Forcing notions in inner models

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Abstract

There is a partial order $\mathbb{P}$ preserving stationary subsets of ω_1 and forcing that every partial order in the ground model V that collapses a sufficiently large ordinal to ω_1 over V also collapses ω_1 over $V^{\mathbb{P}}$. The proof of this uses a coding of reals into ordinals by proper forcing discovered by Justin Moore and a symmetric extension of the universe in which the Axiom of Choice fails. Also, using one feature of the proof of the above result together with an argument involving the stationary tower it is shown that sometimes, after adding one Cohen real c, there are, for every real a in $V[c]$, sets A and B such that c is Cohen generic over both $L[A]$ and $L[B]$ but a is constructible from A together with B.

1 On the nonexistence of nice forcing notions, in a ground model, collapsing a given ordinal

This first section is devoted to showing that one can always extend the universe in a nice way (preserving stationary subsets of ω_1) so that in the extension there are no 'nice' partial orders $\mathcal{Q}$ in V collapsing, when forcing with them over V, some given ordinal α to be of size $\aleph_1$. Here 'nice' means that $\mathcal{Q}$ preserves ω_1 when forcing with it over the extension. In particular, the collapse of α to ω_1 with countable conditions from the point of view of V will necessarily collapse ω_1 over the extension. It is also shown that some restriction on the niceness of the extension in the result mentioned above is necessary. Specifically, one cannot expect to produced the desired conclusion after forcing with a reasonable partial order in the sense of [Fo-M].

The proof of the main theorem proceeds by contradiction in the following way: Some forcing construction is carried out for which there is some intermediate model N of ZF in which there is no well–order of the reals. Assuming the result fails (that is, that there is a 'nice' partial order in V collapsing, over V, the given ordinal α to ω_1), a well–order of the reals can be nonetheless defined in N.

Theorem 1.1 *There are a partial order $\mathbb{P}$ and an ordinal α such that*

(1) Forcing with $\mathbb{P}$ preserves all stationary subsets of ω_1.

(2) $V^{\mathbb{P}} \models \alpha < \omega_2$

(3) In $V^{\mathbb{P}}$ it holds that for every partial order $\mathcal{Q}$ in V, if forcing with $\mathcal{Q}$ over V collapses α to be of size $\aleph_1$, then forcing with $\mathcal{Q}$ over $V^{\mathbb{P}}$ collapses ω_1.

Proof: Start by adding one Cohen real c. It is a well–known fact due to Cohen that after forcing with $2^{<\omega}$ there is a symmetric extension N of V in which there is an infinite set X of reals which is also Dedekind–finite, i.e. such that there is no one–to–one map from ω into X (see for example [J1], Section 5.3 or [J2], Theorem 14.36). Thus we may fix a transitive model N of ZF, $V \subseteq N \subseteq V[c]$, so that in N there is no well–order of the reals. I will show how to build a stationary–set–preserving forcing iteration $\mathbb{P}$ of $V[c]$ and how to find an ordinal α which is forced by $\mathbb{P}$ to be less than ω_2 and such that, under the assumption that $\mathcal{Q} \in N$ and G are such that

(a) G is $\mathbb{P}$–generic over $V[c]$,

(b) $N \models \mathcal{Q}$ forces $|\alpha| = \aleph_1$, and

(c) $V[c][G] \models \mathcal{Q}$ preserves ω_1,

there is, in N, a definable well-order of the reals. This contradiction will prove the theorem.

I need to fix the following objects in V:

(i) A ladder system on ω_1, that is, a sequence

$$\vec{C} = (C_\xi \,:\, \xi < \omega_1,\, \xi \text{ a limit ordinal})$$

such that each C_ξ is a cofinal subset of ξ of order type ω.

(ii) A sequence $(S_n : n < \omega)$ of pairwise disjoint stationary subsets of ω_1.

Let $\langle r_i : i < \lambda \rangle$ be an enumeration in $V[c]$ of the infinite reals $r \subseteq \omega$ in N for $\lambda = (2^{\aleph_0})^V$. $\mathbb{P}$ will be a three–stage iteration. The first stage $\mathcal{C}_0$ of the iteration will be a proper poset coding each r_i into an ordinal α_i in a certain way to be made precise below. The second stage of the iteration codes all relevant objects added by $\mathcal{C}_0$ into a real x by ccc almost disjoint forcing. Finally we shoot a club through a projective stationary subset of $[\kappa]^{\aleph_0}$, for a suitable cardinal κ, consisting of sets coding x in yet another way. α will be this κ.

$\mathcal{C}_0$ will be a countable support iteration $(\mathbb{P}_i : i \leq \lambda + 1)$ based on $(\dot{\mathbb{Q}}_i : i < \lambda + 1)$. Suppose $i < \lambda$ and suppose $\mathbb{P}_i$ has been defined.

In $V[c]^{\mathbb{P}_i}$, $\dot{\mathbb{Q}}_i$ is a name for the forcing $\mathbb{Q}_r$ given by the following lemma, recently proven by Justin Moore ([Mo]), where $r = r_i$.

Lemma 1.2 *(Moore) Suppose $(C_\xi : \xi < \omega_1, \xi$ a limit ordinal$)$ is a ladder system on ω_1, $(S_n : n < \omega)$ is a sequence of pairwise disjoint stationary subsets of ω_1 and $r \subseteq \omega$. There is a partial order $\mathbb{Q}_r$ such that*

(1) $\mathbb{Q}_r$ is proper,

(2) $\mathbb{Q}_r$ forces the existence of an ordinal α_r and a strictly $\subseteq$–increasing and $\subseteq$–continuous sequence $\langle X^r_\nu : \nu < \omega_1 \rangle$ of countable subsets of α_r such that

(a) $\bigcup_\nu X^r_\nu = \alpha_r$, and

(b) for every limit ordinal $\nu < \omega_1$ there is some $\nu_0 < \nu$ such that for all ξ, $\nu_0 < \xi < \nu$,

$$X^r_\xi \cap \omega_1 \in \bigcup_{n \in r} S_n \text{ iff } w(X^r_\xi \cap \omega_1, X^r_\nu \cap \omega_1) < w(X^r_\xi, X^r_\nu)$$

where, given two countable sets $X \subseteq Y$ of ordinals with $ot(Y)$ a limit ordinal, $w(X, Y)$ is the cardinality of $sup(X) \cap \pi_Y^{-1}\text{“}C_{ot(Y)}$ (and π_Y is the transitive collapse of Y).

Note that the correspondence $r \longrightarrow \alpha_r$ yields a coding of reals in the sense that if r, $r' \subseteq \omega$ are distinct and α, α' are such that there are continuous decompositions of $[\alpha]^{\aleph_0}$ and $[\alpha']^{\aleph_0}$ satisfying (b) from Lemma 1.2 for r and r', respectively, then necessarily $\alpha \neq \alpha'$.

Finally, to make sure that $sup_{i<\lambda}\alpha_{r_i}$ and λ are both forced to be less than ω_2, let $\dot{\mathbb{Q}}_\lambda$ be a name for the σ–closed collapse of λ to ω_1. (Note that $\dot{\mathbb{Q}}_\lambda$ indeed collapses $sup_{i<\lambda}\alpha_{r_i}$ to ω_1 since all α_{r_i} are of size $\aleph_1$.) This completes the definition of $\mathcal{C}_0$. Since all $\dot{\mathbb{Q}}_i$ are names for proper posets, $\mathcal{C}_0$ is proper.

In $V[c]^{\mathcal{C}_0}$, let A be a subset of ω_1 such that

$$(\langle X^{r_i}_\xi \,:\, \xi < \omega_1\rangle \,:\, i < \lambda) \in L[A]$$

The second stage of the iteration, $\mathcal{C}_1$, will be the ccc forcing for adding a real x coding A with respect to some sequence $\langle y_\nu \,:\, \nu < \omega_1\rangle$ in V consisting of pairwise almost disjoint reals (that is, $x \cap y_\nu$ is infinite if and only if $\nu \in A$).

Fix a regular cardinal $\kappa \geq \omega_2$ in V such that $2^{<\omega} * (\mathcal{C}_0 * \mathcal{C}_1)$ has the κ–chain condition. Let $\overrightarrow{D} = \langle D_\xi \,:\, \xi \in \kappa \cap cf(\omega)\rangle$ be a club–guessing sequence in V for $\kappa \cap cf(\omega)$. This means that

(1) D_ξ is a cofinal subset of ξ for each $\xi < \kappa$ of countable cofinality, and

(2) for each club $D \subseteq \kappa$ there is some $\xi < \kappa$ of countable cofinality such that $D_\xi \subseteq D$.

The existence of such a sequence is a well–known result of Shelah. We may clearly assume that each D_ξ has order type ω. Also, note that since $2^{<\omega} * (\mathcal{C}_0 * \mathcal{C}_1)$ has the κ–chain condition, every club of κ in $V[c]^{\mathcal{C}_0 * \mathcal{C}_1}$ includes a club in V, and therefore $\overrightarrow{D}$ is still a club–guessing sequence for $\kappa \cap cf(\omega)$ in $V[c]^{\mathcal{C}_0 * \mathcal{C}_1}$.

Let $(D_\xi(n) \,:\, n < \omega)$ be, for each $\xi < \kappa$ of countable cofinality, the increasing enumeration of D_ξ. Consider the set

$$\mathcal{S}_x = \{X \in [\kappa]^{\aleph_0} \,:\, (\forall n < \omega) X \cap [D_{sup(X)}(n),\, D_{sup(X)}(n+1)) \neq \emptyset \text{ iff } n \in x\}$$

where, given two ordinals $\alpha < \beta$, $[\alpha, \beta)$ denotes the interval of ordinals γ such that $\alpha \leq \gamma < \beta$.

The following strengthening of being a stationary set is defined in [F-J].

Definition 1.1 *Given a set $\mathcal{X}$ such that $\omega_1 \subseteq \mathcal{X}$, $A \subseteq [\mathcal{X}]^{\aleph_0}$ is said to be projective stationary if and only if $\{X \in A \,:\, X \cap \omega_1 \in T\}$ is a stationary subset of $[\mathcal{X}]^{\aleph_0}$ for each stationary subset T of ω_1.*

Given a set $A \subseteq [\mathcal{X}]^{\aleph_0}$ there is a natural forcing notion $\mathbb{P}_A$ for adding an ω_1–club of $[\mathcal{X}]^{\aleph_0}$ consisting of members of A: A condition of $\mathbb{P}_A$ is a $\subseteq$–continuous $\subseteq$–chain of members of A of length some successor countable ordinal, and extension in $\mathbb{P}_A$ is reverse inclusion. The relevance of the notion of projective stationary set in the context of shooting clubs is shown by the following easily verified fact.

Fact 1.3 *([F-J]) Let $\mathcal{X}$ be a set including ω_1 and let $A \subseteq [\mathcal{X}]^{\aleph_0}$.*

(a) If A is a stationary subset of $[\mathcal{X}]^{\aleph_0}$, then $\mathbb{P}_A$ forces the existence of an $\subseteq$–increasing and $\subseteq$–continuous sequence $(X_\nu)_\nu$ of elements of A such that $\mathcal{X} = \bigcup_{\nu<\omega_1} X_\nu$.

(b) A is a projective stationary subset of $[\mathcal{X}]^{\aleph_0}$ if and only if forcing with $\mathbb{P}_A$ preserves stationary subsets of ω_1.

Claim 1.4 *In $V[c]^{\mathcal{C}_0 * \mathcal{C}_1}$, $\mathcal{S}_x$ is a projective stationary subset of $[\kappa]^{\aleph_0}$.*

It should be pointed out that the sets $\mathcal{S}_r$ (for any real r) and the fact that they are projective stationary were previously considered by Todorčević in a different context.

Proof: Work in $V[c]^{\mathcal{C}_0 * \mathcal{C}_1}$. Fix a function $F : [\kappa]^{<\omega} \longrightarrow \kappa$ and a stationary $T \subseteq \omega_1$. We want to find some $X \in [\kappa]^{\aleph_0}$ such that $F``[X]^{<\omega} \subseteq X$, $X \cap \omega_1 \in T$ and $X \in \mathcal{S}_x$. Finding such a set will involve the following games $\mathcal{G}^F_\nu$ ($\nu < \omega_1$).

Definition 1.2 *Given a countable ordinal ν, $\mathcal{G}^F_\nu$ is the following game of length ω with two players I and II. The two players collaborate in building a sequence*

$$\langle \langle I_0, \xi_0\rangle, \eta_0, \langle I_1, \xi_1\rangle, \eta_1, \ldots \rangle$$

such that for all k,

- *I_k is a bounded interval of ordinals in κ and $\xi_k \in I_k$,*
- *$\eta_k < \kappa$,*
- *$\eta_k < min(I_{k+1})$*

Player I chooses the pairs $\langle I_k, \xi_k\rangle$ and player II chooses the ordinals η_k. Player I wins if and only if, letting Y be the closure of $\nu \cup \{\xi_k : k < \omega\}$ under F,

- $Y \cap \omega_1 = \nu$, *and*
- $Y \subseteq \nu \cup \bigcup_{k<\omega} I_k$

These games were considered previously by Veličković in [V]. Let

$$B_F = \{\nu < \omega_1 : \text{ player } I \text{ has a winning strategy in } \mathcal{G}^F_\nu\}$$

Subclaim 1.5 *B_F includes a club.*

Proof: This follows from the fact that κ is a regular cardinal bigger than ω_1 (see [V], Lemma 3.7). □

Now fix any $\nu \in B_F \cap T$ and let σ be a winning strategy for player I in $\mathcal{G}^F_\nu$. Let θ be a large enough cardinal and let $G : [\kappa]^{<\omega} \longrightarrow \kappa$ be such that for every subset X of κ closed under G, every Skolem function h for $\langle H_\theta, \in, \sigma, \nu, F, <_\theta\rangle$ (where $<_\theta$ is some fixed well-order of H_θ) and every $\vec{x} \in X^{<\omega}$, if $h(\vec{x}) \in \kappa$, then $h(\vec{x}) \in X$.

Let $D = \{\beta < \kappa : G``[\beta]^{<\omega} \subseteq \beta\}$. Since $\overrightarrow{D}$ is a club–guessing sequence, we may fix some $\xi < \kappa$ such that $cf(\xi) = \omega$ and $D_\xi \subseteq D$. Since each $D_\xi(n)$ is closed under G, we can fix for each n some $N_n \preccurlyeq H_\theta$ such that $\sigma \in N_n$ and $N_n \cap \kappa = D_\xi(n)$.

Now consider a play

$$\langle\langle I_0, \xi_0\rangle, \eta_0, \langle I_1, \xi_1\rangle, \eta_1, \ldots\rangle$$

of $\mathcal{G}^F_\nu$ in which I plays according to σ and II forces I to play in such a way that, letting Y be the F–closure of $\nu \cup \{\xi_k : k < \omega\}$, for all n, $Y \cap [C_\xi(n), C_\xi(n{+}1))$ is nonempty exactly when $n \in x$. That such a play exists is ensured by the facts that each N_n contains σ and that σ is a winning strategy for I. Suppose for example that $0 \in x$, $1 \notin x$ and $2 \in x$. I starts playing according to σ inside N_0. Then II plays some ordinal in $\kappa \cap (N_1 \backslash N_0)$. Now player I applies σ inside N_1. In the next move, player II plays an ordinal in $\kappa \cap (N_3 \backslash N_2)$. Then I plays according to σ inside N_3.

Since x is infinite, $sup(Y) = \xi$. Thus, Y is a set in $\mathcal{S}_x$ closed under F and whose intersection with ω_1 is in T. □

The final stage of our iteration is $\mathbb{P}_{\mathcal{S}_x}$. By Fact 1.3, $\mathbb{P}_{\mathcal{S}_x}$ preserves stationary subsets of ω_1 in $V[c]^{\mathcal{C}_0 * \mathcal{C}_1}$, and therefore $2^{<\omega} * \mathbb{P}$ preserves stationary subsets of ω_1 in V.

Let G be $\mathbb{P}$–generic over $V[c]$, let $\alpha = \kappa$ and suppose $\mathcal{Q} \in N$ is a partial order satisfying (b) and (c) at the beginning of this proof. Given an infinite real $r \subseteq \omega$ in N let β_r be, in N, the first ordinal β such that $\mathcal{Q}$ forces that β codes r with respect to $\overrightarrow{C}$ and $(S_n)_{n<\omega}$ in the sense of Lemma 1.2, that is, $\mathcal{Q}$ forces that there is a continuous and increasing decomposition $(X_\nu)_{\nu<\omega_1}$ of $[\beta]^{\aleph_0}$ such that for every limit ordinal $\nu < \omega_1$ there is some $\nu_0 < \nu$ such that for all ξ, $\nu_0 < \xi < \nu$,

$$X_\xi \cap \omega_1 \in \bigcup_{n\in r} S_n \text{ iff } w(X_\xi \cap \omega_1, X_\nu \cap \omega_1) < w(X_\xi, X_\nu)$$

where w is as in Lemma 1.2. Let us see that the mapping $r \longrightarrow \beta_r$, which belongs to N thanks to the definability over models of ZF of the forcing relation, is defined for all infinite reals in N. This will yield the desired contradiction, since the mapping is trivially one–to–one (see the first paragraph after Lemma 1.2), and so there will be then a well–order of $\mathbb{R}^N$ in N.

Let $r \in N$ be an infinite set of integers. Let $i < \lambda$ be such that $r = r_i$. It will suffice to show that every condition in $\mathcal{Q}$ forces over N that α_i codes r in the sense of Lemma 1.2 with respect to $\overrightarrow{C}$ and $(S_n)_{n<\omega}$. Suppose otherwise and let $q \in \mathcal{Q}$ be a condition forcing over N that α_i does not code r. Let H be any filter of $\mathcal{Q}$ containing q which is generic for $\mathcal{Q}$ over $V[c][G]$. Since H is in particular $\mathcal{Q}$–generic over N, we may fix a club $\{Y_\nu : \nu < \omega_1\}$ of $[\alpha]^{\aleph_0}$ belonging to $N[H]$. Let $\{X_\nu : \nu < \omega_1\}$ be a club of $[\alpha]^{\aleph_0}$ in $V[c][G]$ consisting of members of $\mathcal{S}_x$. Since $\mathcal{Q}$ preserves ω_1 over $V[c][G]$, $\{X_\nu : \nu < \omega_1\} \cap \{Y_\nu : \nu < \omega_1\}$ is a club of $[\alpha]^{\aleph_0}$ in $V[c][G][H]$. Therefore there are ν, ν' such that $X_\nu = Y_{\nu'} \in N[H]$. But then, x can be decoded from $Y_{\nu'}$ and from $\overrightarrow{D}$ inside $N[H]$, and $\langle X^r_\xi : \xi < \omega_1\rangle$ can be decoded from x and $\langle y_\nu : \nu < \omega_1\rangle$. Thus, in $N[H]$ there is after all a club of $[\alpha_i]^{\aleph_o}$ witnessing that α_i codes r with respect to $\overrightarrow{C}$ and $(S_n)_{n<\omega}$. This contradiction finishes the proof. $\square$

It should be remarked that there is no proper forcing $\mathbb{P}$ yielding the conclusion of Theorem 1.1. In fact, there is no such *reasonable* forcing, where being reasonable is the following weakening of properness defined in [Fo-M].

Definition 1.3 *A partial order $\mathbb{P}$ is reasonable if and only if $([\alpha]^{\aleph_0})^V$ is a stationary subset of $[\alpha]^{\aleph_0}$ in $V[G]$ for every V–generic $G \subseteq \mathbb{P}$.*

This is a consequence of the following fact.

Theorem 1.6 *Let α be an uncountable ordinal. Then there is a proper partial order $\mathcal{Q}$ collapsing α to ω_1 such that whenever $\mathbb{P}$ is a reasonable partial order, $\mathcal{Q}$ preserves ω_1 in $V^{\mathbb{P}}$.*

Proof: Let $\kappa > \alpha$ be a cardinal. Define $\mathcal{Q}$ to consist of all finite functions $q \subseteq \omega_1 \times [H_\kappa]^{\aleph_0}$, ordered by reverse inclusion, such that $q(\alpha) \in q(\beta)$ for all $\alpha < \beta$ in the domain of q. It is clear that forcing with $\mathcal{Q}$ collapses H_κ to ω_1. To see that $\mathcal{Q}$ is proper, note that for every countable $M \preccurlyeq H_\theta$ (for any large enough θ) containing κ and for every condition q of $\mathcal{Q}$ in M, $q \cup \{\langle M \cap \omega_1, M \cap H_\kappa\rangle\}$ is an $(M, \mathcal{Q})$–generic condition extending q. This is due to the fact that $r \restriction (M \cap \omega_1) \in M$ for every condition r extending q. Now suppose $\mathbb{P}$ is a reasonable forcing. Let G be $\mathbb{P}$–generic, let q be any condition in $\mathcal{Q}$ and let $\dot{F}$ be a $\mathcal{Q}$–name in $V[G]$ for an ω–sequence of countable ordinals. Let $\theta > \kappa$ be some large enough cardinal and let E be a club of $[H_\kappa^V]^{\aleph_0}$ in $V[G]$ consisting of the intersection with H_κ^V of countable elementary substructures of H_θ containing $\mathcal{Q}$, q and $\dot{F}$. Since $\mathbb{P}$ is reasonable, E remains stationary in $V[G]$, and so there are countable $M \preccurlyeq H_\kappa^V$ and $N \preccurlyeq H_\theta$ such that

(a) M belongs to V,

(b) $\mathcal{Q}$, q, $\dot{F} \in N$,

(c) $N \cap H_\kappa^V = M$.

But then, $q \cup \{\langle M \cap \omega_1, M\rangle\}$ is in $V[G]$ an $(N, \mathcal{Q})$–generic condition extending q. Hence, q does not force over $V[G]$ that the range of $\dot{F}$ is not bounded by $M \cap \omega_1$.
□

A similar argument shows that if $\mathbb{P}$ is proper, then forcing with $\mathcal{Q}$ over $V^{\mathbb{P}}$ preserves in fact all stationary subsets of ω_1 in V. This is due to the fact that the set of countable $M \preccurlyeq H_\kappa$ such that $M \cap \omega_1 \in S$ remains then a stationary subset of $[H_\kappa^V]^{\aleph_0}$ in $V^{\mathbb{P}}$ for every stationary $S \subseteq \omega_1$.

2 Adding a real that is generic over two models but can be decoded from them together

In this section I will show that, under a suitable large cardinal hypothesis, after adding a Cohen real c there are, for every real a in $V[c]$, sets A and B such that c is Cohen generic over both $L[A]$ and $L[B]$ but a is in $L[A, B]$. Furthermore, one can take one of these two sets to be a fixed club–guessing sequence for $\omega_2 \cap cf(\omega)$ in V.

The proof of this result relies on the use of a relevant version of the set $\mathcal{S}_x$ from the proof of Theorem 1.1. It also uses the fact that if δ is a Woodin cardinal and the second uniform indiscernible (u_2) is ω_2, then the image J of ω_2 under the elementary embedding derived from forcing with the countable stationary tower restricted to V_δ (denoted by $\mathbb{Q}_{<\delta}$) is already in the ground model, and moreover that for every partial order $\mathbb{P}$ in V_δ that preserves ω_1, J is still the image of ω_2^V under the elementary embedding coming from forcing with $\mathbb{Q}_{<\delta}$ over $V^{\mathbb{P}}$.

Theorem 2.1 *Suppose δ is a Woodin cardinal and $u_2 = \omega_2$. Let $\overrightarrow{D} = \langle D_\xi : \xi \in \omega_2 \cap cf(\omega)\rangle$ be a club–guessing sequence for $\omega_2 \cap cf(\omega)$. Then, if c is Cohen generic over V, in $V[c]$ it holds that for every real a there is a countable sequence $\vec{r} = \langle r_i : i < \lambda\rangle$ of reals, all of them belonging to V, such that*

(1) c is Cohen generic over $L[\vec{r}]$, and

(2) $a \in L[\vec{r}, \overrightarrow{D}]$

It may be worth noting that the hypothesis of the theorem is consistent with $2^{\aleph_0}$ being arbitrarily large. To see this, suppose $\kappa < \lambda < \delta$ are regular cardinals, κ is supercompact and δ is Woodin. Force Martin's Maximum with a forcing included in V_κ as in [Fo-M-S]. It is a result from [Fo-M-S] that Martin's Maximum implies that the nonstationary ideal on ω_1 is saturated and that its saturation is preserved under ccc forcing. Hence we may go to a ccc forcing extension in which $2^{\aleph_0}$ is λ and the nonstationary ideal remains saturated. Since in this extension there are measurable cardinals, by a result of Woodin ([W], Theorem 3.17), $u_2 = \omega_2$ holds in it. Furthermore, δ remains a Woodin cardinal there.

Proof: Given a limit ordinal δ, $\mathbb{Q}_{<\delta}$ is the set of stationary sets in V_δ consisting of countable sets,[1] ordered by setting $b \leq a$ if and only if $\cup a \subseteq \cup b$ and $b \cap \cup a \subseteq a$. Fact 2.2 states the properties of $\mathbb{Q}_{<\delta}$ that we shall need. All results stated there, as well as the definition of the stationary tower, are due to Woodin.

Fact 2.2 *Suppose δ is a Woodin cardinal and suppose G is $\mathbb{Q}_{<\delta}$–generic over V. Then, if $j : V \longrightarrow M$ is the canonical elementary embedding coming from G, it holds that*

(a) the critical point of j is ω_1 and $j(\omega_1) = \delta$,

(b) M is a transitive class definable from V and G, and

(c) given any stationary $a \in \mathbb{Q}_{<\delta}$, $a \in G$ if and only if $j``\cup a \in j(a)$.

It should be noted that (a) and (b) are true in more general cases and that (c), as well as the fact that the critical point of j is ω_1, are true in general.

[L] contains thorough proofs of everything stated in Fact 2.2. We shall use the following general fact.

Fact 2.3 *Suppose $u_2 = \omega_2$ and δ is a Woodin cardinal and let $\mathbb{P} \in V_\delta$ be a forcing notion preserving ω_1. Suppose H is $\mathbb{P}$–generic over V, G is $(\mathbb{Q}_{<\delta})^{V[H]}$–generic over $V[H]$ and $j : V[H] \longrightarrow M$ is the corresponding elementary embedding. Then, $j``\omega_2^V$ belongs to $V$. In fact, $j``\omega_2^V$ is the set of all $t^{L[x]}(\delta)$, where x is a real from V and $t(y)$ is a Skolem term in $L[x]$ for an ordinal.*

Proof: Since Woodinness is preserved under small forcing, we know that (a)–(b) of Fact 2.2 hold for j and M. Moreover, since $u_2 = \omega_2$ holds in V, every ordinal less than ω_2^V is of the form $t^{L[x]}(\omega_1^V)$ for some real x in V and some Skolem term in $L[x]$ for some ordinal. Thus, it follows from the fact that $\omega_1^V = \omega_1^{V[H]}$, together with Fact 2.2 (a) for j and M and with the elementarity of j, that $j``\omega_2^V$ is the set of all $t^{L[x]}(\delta)$, where x is a real from V and $t(y)$ is a Skolem term in $L[x]$ for an ordinal. $\square$

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[1] That is, $\mathbb{Q}_{<\delta}$ consists of all sets a in V_δ such that each member of a is a countable set and such that a is a stationary subset of $[\cup a]^{\aleph_0}$.

Since $2^{<\omega}$ is ccc, we know by the proof of Theorem 1.1 that

$$\mathcal{S}_a = \{X \in [\omega_2]^{\aleph_0} : (\forall n < \omega)[D_{sup(X)}(n), D_{sup(X)}(n+1)) \cap X \neq \emptyset \text{ iff } n \in a\}$$

is a stationary subset of $[\omega_2]^{\aleph_0}$. In particular it is a condition in $\mathbb{Q}_{<\delta}$. Therefore we may force with $\mathbb{Q}_{<\delta} \restriction \mathcal{S}_a$ over $V[c]$ and obtain an elementary embedding $j : V[c] \longrightarrow M$ such that $j``\omega_2 \in j(\mathcal{S}_a)$. Note that $a$ can be decoded from $j``\omega_2$ and $j(\overrightarrow{D})$. Let $\vec{r} = \langle r_i : i < \lambda \rangle$ be an enumeration in V of the reals. By Fact 2.3, $j``\omega_2 \in L[\vec{r}]$, and so $a \in L[\vec{r}, j(\overrightarrow{D})]$. Of course c is Cohen generic over $L[\vec{r}] \subseteq V$. Finally, λ is countable in M and each r_i is in $j(\mathbb{R}^V)$. Now the desired result follows from the elementarity of j. $\Box$

P. Welch has observed that the existence of $0^\sharp$ suffices to perform the action expressed in the title of this section. I thank him for letting me include this result here.

Theorem 2.4 *(Welch) In $L[0^\sharp]$ there are reals a, b and c such that*

(1) c is Cohen generic over both $L[a]$ and $L[b]$, and

(2) $c^\sharp \in L[a,b]$[2]

Proof: Work in $L[0^\sharp]$. Let a be a real coding a bijection between ω and ω_1^L so that $\omega_1^{L[a]}$ is countable (for example pick $a \in L[h]$ where h is $Coll(\omega, \omega_1^L)$–generic over L). In $L[a]$ let $\langle D_n : n < \omega \rangle$ be an enumeration of all dense and open subsets of Cohen forcing in L and let $T = \{p_s : s \in 2^{<\omega}\}$ be a tree of Cohen conditions such that $p_{s\frown\langle 0\rangle}$ and $p_{s\frown\langle 1\rangle}$ are incompatible for each $s \in 2^{<\omega}$ and such that each p_s is in $D_{|s|}$. Pick a Cohen generic b over L such that $p_s \in b$ if and only if s is an initial segment of the characteristic function of $0^\sharp$. Note that $0^\sharp \in L[b,T] \subseteq L[a,b]$. Since b is Cohen generic over L, $\omega_1^{L[b]}$ is countable in $L[0^\sharp]$, so we may fix a real c there which is Cohen generic over $L[a]$ and $L[b]$. Again, since c is in a set–generic extension of L, $c^\sharp \in L[0^\sharp]$, and therefore $c^\sharp \in L[a,b]$. $\Box$

[2] Note that in Theorem 2.1 a can be taken to be $c^\sharp$.

References

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