forcing axiom holds in $V^{\mathcal{P}}$. This is because, by Theorem 2.1 (3), $\mathcal{P}_\kappa/\dot{G}_\xi$ is forced, for each $\xi < \kappa$, to be semiproper, and so in particular to preserve stationary subsets of ω_1. On the other hand, $\mathcal{P}$ forces, for each i, that S_i has guessing density a pair (β_0, β_1) with $\eta_{\xi_i} + 1 \leq \beta_0$ and $\beta_1 \leq \omega^{\eta_{\xi_i}\cdot\omega}$. This finishes the proof of Theorem 5.4.

Note that, although the proof of Theorem 2.1 is robust enough to accommodate arbitrary proper posets in it, it breaks down completely if we intend to make it work with arbitrary semiproper posets.[55] In fact, there is no hope that a version of the construction for Theorem 2.1 can produce a model of BMM, as BMM implies $\omega_1 \leq \beta_0$ whenever (β_0, β_1) is the guessing density of a stationary subset of ω_1 (Fact 1.1).

In a previous version of this paper I was asking whether the assumption that there exists is a supercompact cardinal (or some other reasonable large cardinal assumption) implies that it is possible to force in such a way that Martin's Maximum (MM) holds in the extension, together with the existence of a well–order of $H(\omega_2)$ definable, over $\langle H(\omega_2), \in\rangle$, by a parameter–free formula (or by a formula with only a real number as parameter). Concerning this question, P. Larson ([L]) has recently forced, over a model with a supercompact limit of supercompact cardinals, in such a way that, in the extension, $MM^{+\omega}$ holds and there is a well–order of $H(\omega_2)$ definable over $\langle H(\omega_2), \in\rangle$ by a formula without parameters. $MM^{+\omega}$ is the strengthening of MM saying that for every poset $\mathcal{Q}$ preserving stationary subsets of ω_1, every set $\{D_i : i < \omega_1\}$ of dense subsets of $\mathcal{Q}$ and every set $\{\tau_n : n < \omega\}$ of $\mathcal{Q}$–names for stationary subsets of ω_1 there is a filter $G \subseteq \mathcal{Q}$ such that $G \cap D_i \neq \emptyset$ for every $i < \omega_1$ and such that $\{\alpha < \omega_1 : (\exists p \in G)(p \Vdash_{\mathcal{Q}} \alpha \in \tau_n)\}$ is stationary for every $n < \omega$.[56] And MM^{++} is the strengthening of $MM^{+\omega}$ incorporating sets of size $\aleph_1$ – instead of just countable size – of names for stationary subsets of ω_1.

In Larson's model, MM^{++} fails necessarily. As far as I know, the follow-

[55] As then we would lose the necessary control, in the relevant ground models, on the order types of the structures for which we can find $(\tilde{N}, \mathcal{Q})$–generic conditions (for the relevant $\mathcal{Q}$'s).

[56] By a result in [S], $MM^{+\omega}$ does not imply PFA^{++}.

ing questions remain open.

Questions 5.1 *Assume some reasonable large cardinal hypothesis. Is it possible to force in such a way that MM^{++} holds in the extension, together with the existence of a well–order of $H(\omega_2)$ definable, over $\langle H(\omega_2), \in\rangle$, by a formula without parameters (or even by a formula with only a real number as parameter)?*

Does MM^{++} imply that there is a well–order of $H(\omega_2)$ definable, over $\langle H(\omega_2), \in\rangle$, by a formula with at most a real number as parameter?

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Institució Catalana de Recerca i Estudis Avançats (ICREA)
and
Dept. de Lògica, Història i Filosofia de la Ciència,
Universitat de Barcelona,
Montalegre 6, 08001 Barcelona

http://www.icrea.es/pag.asp?id=David.Aspero
david.aspero@icrea.es