THE CONSISTENCY OF A CLUB–GUESSING FAILURE AT THE SUCCESSOR OF A REGULAR CARDINAL

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Abstract. I answer a question of Shelah by showing that if κ is a regular cardinal such that $2^{<\kappa} = \kappa$, then there is a $<\kappa$–closed partial order preserving cofinalities and forcing that for every club–sequence $\langle C_\delta \mid \delta \in \kappa^+ \cap \mathrm{cf}(\kappa)\rangle$ with $\mathrm{ot}(C_\delta) = \kappa$ for all δ there is a club $D \subseteq \kappa^+$ such that $\{\alpha < \kappa \mid \{C_\delta(\alpha+1), C_\delta(\alpha+2)\} \subseteq D\}$ is bounded for every δ. This forcing is built as an iteration with $<\kappa$–supports and with symmetric systems of submodels as side conditions.

1. Introduction

The purpose of this paper is to present a consistency results at κ^+ for an arbitrarily fixed regular cardinal κ satisfying $2^{<\kappa} = \kappa$. This result is obtained by a variant of the method of iterated forcing with finite supports and (finite) symmetric systems of submodels as side conditions introduced in [2] (see also [3]). This is the variant of that method in which one considers supports of size less than κ, rather than just finite, and systems, also of size less than κ, consisting of κ–sized structures closed under $<\kappa$–sequences. All iterands in these constructions, as well as the resulting iteration, are $<\kappa$–closed. I will say something more about the method used to prove the main result in this paper in a moment, but first I will introduce the result itself.

Given a set of ordinals S, $\vec{C} = \langle C_\delta \mid \delta \in S\rangle$ is a club–sequence if C_δ is a closed and unbounded (club) subset of δ for every $\delta \in S$. Club–guessing principles are well–studied weakenings of $\diamondsuit_\kappa$, for a cardinal κ, in which the guessing object is a club–sequence defined on (some subset of) κ and in which the relevant guessing applies to closed and unbounded, rather than arbitrary, subsets of κ. It is well–known that, whereas the truth of these principles on ω_1 is easy to manipulate by forcing, many instances of club–guessing at a regular cardinal $\kappa \geq \omega_2$ are provable in ZFC (cf. [8]). The following theorem of Shelah is such

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a result ([9], Claim 3.3; see also [13] for a nicely written proof of this theorem).

Theorem 1.1. *(Shelah) Let $\kappa \geq \omega_1$ be a regular cardinal. Then for every stationary $S \subseteq \kappa^+ \cap \mathrm{cf}(\kappa)$ there is a club–sequence $\langle C_\delta \mid \delta \in S\rangle$ such that for all $\delta \in S$,*

- $\mathrm{ot}(C_\delta) = \kappa$, *and*
- $\mathrm{cf}(C_\delta(\alpha+1)) = \kappa$ *for all* $\alpha < \kappa$,

and such that for every club $D \subseteq \kappa^+$ there is some $\delta \in S$ (equivalently, stationary many $\delta \in S$) such that

$$\{\alpha < \kappa \mid C_\delta(\alpha+1) \in D\}$$

is stationary.

In the statement of Theorem 1.1, and throughout the paper, given a set C of ordinals and an ordinal ξ, I am denoting by $C(\xi)$ the ξ-th member of the strictly increasing enumeration of C.

The following corollary is a straightforward consequence of Theorem 1.1.

Corollary 1.2. *Let $\kappa \geq \omega_1$ be a regular cardinal. Then for every stationary $S \subseteq \kappa^+ \cap \mathrm{cf}(\kappa)$ and every $\rho < \kappa$ there is a club–sequence $\langle C_\delta \mid \delta \in S\rangle$ such that for all $\delta \in S$,*

- $\mathrm{ot}(C_\delta) = \kappa$, *and*
- $\mathrm{cf}(C_\delta(\alpha+\rho)) = \kappa$ *for all* $\alpha < \kappa$,

and such that for every club $D \subseteq \kappa^+$ there is some $\delta \in S$ (equivalently, stationary many $\delta \in S$) such that

$$\{\alpha < \kappa \mid C_\delta(\alpha+\rho) \in D\}$$

is stationary.

The following question appears in [11] as Question 5.4 (cf. [7], Question 13).

Question 1.3. *Is it true in* ZFC *that for every regular cardinal $\kappa \geq \omega_1$ there is a club–sequence $\vec{C} = \langle C_\delta \mid \delta \in \kappa^+ \cap \mathrm{cf}(\kappa)\rangle$ with $\mathrm{ot}(C_\delta) = \kappa$ for all δ and such that for every club $D \subseteq \kappa^+$ there is some δ such that*

$$\{\alpha < \kappa \mid \{C_\delta(\alpha+1), C_\delta(\alpha+2)\} \subseteq D\}$$

is stationary?

According to Shelah in [11], if there is a club–sequence as in the above question on $\kappa^+ \cap \mathrm{cf}(\kappa)$ and GCH holds, then there is a κ^+–Souslin tree. In particular, an affirmative answer to Question 1.3 would yield an affirmative answer to the following well–known open question (see e.g. [5] or [7]).

Question 1.4. *Does* GCH *imply that there is an ω_2–Souslin tree?*

As mentioned also in [11], using the methods from [12] it is possible to provide a negative answer to the easier form of Question 1.3 where we fix a stationary $S \subseteq \kappa^+ \cap \mathrm{cf}(\kappa)$ with $(\kappa^+ \cap \mathrm{cf}(\kappa)) \setminus S$ also stationary and we ask that $\vec{C}$ be defined on S rather than on all of $\kappa^+ \cap \mathrm{cf}(\kappa)$:

Theorem 1.5. *(Shelah) Suppose $\kappa \geq \omega_1$ is a regular cardinal such that $\kappa^{<\kappa} = \kappa$, $S \subseteq \kappa^+ \cap \mathrm{cf}(\kappa)$ is stationary and $S' = (\kappa^+ \cap \mathrm{cf}(\kappa)) \setminus S$ is also stationary. Then the following holds in a generic extension preserving the stationarity of both S and S' and not adding new $<\kappa$–sequences of ordinals.*

1. *For every club–sequence $\langle C_\delta \mid \delta \in S\rangle$ such that $\mathrm{ot}(C_\delta) = \kappa$ for all δ there is a club $D \subseteq \kappa^+$ such that*
$$\{\alpha < \kappa \mid \{C_\delta(\alpha+1), C_\delta(\alpha+2)\} \subseteq D\}$$
is bounded for all $\delta \in S$.
2. *For every club–sequence $\langle C_\delta \mid \delta \in S\rangle$, if it holds for all δ that*
   (a) *$\mathrm{ot}(C_\delta) = \kappa$ and that*
   (b) *$\mathrm{cf}(C_\delta(\alpha+1)) < \kappa$ for all α,*
   then there is a club $D \subseteq \kappa^+$ such that
$$\{\alpha < \kappa \mid C_\delta(\alpha+1) \in D\}$$
is bounded for all $\delta \in S$.

The main result in this paper is the following.

Theorem 1.6. *Let $\omega_1 \leq \kappa < \kappa^{++} \leq \theta$ be regular cardinals such that $2^{<\kappa} = \kappa$, $2^\kappa = \kappa^+$ and $2^{<\theta} = \theta$. Then there is a partial order $\mathcal{P}$ with the following properties.*

1. *$\mathcal{P}$ is $<\kappa$–closed.*
2. *There is some $\Phi \in H(\theta^+)$ such that $\mathcal{P}$ is proper with respect to all $N \preccurlyeq H((2^\theta)^+)$ such that $\mathcal{P}, \Phi \in N$, $|N| = \kappa$ and ${}^{<\kappa}N \subseteq N$.*
3. *$\mathcal{P}$ is κ^{++}–Knaster.*
4. *$\mathcal{P}$ forces that for every club–sequence $\langle C_\delta \mid \delta \in \kappa^+ \cap \mathrm{cf}(\kappa)\rangle$ with $\mathrm{ot}(C_\delta) = \kappa$ for all δ there is a club $D \subseteq \kappa^+$ such that*
$$\{\alpha < \kappa \mid \{C_\delta(\alpha+1), C_\delta(\alpha+2)\} \subseteq D\}$$
is bounded in κ for all δ.
5. *$\mathcal{P}$ forces that for every club–sequence $\langle C_\delta \mid \delta \in \kappa^+ \cap \mathrm{cf}(\kappa)\rangle$, if for all δ,*
   (a) *$\mathrm{ot}(C_\delta) = \kappa$, and*
   (b) *$\mathrm{cf}(C_\delta(\alpha+1)) < \kappa$ for all $\alpha < \kappa$,*

then there is a club $D \subseteq \kappa^+$ such that

$$\{\alpha < \kappa \mid C_\delta(\alpha + 1) \in D\}$$

is bounded in κ for all δ.

(6) *$\mathcal{P}$ forces $2^\mu = \theta$ for every $\mu \in [\kappa, \theta)$.*

The classical notion of properness can be extended to structures which are not necessarily countable. Specifically, conclusion (2) in Theorem 1.6 says that if $N \preccurlyeq H((2^\theta)^+)$ is such that $|N| = \kappa$, ${}^{<\kappa}N \subseteq N$, and $\mathcal{P}$, $\Phi \in N$, then for every $q \in \mathcal{P} \cap N$ there is an extension q' of q which is $(N, \mathcal{P})$–generic, i.e., such that q' forces $E \cap \dot{G} \cap N \neq \emptyset$ for every dense subset E of $\mathcal{P}$ belonging to N. Parts of the standard theory of properness (but not all of it)[1] extend to the general setting. In particular, if χ is a cardinal, $T \subseteq \chi$, and $\mathbb{P}$ is a partial order wich is proper for a stationary class of structures N such that $N \cap \chi \in T$, then forcing with $\mathbb{P}$ preserves the stationarity of T. Note that $2^{<\kappa} = \kappa$ implies that for every stationary $T \subseteq \kappa^+ \cap \mathrm{cf}(\kappa)$ and every cardinal $\rho \geq \kappa^+$, the set of $N \preccurlyeq H(\rho)$ such that $|N| = \kappa$, ${}^{<\kappa}N \subseteq N$ and $N \cap \kappa^+ \in T$ is stationary. Also, recall that, for a cardinal λ, a partial order $\mathbb{P}$ is λ–Knaster if for every $X \subseteq \mathbb{P}$ of size λ there is a subset of X of size λ consisting of pairwise compatible conditions.

It follows that every forcing satisfying (1)–(3) from Theorem 1.6 preserves all cofinalities and all stationary subsets of $\kappa^+ \cap \mathrm{cf}(\kappa)$. In particular, Theorem 1.6 answers Shelah's Question 1.3 negatively, but it also shows that 2^κ, where κ is the regular cardinal for which we kill the relevant club–guessing at κ^+, can be arbitrarily large.

As I briefly mentioned at the beginning, Theorem 1.6 is proved by building a certain forcing iteration with supports of size less than κ and with certain systems of submodels as side conditions.[2] These submodels are 'active', as side conditions, at an initial segment of stages of the iteration, as indicated by markers associated to them. This type of forcing construction was first used in [2] and afterwards in [3], but only for $\kappa = \omega$. Te reader can find in [2] the appropriate background and general motivation for this kind of construction, so I will not go into that here.

The present construction is in spirit quite similar to the one in [2], replacing of course ω by κ everywhere and, as one would also expect, looking at submodels of size κ closed under $<\kappa$–sequences except of

[1] It is well–known that there can be no general preservation theorem for properness with respect to uncountable models (on the other hand, properness with respect to countable models is always preserved under countable support iterations).

[2] These are elementary submodels of $(H(\theta), \in, T)$ for some cardinal θ and some suitable predicate $T \subseteq H(\theta)$.

countable submodels. One the other hand, the present construction has a couple of features which were not present in the constructions from [2] or [3]:

The present construction can be viewed as an iteration, with a suitable type of symmetric systems of structures as side conditions, in which at each stage a certain book–keeping function feeds us a club–sequence $\vec{C}$ on $\kappa^+ \cap \mathrm{cf}(\kappa)$, and we shoot a certain club of κ^+ which will destroy (part of) the potential guessing character of $\vec{C}$. However, it is important for the proof to go through – and, specifically, for the properness proof (Lemma 3.12) – that we destroy these cub–sequences rather slowly: More specifically, we start by fixing a partition $(S_\rho)_{\rho<\kappa}$ of $\kappa^+ \cap \mathrm{cf}(\kappa)$ into stationary sets and for every specific stage β of the iteration we make sure that there is exactly one $\rho < \kappa$ such that the club D_β we add at that stage is asked to "kill" the relevant $\vec{C}$ picked at that stage only for those $\delta \in D_\beta$ which are in S_ρ (in the end, the club witnessing that $\vec{C}$ has been "killed" everywhere, and not just on some stationary set $S \subseteq \kappa^+ \cap \mathrm{cf}(\kappa)$, is the intersection of κ–many of the clubs D_β explicitly added along the iteration). Nothing like this was needed in the constructions from [2] or [3]. It looks difficult to convey in few word why such a move is needed here; I will simply refer the reader to the actual proof (specifically, see the proof of Lemma 3.12, and particularly the part of it when $t^* \in A$ is found in M^* and shown to be compatible with t).

Another sense in which the present construction differs from the one in [2] and also the one in [3] is the following: An essential feature of some of the proofs in [2] and [3] is that they are by induction. For example, in the proofs of properness, if $(\mathcal{P}_\alpha \mid \alpha < \theta)$ is the corresponding iteration, one proves for every $\alpha \leq \theta$ that if $\mathcal{P}_\beta$ has the relevant form of properness for all $\beta < \alpha$, then this is true for $\mathcal{P}_\alpha$ itself.[3] In order to run this type of proof it is crucial that the supports be finite, or otherwise the induction breaks down at stages of countable cofinality. However, given the nature of the present approach, finite support do not work as we want to have a $<\kappa$–closed forcing in the end[4] (we need $<\kappa$–supports instead), and the type of inductive approach from [2] and [3] completely breaks down. Instead, here one proves the relevant properness lemma by a direct construction.

[3]This type of inductive argument seems to be unavoidable when the goal is to build a model of a reasonably general forcing axiom (as in [2] and [3]).

[4]The reason we want a $<\kappa$–closed forcing is that we want to preserve all cardinals $\lambda \leq \kappa$.

The notation in this paper is fairly standard (see e.g. [4] or [6]), but I will also use pieces of notation that are not so standard and that will be introduced at the appropriate place. Given a set N, if $N \cap |N|^+$ is an ordinal, then I will usually denote this ordinal by δ_N. Also, if $q = (F, \Delta)$, F is a function, Δ consists of pairs (N, τ), where τ is an ordinal, and β is an ordinal, *the restriction of q to β,* denoted by $q|_\beta$, is the ordered pair $(F \restriction \beta, \Delta')$, where Δ' consists of all pairs (N, τ') with $(N, \tau) \in \Delta$ and $\tau' = \min\{\tau, \mathsf{sup}(N \cap (\beta + 1))\}$.

The rest of the paper is structured as follows. In Section 2 I adapt the notion of symmetric system from [2] to the present context and present the relevant amalgamation lemmas. In Subsection 3.1 I introduce a forcing notion for destroying an instance of the club–guessing we are looking at here and prove the relevant density lemmas for it. Then, in Subsection 3.2, I first construct the forcing $\mathcal{P}$ witnessing Theorem 1.6 and then prove a sequence of lemmas which together will prove the theorem.

2. Symmetric systems of submodels

Let us fix two arbitrary regular cardinals $\lambda < \chi$ for this section. In Section 2 of [2] we consider a certain natural notion of symmetric system of submodels and prove its relevant properties.[5] The notion of symmetric system admits a natural generalisation to higher cardinalities, which is the one I will consider next. The theory of finite symmetric systems as developed in [2] goes through in the general setting with just notational changes.

Definition 2.1. Let $T \subseteq H(\chi)$ and let $\mathcal{N} \subseteq \mathcal{P}(H(\chi))$ be such that $|\mathcal{N}| < \lambda$. $\mathcal{N}$ *is a symmetric λ–T–system* if and only if the following holds.

(A) For every $N \in \mathcal{N}$, $(N, \in, T) \preccurlyeq (H(\chi), \in, T)$, $|N| = \lambda$, $\lambda \in N$, and ${}^{<\lambda}N \subseteq N$.

(B) Given N and N' in $\mathcal{N}$, if $\delta_N = \delta_{N'}$, then there is a (unique) isomorphism
$$\Psi_{N,N'} : (N, \in, T) \longrightarrow (N', \in, T)$$
Furthermore, $\Psi_{N,N'}$ is the identity on $N \cap N'$.

(C) For all N_0, N_1 and N_1' in $\mathcal{N}$, if $N_0 \in N_1$ and $\delta_{N_1} = \delta_{N_1'}$, then $\Psi_{N_1,N_1'}(N_0) \in \mathcal{N}$.

(D) For all N_0 and N_1 in $\mathcal{N}$, if $\delta_{N_0} < \delta_{N_1}$, then there is some $N_1' \in \mathcal{N}$ such that $\delta_{N_1'} = \delta_{N_1}$ and $N_0 \in N_1'$.

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[5]This notion was not new. [2] contains older references in the literature where this same type of system appears.

In the statement of condition (B) and throughout the paper, if N and N' are such that there is a unique isomorphism $\Psi : (N, \in) \longrightarrow (N', \in)$, I will tend to denote this isomorphism by $\Psi_{N,N'}$. Also, I will occasionally refer to 'symmetric λ–systems' or even 'symmetric systems', without mention of T and/or λ, in contexts where these parameters are either understood or not relevant.

The proof of the following fact is immediate. It will be used in the proof of Lemma 3.12.

Fact 2.2. *For every $T \subseteq H(\chi)$ as in Definition 2.1 and every λ–T–symmetric system $\mathcal{N}$, if N_0, N_1 are in $\mathcal{N}$ and $\delta_{N_0} < \delta_{N_1}$, then there is some $N_0' \in \mathcal{N} \cap N_1$ such that $\delta_{N_0'} = \delta_{N_0}$ and such that $N_0 \cap N_1 = N_0 \cap N_0'$.*

Proof. Let $N_1' \in \mathcal{N}$ be as given by (D) in Definition 2.1 for the pair N_0, N_1, and let $N_0' = \Psi_{N_1', N_1}(N_0)$. □

Our main amalgamation lemmas for λ–T–symmetric systems will be the following. The proofs of these lemmas are identical to the proofs of the corresponding lemmas in [2], with just the obvious notational changes.

Lemma 2.3. *Let $T \subseteq H(\chi)$, let $\mathcal{N}$ be a λ–T–symmetric system, and let $N \in \mathcal{N}$. Then the following hold.*

(i) *$\mathcal{N} \cap N$ is also a λ–T–symmetric system.*
(ii) *If $\mathcal{W} \subseteq N$ is a λ–T–symmetric system and $\mathcal{N} \cap N \subseteq \mathcal{W}$, then*

$$\mathcal{V} := \mathcal{N} \cup \{\Psi_{N,N'}(W) \, : \, W \in \mathcal{W},\, N' \in \mathcal{N},\, \delta_{N'} = \delta_N\}$$

is a λ–T–symmetric system.

Lemma 2.4. *Let $T \subseteq H(\chi)$, and suppose $\mathcal{N}_0 = \{N_i^0 \, : \, i < \mu\}$ and $\mathcal{N}_1 = \{N_i^1 \, : \, i < \mu\}$ are λ–T–symmetric systems for some $\mu < \lambda$. Suppose that $(\bigcup \mathcal{N}_0) \cap (\bigcup \mathcal{N}_1) = R$ and that there is an isomorphism Ψ between the structures*

$$\langle \bigcup_{i<\mu} N_i^0, \in, T, R, N_i^0 \rangle_{i<\mu}$$

and

$$\langle \bigcup_{i<m} N_i^1, \in, T, R, N_i^1 \rangle_{i<\mu}$$

fixing R. Then $\mathcal{N}_0 \cup \mathcal{N}_1$ is a λ–T–symmetric system.

3. The proof

3.1. **Killing a club–sequence.** I will start by introducing the following notion of rank (cf. the definition in [1] or [2]): Given a regular cardinal λ and an ordinal δ, we define the *λ–Cantor–Bendixson rank of* δ, $rank^{\lambda}(\delta)$, by specifying that

- $rank^{\lambda}(\delta) \geq 1$ if and only if δ is a limit point of ordinals of cofinality λ, and that
- if $\mu > 1$, $rank^{\lambda}(\delta) \geq \mu$ if and only if for every $\eta < \mu$, δ is a limit of ordinals ϵ such that $\mathrm{cf}(\epsilon) = \lambda$ and $rank^{\lambda}(\epsilon) \geq \eta$.

If $(\xi_i)_{i<\lambda}$ and $(\rho_i)_{i<\lambda}$ are increasing sequences of ordinals and $\rho_i \leq \mathrm{rank}^{\lambda}(\xi_i)$ for all i, then $\mathrm{rank}^{\lambda}(\mathsf{sup}\{\xi_i \mid i < \lambda\}) \geq \mathsf{sup}\{\rho_i \mid i < \lambda\}$. In particular, if N is an elementary substructure of some $H(\chi)$, $\lambda \in N$ and δ_N exists, then $\mathrm{rank}^{\lambda}(\delta_N) = \delta_N$. This is true because, by correctness of N inside $H(\chi)$, letting $\mu = \min((N \cap \chi) \setminus \delta_N)$ if $(N \cap \chi) \setminus \delta_N \neq \emptyset$ and $\mu = \chi$ otherwise, for every $\xi \in \mu \cap N$ there is some $\xi' \geq \xi$ in N such that $\mathrm{rank}^{\lambda}(\xi') = \xi'$, and therefore there is an increasing sequence $(\xi_i)_{i<\mathrm{cf}(|N|)}$ of ordinals such that $\mathsf{sup}\{\xi_i \mid i < |N|\} = \delta_N$ and such that for all i, $\xi_i \in N \cap \delta_N$ and $\mathrm{rank}^{\lambda}(\xi_i) = \xi_i$.

Definition 3.1. Given a regular cardinal λ, a function $f \subseteq \lambda^+ \times \lambda^+$ is a *λ–approximation* if the following holds.

(a) $|f| < \lambda$
(b) f is strictly increasing.
(c) For every $\xi \in \mathrm{dom}(f)$, $\mathrm{rank}^{\lambda}(f(\xi)) \geq \lambda + \xi$.
(d) For every $\xi \in \mathrm{dom}(f)$,
  (d.1) if ξ is a nonzero limit ordinal such that $\mathrm{cf}(\xi) < \lambda$, then $\xi = \mathsf{sup}(\mathrm{dom}(f \restriction \xi))$ and $f(\xi) = \mathsf{sup}(f``\xi)$, and
  (d.2) if ξ is either a successor ordinal or a limit ordinal such that $\mathrm{cf}(\xi) = \lambda$, then $\mathrm{cf}(f(\xi)) = \lambda$.

It follows of course from (b) together with (d.1) and (d.2) that if f is a λ–approximation and $\xi \in \mathrm{dom}(f)$ is any nonzero limit ordinal, then $\mathrm{cf}(f(\xi)) = \mathrm{cf}(\xi)$.

The following fact is an immediate consequence of the definitions.

Fact 3.2. *For every regular cardinal λ, if $\mathcal{F}$ is a collection of size less than λ consisting of λ–approximations and every two members of $\mathcal{F}$ are compatible as functions, then $\bigcup \mathcal{F}$ is a λ–approximation.*

Given a club–sequence $\vec{C} = \langle C_\delta \mid \delta \in \kappa^+ \cap \mathrm{cf}(\kappa)\rangle$ with $\mathrm{ot}(C_\delta) = \kappa$ for all δ, let $\mathcal{Q}$ be the following partial order: A condition in $\mathcal{Q}$ is an ordered pair (f, π) satisfying the following conditions:

(1) $f \subseteq \kappa^+ \times \kappa^+$ is a κ–approximation.
(2) π is a function with $\mathrm{dom}(\pi) \subseteq \mathrm{dom}(f) \cap \mathrm{cf}(\kappa)$ and $\pi(\xi) \in f(\xi)$ for all $\xi \in \mathrm{dom}(\pi)$.
(3) Let $\xi \in \mathrm{dom}(\pi)$ and $\zeta \in \mathrm{dom}(f) \cap \xi$. Suppose
  (i) $\sigma_0^\zeta = \max(C_{f(\xi)} \cap f(\zeta))$ in case $\max(C_{f(\xi)} \cap f(\zeta))$ exists.
  (ii) $\sigma_1^\zeta = \min\{\sigma \in C_{f(\xi)} \mid \sigma \geq f(\zeta)\}$,
  (iii) $\sigma_2^\zeta = \min\{\sigma \in C_{f(\xi)} \mid \sigma > \sigma_1^\zeta\}$, and
  (iv) $\pi(\xi) < f(\zeta)$
  Then the following holds.
  (3.0) If σ_0^ζ exists, is a successor point of $C_{f(\xi)}$, and $\pi(\xi) < \sigma_0^\zeta$, then there is some $\zeta_0 < \zeta$ such that $\{\zeta_0, \zeta_0 + 1\} \subseteq \mathrm{dom}(f)$, $f(\zeta_0) < \sigma_0^\zeta$ and $f(\zeta_0 + 1) > \sigma_0^\zeta$.
  (3.1) If σ_1^ζ is a successor point of $C_{f(\xi)}$, then
   - if $f(\zeta) < \sigma_1^\zeta$, then $\zeta + 1 \in \mathrm{dom}(f)$ and $f(\zeta+1) > \sigma_1^\zeta$, and
   - if ζ is a limit ordinal with $\mathrm{cf}(\zeta) < \kappa$, then $f(\zeta) < \sigma_1^\zeta$.
  (3.2) There is some $\zeta_2 \geq \zeta$ with $\{\zeta_2, \zeta_2+1\} \subseteq \mathrm{dom}(f)$, $f(\zeta_2) < \sigma_2^\zeta$ and $f(\zeta_2 + 1) > \sigma_2^\zeta$.

Given $\mathcal{Q}$–conditions (f_0, π_0) and (f_1, π_1), (f_1, π_1) extends (f_0, π_0) if $f_0 \subseteq f_1$ and $\pi_0 \subseteq \pi_1$.

Given a club–sequence $\vec{C}$ as above, I will denote the corresponding forcing $\mathcal{Q}$ by $\mathcal{Q}_{\vec{C}}$.

The following simple observation is an immediate consequence of (3.2) in the above definition together with (d.1) in Definition 3.1.

Fact 3.3. *If $\vec{C} = \langle C_\delta \mid \delta \in \kappa^+ \cap \mathrm{cf}(\kappa)\rangle$ is a club–sequence with $\mathrm{ot}(C_\delta) = \kappa$ for all δ, $(f, \pi) \in \mathcal{Q}_{\vec{C}}$, $\xi \in \mathrm{dom}(\pi)$, and there is some $\zeta \in \mathrm{dom}(f \restriction \xi)$ such that $f(\zeta) > \pi(\xi)$, then $\mathsf{sup}(f``\xi)$ is a limit point of $C_{f(\xi)}$.*

Lemma 3.4. *Let $\vec{C} = \langle C_\delta \mid \delta \in \kappa^+ \cap \mathrm{cf}(\kappa)\rangle$ be a club–sequence such that $\mathrm{ot}(C_\delta) = \kappa$ for all δ, and let (f, π) be a condition in $\mathcal{Q}_{\vec{C}}$. Then the following holds.*

(a) *For every $\xi \in \mathrm{dom}(f)$ such that $\mathrm{cf}(\xi) = \kappa$ there is an extension (f, π') of (f, π) such that $\xi \in \mathrm{dom}(\pi')$.*
(b) *For every limit ordinal $\xi \in \mathrm{dom}(f)$, $\xi' \in \xi$ and $\alpha \in f(\xi)$ there are $\xi'' \in (\xi', \xi)$, $\beta \in (\alpha, f(\xi))$ and an extension (f', π) of (f, π) such that $\xi'' \in \mathrm{dom}(f')$ and $f'(\xi'') = \beta$.*
(c) *For every $\xi \in \kappa^+ \setminus \mathrm{dom}(f)$ there is an extension (f', π) of (f, π) such that $\xi \in \mathrm{dom}(f')$. Furthermore, if*
  (1) $\mathrm{rank}^\kappa(\xi) = \xi = \kappa + \xi$,
  (2) $\mathrm{cf}(\xi) = \kappa$,

(3) $\mathrm{range}(f \restriction \xi) \subseteq \xi$,
(4) $\mathrm{cf}(\min(\mathrm{dom}(f) \setminus \xi)) = \kappa$ *in case* $\min(\mathrm{dom}(f) \setminus \xi)$ *exists, and*
(5) $[\xi,\, f(\xi^*)) \cap C_{f(\rho)} = \emptyset$ *whenever* $\rho \in \mathrm{dom}(\pi)$ *is such that* $\rho > \xi^* = \min(\mathrm{dom}(f) \setminus \xi)$ *in case* $\min(\mathrm{dom}(f) \setminus \xi)$ *exists,*

then f' can be taken so that $f'(\xi) = \xi$.

Proof. Part (a) is obvious: Since $|f| < \kappa$ and $\mathrm{cf}(f(\xi)) = \kappa$, it suffices to let $\pi'(\xi) < f(\xi)$ be such that $f``\xi \subseteq \pi'(\xi)$.

For part (b) we may assume $\mathrm{cf}(\xi) = \kappa$ as otherwise we are done by (d.1) in the definition of κ–approximation. Assume $\mathrm{dom}(f \restriction \xi) \neq \emptyset$, and let $\bar{\xi} = \mathsf{sup}(\mathrm{dom}(f \restriction \xi))$ (the case $\mathrm{dom}(f \restriction \xi) = \emptyset$ is easier). We may also assume that $\max(\mathrm{dom}(f \restriction \xi))$ does not exist (otherwise the proof is again easier). Pick any successor ordinal $\xi'' \in (\xi',\, \xi)$ such that $\bar{\xi} + 1 < \xi''$. Since $\mathrm{cf}(f(\xi)) = \kappa$ and $\mathrm{rank}^\kappa(f(\xi)) > \kappa$, we may find $\beta \in (\alpha,\, f(\xi)) \setminus C_{f(\xi)}$ of cofinality κ such that

(i) β is above all members of $C_\delta \cap f(\xi)$ for all $\delta \in f``(\xi,\, \kappa^+)$ of cofinality $\kappa$ and above at least two $\sigma \in C_{f(\xi)}$ such that $\sigma > \mathsf{sup}(f``\xi)$, and
(ii) $\mathrm{rank}^\kappa(\beta) \geq \kappa + \xi''$.

Let also $\beta_0 \in (\alpha,\, \beta)$ of cofinality κ be such that

(iii) β_0 is above all members of $C_\delta \cap f(\xi)$ for all $\delta \in f``(\xi,\, \kappa^+)$ of cofinality κ,
(iv) $\mathrm{rank}^\kappa(\beta_0) \geq \kappa + \bar{\xi} + 1$, and such that
(v) β_0 is above $\max(C_{f(\xi)} \cap \beta)$.[6]

Finally, let $(\delta_n)_{1 \leq n < \omega}$ be a strictly increasing sequence of ordinals in $(\beta,\, f(\xi))$ of cofinality κ such that $|(\beta,\, \delta_1) \cap C_{f(\xi)}| \geq 2$ and such that for all $n \geq 1$,

(vi) $\delta_n \notin C_{f(\xi)}$,
(vii) $\mathrm{rank}^\kappa(\delta_n) \geq \kappa + \xi'' + n$, and
(viii) $|(\delta_n,\, \delta_{n+1}) \cap C_{f(\xi)}| \geq 2$

This sequence exists since $C_{f(\xi)}$ is cofinal in $f(\xi)$ and of order type κ and since $\mathrm{rank}^\kappa(f(\xi)) > \kappa$. Now we may extend f to a function f' with domain $\mathrm{dom}(f) \cup \{\bar{\xi}, \bar{\xi} + 1\} \cup \{\xi'' + n \mid n < \omega\}$ which send $\bar{\xi}$ to $\mathsf{sup}(f``\bar{\xi})$, $\bar{\xi} + 1$ to β_0, ξ'' to β and, for every integer $n > 0$, sends $\xi'' + n$ to δ_n. Let us check that (f', π) is a condition in $\mathcal{Q}_{\vec{C}}$:

By the choice of β_0 in (iii) it is clear that for every γ in the set $\{\mathsf{sup}(f``\bar{\xi}), \beta_0, \beta\} \cup \{\delta_n \mid 0 < n < \omega\}$, the addition of γ to the range of f will not cause any problem with condition (3) in the definition of $\mathcal{Q}_{\vec{C}}$–condition for those $\rho \in \mathrm{dom}(\pi)$ above ξ such that $\pi(\rho) < \gamma$.

[6]Note that $\max(C_{f(\xi)} \cap \beta)$ indeed exists since $\mathrm{cf}(\beta) = \kappa$.

This follows from $\pi(\rho) < f(\xi)$ together with the fact that $\max(C_{f(\rho)} \cap f(\xi)) = \max(C_{f(\rho)} \cap \gamma)$ if $\gamma \geq \beta_0$, that γ is a limit point of $C_{f(\rho)}$ if $\gamma = \mathsf{sup}(f``\bar{\xi})$ by (3.2) applied to ρ and to a tail of $\mathrm{dom}(f \restriction \bar{\xi})$, and that $\min\{\sigma \in C_{f(\rho)} \mid \sigma \geq f(\xi)\} = \min\{\sigma \in C_{f(\rho)} \mid \sigma \geq \gamma\}$ if $\gamma \geq \beta_0$.

As to condition (3) for ξ in case $\xi \in \mathrm{dom}(\pi)$, we show, for γ as above, that the corresponding instance of (3.0)–(3.2) holds. To start with, note that if $\mathsf{sup}(f``\bar{\xi}) > \pi(\xi)$, then $\mathsf{sup}(f``\bar{\xi})$ is a limit point of $C_{f(\xi)}$ by Lemma 3.3, which means that (3.0) holds for the pair ξ, $\bar{\xi}$, and that $\{\bar{\xi}, \bar{\xi}+1\}$ witnesses the relevant instances of (3.1) and (3.2) for the pair ξ, $\bar{\xi}$ by (i) and (v). For $\gamma \geq \beta_0$, the corresponding instances of (3.0)–(3.2) hold immediately by construction.

The first part of (c) can be easily established by arguing as in the proof of (b) and can be left as an exercise for the reader (in the case when ξ is a limit ordinal with $\mathrm{cf}(\xi) < \kappa$, one has to make sure of course that $\mathrm{dom}(f' \restriction \xi)$ is cofinal in ξ and $\mathsf{sup}(\mathrm{range}(f' \restriction \xi)) = f'(\xi)$).

I will just sketch the proof of the second part of (c). Suppose $\xi \notin \mathrm{dom}(f)$ satisfies the hypotheses (1)–(5). We have to show that there is an extension (f', π) of (f, π) such that $\xi \in \mathrm{dom}(f')$ and $f'(\xi) = \xi$. We may assume that $f \restriction \xi$ is nonempty and does not have a maximum and also that $\xi^* = \min(\mathrm{dom}(f) \setminus \xi)$ exists (the proof in the other cases is easier). Let $\bar{\xi} = \mathsf{sup}(\mathrm{dom}(f \restriction \xi))$ and let $\beta_0 < \xi$ be above $\max(C_{f(\rho)} \cap \xi)$ for every $\rho \in \mathrm{dom}(\pi)$ such that $\rho \geq \xi^*$. Since $\mathrm{cf}(\xi^*) = \kappa$ and $\mathrm{rank}^{\kappa}(f(\xi^*)) > \kappa$, we may pick a strictly increasing sequence $(\delta_n)_{1 \leq n < \omega}$ of ordinals of cofinality κ in $(\xi, f(\xi^*)) \setminus C_{f(\xi^*)}$ such that $|(\xi, \delta_1) \cap C_{f(\xi^*)}| \geq 2$ and such that for all n,

(i) $\mathrm{rank}^{\kappa}(\delta_n) \geq \kappa + \xi + n$, and
(ii) $|(\delta_n, \delta_{n+1}) \cap C_{f(\xi^*)}| \geq 2$.

Now it is easy to verify as in the proof of part (b) that f' is as desired, where $\mathrm{dom}(f') = \mathrm{dom}(f) \cup \{\bar{\xi}, \bar{\xi}+1\} \cup \{\xi + n \mid n < \omega\}$ extends f, sends $\bar{\xi}$ to $\mathsf{sup}(f``\bar{\xi})$, $\bar{\xi}+1$ to $\beta_0$, $\xi$ to itself, and send each $\xi + n$, for $n > 0$, to $\delta_n$. The verification that the act of adding either $\mathsf{sup}(f``\bar{\xi})$, β_0, or any δ_n, to the range of f' does not interfere with condition (3) in the definition of $\mathcal{Q}_{\vec{C}}$ relative to either ξ^*, if $\xi^* \in \mathrm{dom}(\pi)$, or relative to any $\rho \in \mathrm{dom}(\pi)$ above ξ^* is as in the proof of part (b). Hence we just need to argue that adding ξ does not cause trouble either. Note that $\max(C_{f(\xi^*)} \cap \xi) = \max(C_{f(\xi^*)} \cap \beta_0)$ and therefore (3.0) holds for the pair ξ^*, ξ since it holds for the pair ξ^*, $\bar{\xi}+1$ as witnessed by $\{\bar{\xi}, \bar{\xi}+1\}$ (cf. the proof of part (b)). Also, if $\rho \in \mathrm{dom}(\pi)$ is above ξ^*, $\max(C_{f(\rho)} \cap \xi)$ exists and is a successor ordinal, and $\max(C_{f(\rho)} \cap \xi) > \pi(\rho)$, then (3.0) holds for ρ, ξ because it holds for ρ, $\bar{\xi}+1$ as witnessed again by the pair $\{\bar{\xi}, \bar{\xi}+1\}$ (again cf. the proof of part (b)). The relevant instances of

(3.1) and (3.2) are satisfied automatically, as witnessed by $\{\xi, \xi+1\}$, since $\mathrm{cf}(\xi) = \kappa$ and by the choice of δ_1. Finally, let $\rho \in \mathrm{dom}(\pi)$, $\rho > \xi^*$, such that $\pi(\rho) < \xi$, let $\sigma = \min(C_{f(\rho)} \setminus \xi)$ and let $\sigma' = \min(C_{f(\rho)} \setminus (\sigma+1))$. Since $[\xi,\, f(\xi^*)) \cap C_{f(\rho)} = \emptyset$, $\sigma = \min(C_{f(\rho)} \setminus f(\xi^*))$ and $\sigma' = \min(C_{f(\rho)} \setminus (\sigma+1))$. But then (3.1) and (3.2) hold for ρ, ξ because they hold for ρ, ξ^*. $\square$

The following lemma is easily proved by an easier version of the proof of Lemma 3.4.

Lemma 3.5. *Let $\vec{C} = \langle C_\delta \mid \delta \in \kappa^+ \cap \mathrm{cf}(\kappa)\rangle$ be a club–sequence such that $\mathrm{ot}(C_\delta) = \kappa$ for all δ, let (f, π) and (f', π') be $\mathcal{Q}_{\vec{C}}$–conditions, and suppose there are $\eta < \xi$ such that*

- *$\xi \in \mathrm{dom}(f)$, $\mathrm{cf}(\xi) = \kappa$, and $f(\xi) = \xi$,*
- *$\eta > \max(C_\delta \cap \xi)$ for every $\delta \in \mathrm{range}(f)$ above ξ of cofinality κ,*
- *$\xi \notin \mathrm{dom}(\pi)$,*
- *$\mathrm{dom}(f') \subseteq \xi$, and*
- *$(f' \restriction \eta, \pi' \restriction \eta) = (f \restriction \xi, \pi \restriction \xi)$.*

Then $(f \cup f', \pi \cup \pi')$ is a $\mathcal{Q}_{\vec{C}}$–condition.

Lemma 3.6. *Let $\vec{C} = \langle C_\delta \mid \delta \in \kappa^+ \cap \mathrm{cf}(\kappa)\rangle$ be a club–sequence such that $\mathrm{ot}(C_\delta) = \kappa$ for all δ. Then the following holds.*

1. *$\mathcal{Q}_{\vec{C}}$ is $<\kappa$–directed closed. In fact, if $\{(f_i, \pi_i) \mid i < \mu\} \subseteq \mathcal{Q}_{\vec{C}}$ is a directed set of size less than κ, then $(\bigcup_{i<\mu} f_i, \bigcup_{i<\mu} \pi_i)$ is the greatest lower bound of $\{(f_i, \pi_i) \mid i < \mu\}$.*
2. *If G is generic for $\mathcal{Q}_{\vec{C}}$, then $F = \bigcup\{f \mid (\exists \pi)((f, \pi) \in G)\}$ is the enumerating function of a club $D \subseteq (\kappa^+)^{\mathbf{V}}$ such that*
   (a) *$\{\alpha < \kappa \mid \{C_\delta(\alpha+1), C_\delta(\alpha+2)\} \subseteq D\}$ is bounded in δ for every δ, and such that*
   (b) *if, in addition, $\mathrm{cf}(C_\delta(\alpha+1)) < \kappa$ for all $\alpha < \kappa$, then $\{\alpha < \kappa \mid C_\delta(\alpha+1) \in D\}$ is bounded in δ for every δ.*

Proof. The first part of Lemma 3.6 follows immediately from Fact 3.2. The second part is a consequence of Lemma 3.4 together with clause (3) in the definition of $\mathcal{Q}$–condition: By parts (b) and (c) of Lemma 3.4, F is the enumerating function of a club of κ^+. By part (a), if $\xi \in \kappa^+ \cap \mathrm{cf}(\kappa)$, then for a tail of $\zeta < \xi$ it holds that if

(i) $\sigma_0^\zeta = \max(C_{F(\xi)} \cap F(\zeta))$ in case $\max(C_{F(\xi)} \cap F(\zeta))$ exists,
(ii) $\sigma_1^\zeta = \min\{\sigma \in C_{F(\xi)} \mid \sigma \geq F(\zeta)\}$, and
(iii) $\sigma_2^\zeta = \min\{\sigma \in C_{F(\xi)} \mid \sigma > \sigma_1^\zeta\}$,

then,

(A0) if σ_0^ζ exists and is a successor point of $C_{F(\xi)}$, then there is some $\zeta_0 < \zeta$ such that $F(\zeta_0) < \sigma_0^\zeta$ and $F(\zeta_0 + 1) > \sigma_0^\zeta$,

(A1) if ζ is a limit ordinal with $\mathrm{cf}(\zeta) < \kappa$ and σ_1^ζ is a successor point of $C_{F(\xi)}$, then $F(\zeta) < \sigma_1^\zeta$, and

(A2) there is some $\zeta_2 \geq \zeta$ with $F(\zeta_2) < \sigma_2^\zeta$ and $F(\zeta_2 + 1) > \sigma_2^\zeta$.

Let ζ be in this tail. It suffices to show that

(1) $\min(C_{F(\xi)} \setminus (F(\zeta)+1)) \notin D$ if $F(\zeta)$ is a successor point of $C_{F(\xi)}$, that

(2) $\max(C_{F(\xi)} \cap F(\zeta)) \notin D$ if $F(\zeta)$ is a double successor point of $C_{F(\xi)}$, and that

(3) $F(\zeta)$ is not a successor point of $C_{F(\xi)}$ if $\mathrm{cf}(C_{F(\xi)}(\alpha)) < \kappa$ for all $\alpha < \kappa$.

For (1), note that if $F(\zeta)$ is a successor point of $C_{F(\xi)}$, then $F(\zeta) = \sigma_1^\zeta$ and $\min(C_{F(\xi)} \setminus (F(\zeta) + 1)) = \sigma_2^\zeta$. But by (A2), $\sigma_2^\zeta \notin D$. For (2), we have that if $F(\zeta)$ is a double successor point of $C_{F(\xi)}$, then $\max(C_{F(\xi)} \cap F(\zeta)) = \sigma_0^\zeta$ exists and is a successor point of $C_{F(\xi)}$. But then $\sigma_0^\zeta \notin D$ by (A0). Finally, for (3), if $\mathrm{cf}(C_{F(\xi)}(\alpha)) < \kappa$ for all α and $F(\zeta)$ is a successor point of $C_{F(\xi)}$, then $\sigma_1^\zeta = F(\zeta)$. But then ζ is a limit ordinal of cofinality less than κ, and therefore $F(\zeta) < \sigma_1^\zeta$ by (A1), which is a contradiction. □

It is worth pointing out that if $\vec{C} = \langle C_\delta \mid \delta \in \kappa^+ \cap \mathrm{cf}(\kappa)\rangle$ is a club–sequence such that $\mathrm{cf}(C_\delta(\alpha+1)) = \kappa$ for all α, then the corresponding form of (3) in the above proof cannot be derived; in fact we cannot rule out that there are ζ which are not limit ordinals of cofinality less than κ such that $F(\zeta)$ is a successor point of $C_{F(\xi)}$. Such ζ will typically come from extending a $\mathcal{Q}_{\vec{C}}$–condition (f, π) by adding to f the pair (ζ, ζ) as in Lemma 3.4 (c) (where ζ has cofinality κ, $f``\zeta \subseteq \zeta$, $\mathrm{rank}^\kappa(\zeta) = \kappa+\zeta = \zeta$, and so on). This situation will come up in the proof of Lemma 3.12 and is the reason why the proof of Theorem 1.6 cannot be adapted to the construction of a model in which for every $\vec{C}$ as above there is a club $D \subseteq \kappa^+$ such that $\{\alpha < \kappa \mid C_\delta(\alpha + 1) \in D\}$ is bounded for all δ. And of course we know by Theorem 1.1 that such a construction is impossible.

3.2. **The main construction.** To start with, let $(S_\rho)_{\rho<\kappa}$ be a partition of $\kappa^+ \cap \mathrm{cf}(\kappa)$ into stationary sets and let $\Phi : \theta \longrightarrow H(\theta)$ be such that for every $a \in H(\theta)$, $\Phi^{-1}(a) \subseteq \theta$ is unbounded. Φ exists by $2^{<\theta} = \theta$. For every $a \in H(\theta)$ let also $(X_\rho^a)_{\rho<\kappa}$ be a partition of $\Phi^{-1}(a)$ into unbounded subsets of θ.

The poset $\mathcal{P}$ witnessing Theorem 1.6 will be $\mathcal{P}_\theta$, where $\langle \mathcal{P}_\alpha \mid \alpha \leq \theta \rangle$ is the sequence of posets to be defined soon.

Given an ordered pair $q = (F, \Delta)$, I will sometimes refer to F and Δ as F_q and Δ_q, respectively. If F_q is a function, $\gamma \in \mathrm{dom}(F_q)$, and $F_q(\gamma)$ is also an ordered pair, I will tend to write $F_q(\gamma)$ as $(f_q(\gamma), \pi_q(\gamma))$. These conventions will be typically applied to conditions or to ordered pairs in the process of becoming conditions.

Fix now $\alpha \leq \theta$ and suppose $\mathcal{P}_\beta$ has been defined for all $\beta < \alpha$. A condition in $\mathcal{P}_\alpha$ is an ordered pair $q = (F, \Delta)$ with the following properties.

(1) $F \subseteq \alpha \times [H(\kappa^+)]^{<\kappa}$ is a function of size less than κ and such that for all $\gamma \in \mathrm{dom}(F)$, $F(\gamma)$ is of the form $(f(\gamma), \pi(\gamma))$, where $f(\gamma) \subseteq \kappa^+ \times \kappa^+$ is a κ–approximation and $\pi(\gamma)$ is a function with $\mathrm{dom}(\pi(\gamma)) \subseteq \mathrm{dom}(f(\gamma)) \cap \mathrm{cf}(\kappa)$ and $\pi(\gamma)(\nu) \in f(\gamma)(\nu)$ for all $\nu \in \mathrm{dom}(\pi(\gamma))$.

(2) Δ is such that
  (i) Δ is a binary relation with $\mathrm{dom}(\Delta)$ a symmetric κ–Φ–system of elementary substructures of $H(\theta)$, and
  (ii) every member of Δ is of the form (N, τ), where $\tau \leq \alpha$ is an ordinal which is in N or is a limit point of ordinals in N.

(3) $q|_\beta \in \mathcal{P}_\beta$ for all $\beta < \alpha$.

(4) Suppose $\beta = \alpha + 1$. If $\Phi(\beta)$ is not a $\mathcal{P}_\beta$–name, then let $\Phi^*(\beta)$ be, say, a $\mathcal{P}_\beta$–name for the Φ–first club–sequence of the form $\langle C_\delta \mid \delta \in (\kappa^+ \cap \mathrm{cf}(\kappa))^{\mathbf{V}} \rangle$ with $\mathrm{ot}(C_\delta) = \kappa$ for all δ. If $\Phi(\beta)$ is a $\mathcal{P}_\beta$–name, then let $\Phi^*(\beta)$ be a $\mathcal{P}_\beta$–name such that $\mathcal{P}_\beta$ forces that $\Phi^*(\beta)$ is a club–sequence as above and that $\Phi^*(\beta) = \Phi(\beta)$ if $\Phi(\beta)$ is such a club–sequence. Also, for every $\delta \in \kappa^+ \cap \mathrm{cf}(\kappa)$, let $\dot{C}^\beta_\delta$ be the canonical name for the δ–th member of $\Phi^*(\beta)$. If $\beta \in \mathrm{dom}(F)$, then the following holds.
  (A) $q|_\beta \Vdash_{\mathcal{P}_\beta} (f(\beta), \pi(\beta)) \in \mathcal{Q}_{\Phi^*(\beta)}$.
  (B) For every N, if $(N, \alpha) \in \Delta_q$, then $\delta_N \in \mathrm{dom}(f(\beta))$ and $\delta_N = f(\beta)(\delta_N)$.
  (C) For every $\nu \in \mathrm{dom}(f(\beta))$, if $\beta \in X^{\Phi(\beta)}_\rho$ and $f(\beta)(\nu) \notin S_\rho$, then $\nu \notin \mathrm{dom}(\pi(\beta))$.

Given two $\mathcal{P}_\alpha$–conditions q, q', q' *extends* q if and only if

- $\mathrm{dom}(F_q) \subseteq \mathrm{dom}(F'_q)$ and, for each $\gamma \in \mathrm{dom}(F_q)$, $f_q(\gamma) \subseteq f_{q'}(\gamma)$ and $\pi_p(\gamma) \subseteq \pi_{q'}(\gamma)$, and
- $\Delta_q \subseteq \Delta_{q'}$

It is clear that $\mathcal{P}_\alpha \subseteq H(\theta)$ for all α and that $\mathcal{P}_\beta \subseteq \mathcal{P}_\alpha$ for all $\beta < \alpha$. The following lemma shows that $\langle \mathcal{P}_\alpha \mid \alpha \leq \theta \rangle$ is a forcing iteration, in the sense that $\mathcal{P}_\beta$ is a complete suborder of $\mathcal{P}_\alpha$ for all $\beta < \alpha$ (Corollary 3.8). The proof is essentially identical to a corresponding proof in [2]. I include this proof here for the readers' benefit, though.

Lemma 3.7. *Let $\beta \leq \alpha \leq \theta$. Suppose $q = (F_q, \Delta_q) \in \mathcal{P}_\beta$, $r = (F_r, \Delta_r) \in \mathcal{P}_\alpha$, and $q \leq_\beta r|_\beta$. Then*

$$r \wedge q := (F_q \cup (F_r \restriction [\beta,\, \alpha)), \Delta_q \cup \Delta_r)$$

is a condition in $\mathcal{P}_\alpha$ extending r.

Proof. The proof is by induction on $\alpha \geq \beta$. The crucial point is the use of the markers τ in the definition of the forcing. New side conditions (N, τ) appearing in Δ_q may well have the property that $N \cap [\beta,\, \alpha) \neq \emptyset$, but they will not impose any problematic requirements – coming from (4) (B) in the definition – on ordinals $\gamma \in \mathrm{dom}(F_r \restriction [\beta,\, \alpha))$. The reason is simply that $\tau \leq \beta$. The details of the proof are as follows.

The case $\alpha = \beta$ of the induction is obvious, so let us start by assuming that $\alpha = \beta^* + 1$, where $\beta^* \geq \beta$. Clearly, $r \wedge q$ satisfies (1) and (2) in the definition of $\mathcal{P}_{\beta^*+1}$. By the induction hypothesis we know that the restriction of $r \wedge q$ to β^*, that is,

$$(r \wedge q)|_{\beta^*} = (F_q \cup (F_r \restriction [\beta,\, \beta^*)), \Delta_q \cup \Delta_{r|_{\beta^*}})$$

is a condition in $\mathcal{P}_{\beta^*}$ extending $r|_{\beta^*}$. Therefore, $r \wedge q$ also satisfies (3). If $\beta^* \notin \mathrm{dom}(F_r)$, then $r \wedge q$ is a condition in $\mathcal{P}_{\beta^*+1}$ since (4) is automatically satisfied. If $\beta^* \in \mathrm{dom}(F_r)$, then $F_{r\wedge q}(\beta^*) = F_r(\beta^*)$, which immediately gives (4) (C) for β^*. Also, $(r \wedge q)|_{\beta^*}$ forces in $\mathcal{P}_{\beta^*}$ that $F_r(\beta^*)$ is in $\mathcal{Q}_{\Phi^*(\beta^*)}$ (since $r|_{\beta^*}$ forces this and $(r \wedge q)|_{\beta^*}$ extends $r|_{\beta^*}$). This gives (4) (A) for $q \wedge_\beta r$ and β^* in this case. Now we check that δ_N is a fixed point of $F_r(\beta^*)$ whenever $(N, \beta^* + 1) \in \Delta_q \cup \Delta_r$. For this, note that $(N, \beta^* + 1) \in \Delta_r$. Hence δ_N is a fixed point of $F_r(\beta^*)$ by (4) for r and β^*. Finally note that the induction hypothesis and the inclusion $\Delta_r \subseteq \Delta_{r\wedge q}$ together imply that $r \wedge q$ extends r.

The case when α is a nonzero limit ordinal follows directly from the induction hypothesis. □

Corollary 3.8. *For all $\beta < \alpha \leq \kappa$, every maximal antichain in $\mathcal{P}_\beta$ is a maximal antichain in $\mathcal{P}_\alpha$, and therefore $\mathcal{P}_\beta$ is a complete suborder of $\mathcal{P}_\alpha$.*

Lemma 3.9 follows immediately from the first part of Lemma 3.6.

Lemma 3.9. *For every $\alpha \leq \theta$, $\mathcal{P}_\alpha$ is $<\kappa$–closed. In fact, if $\lambda < \kappa$ and $(q_\alpha)_{\alpha<\lambda}$ is a decreasing sequence of conditions in $\mathcal{P}_\alpha$, then $q^* =$*

$(F_{q^*}, \bigcup_{\alpha<\lambda} \Delta_{q_\alpha})$, *where* $\mathrm{dom}(F_{q^*}) = \bigcup_{\alpha<\lambda} \mathrm{dom}(F_{q_\alpha})$ *and, for all* $\xi \in \mathrm{dom}(F^*)$,

$$f_{q^*}(\xi) = \bigcup\{f_{q_\alpha}(\xi) \mid \alpha < \lambda,\, \xi \in \mathrm{dom}(F_{q_\alpha})\}$$

and

$$\pi_{q^*}(\xi) = \bigcup\{\pi_{q_\alpha}(\xi) \mid \alpha < \lambda,\, \xi \in \mathrm{dom}(F_{q_\alpha})\},$$

is a greatest lower bound of $\{q_\alpha \mid \alpha < \lambda\}$.

Under the hypotheses of Lemma 3.9, I will call the condition q^* *the canonical greatest lower bound of* $\{q_\alpha \mid \alpha < \lambda\}$.

Lemma 3.10. *For all* $\alpha \leq \theta$, $\mathcal{P}_\alpha$ *is* κ^{++}*–Knaster.*

Proof. The proof uses standard Δ–system arguments. Suppose $(q_i)_{i<\kappa^{++}}$ is a sequence of conditions in $\mathcal{P}_\alpha$. By $2^\kappa = \kappa^+$ we may assume that the collection $\{\bigcup \mathrm{dom}(\Delta_{q_i}) \mid i < \kappa^{++}\}$ forms a Δ–system with root R, i.e., for all distinct i, i', $\bigcup \mathrm{dom}(\Delta_{q_i}) \cap \bigcup \mathrm{dom}(\Delta_{q_{i'}}) = R$.

Again using $2^\kappa = \kappa^+$ we may assume as well that there are $\mu < \kappa$ and enumerations $(N^i_\varsigma)_{\varsigma<\mu}$ of Δ_{q_i} (for $i < \mu$) such that, letting $\mathfrak{N}_i = \langle \bigcup \mathrm{dom}(\Delta_{q_i}), \in, \Phi, R, N^i_\varsigma, \Phi\rangle_{\varsigma<\mu}$ for all i, we have that for all i, $i' < \kappa^{++}$, $\mathfrak{N}_i$ and $\mathfrak{N}_{i'}$ are isomorphic via a unique isomorphism that fixes R.

The first assertion follows from the fact that by $2^\kappa = \kappa^+$ there are at most κ^+–many isomorphism types for such structures. For the second assertion note that, if Ψ is the unique isomorphism between $\mathfrak{N}_i$ and $\mathfrak{N}_{i'}$, then the restriction of Ψ to $R \cap \theta$ has to be the identity on $R \cap \theta$. Since there is a bijection $\varphi : H(\theta) \longrightarrow \theta$ definable in $(H(\theta), \in, \Phi)$, we have that Ψ fixes R if and only if it fixes $R \cap \theta$, and therefore it fixes R.

We may assume as well that $\{\mathrm{dom}(F_{q_i}) \mid i < \kappa^{++}\}$ forms a Δ–system with root ρ and that for all $\gamma \in \rho$, $\delta < \kappa^+$ and all distinct i, i' in κ^{++},

(i) $\bigcup \mathrm{dom}(\Delta_{q_i}) \cap \mathrm{dom}(F_{q_{i'}}) = \bigcup \mathrm{dom}(\Delta_{q_i}) \cap \rho$,
(ii) $(f_{q_i}(\gamma), \pi_{q_i}(\gamma)) = (f_{q_{i'}}(\gamma), \pi_{q_{i'}}(\gamma))$, and
(iii) there is some N such that $(N, \gamma+1) \in \Delta_{q_i|_{\gamma+1}}$ and $\delta_N = \delta$ if and only if there is some N such that $(N, \gamma+1) \in \Delta_{q_{i'}|_{\gamma+1}}$ and $\delta_N = \delta$.

Let $i < i' < \kappa^{++}$. Using the above paragraph, together with the fact that $\bigcup \mathrm{dom}(\Delta_{q_i}) \cap \bigcup \mathrm{dom}(\Delta_{q_{i'}})$ is a symmetric κ–Φ–system by Lemma 2.4, one can easily verify that q_i and $q_{i'}$ are compatible as witnessed by $(F_{q_i} \cup F_{q_{i'}}, \Delta_{q_i} \cup \Delta_{q_{i'}})$. In fact it is not difficult to see by induction on $\beta \leq \alpha$ that the restriction of the pair $(F_{q_i} \cup F_{q_{i'}}, \Delta_{q_i} \cup \Delta_{q_{i'}})$ to β is a condition in $\mathcal{P}_\beta$. $\square$

Lemma 3.11. $\mathcal{P}$ *forces* $2^\lambda = \theta$ *for every* $\mathbf{V}$*–cardinal* $\lambda \in [\kappa, \theta)$.

Proof. It is easy to see that $\mathcal{P}$ adds at least θ–many Cohen subsets of κ. For this, let G be $\mathcal{P}$–generic. Given $\gamma < \theta$, let $f^G_\gamma : (\kappa^+)^{\mathbf{V}} \longrightarrow (\kappa^+)^{\mathbf{V}}$ be the function added by G at the γ-th coordinate, i.e., the union of the functions $f_q(\gamma)$ for $q \in G$ with $\gamma \in \mathrm{dom}(F_q)$. Fix $X_\gamma \subseteq f^G_\gamma(\kappa)$, $X_\gamma \in \mathbf{V}$, such that for every $\eta < f^G_\gamma(\kappa)$ and every $\nu < \kappa$ there are, both in X_γ and in $f^G_\gamma(\kappa) \backslash X_\gamma$, unboundedly many ordinals σ below $f^G_\gamma(\kappa)$ such that $\mathrm{rank}^\lambda(\sigma) \geq \kappa + \nu$. Then, $A_\gamma := \{\nu < \kappa \,:\, f^G_\gamma(\nu + 1) \in X_\gamma\}$ is a Cohen subset of κ by a straightforward density argument. Furthermore, by another density argument, $A_\gamma \neq A_{\gamma'}$ for all distinct γ, γ'.

For the other inequality, note that for every cardinal $\lambda \in [\kappa^+, \theta)$ there are not more than θ^λ–many nice names for subsets of λ by the κ^{++}–c.c. of $\mathcal{P}$. But $\theta^\lambda = \theta$. $\square$

Next comes a crucial properness lemma. Many of the features of our construction are there precisely to make this lemma work. Lemma 3.12 shows that $\mathcal{P}$ is proper with respect to all the relevant submodels. As mentioned shortly in the introduction, there is no general preservation theorem for properness with respect to any reasonable class of uncountable submodels. It is therefore not surprising that, unlike in most proofs of properness in the context of iterated forcing relative to countable submodels (in particular the proofs of properness in [2] and [3]), the proof of Lemma 3.12 is not by induction,[7] but instead proceeds by giving a direct[8] construction.

Lemma 3.12. *$\mathcal{P}$ is proper with respect to all $N^* \preccurlyeq H((2^\theta)^+)$ such that $\mathcal{P}$, $\Phi \in N^*$, ${}^{<\kappa}N^* \subseteq N^*$ and $|N^*| = \kappa$.*

Proof. Let $N = N^* \cap H(\theta)$, and note that ${}^{<\kappa}N \subseteq N$. Let $q \in N \cap \mathcal{P}$. Let F^* be the function with the same domain as F_q such that $F^*(\gamma) = (f_q(\gamma) \cup \{(\delta_N, \delta_N)\}, \pi_q(\gamma))$ for all $\gamma \in \mathrm{dom}(F_q)$ and let q^* be given by $q^* = (F^*, \Delta_q \cup \{(N, \mathsf{sup}(N \cap \theta))\})$. q^* is clearly a condition in $\mathcal{P}$ extending q. Hence it will be enough to show that q^* is $(N^*, \mathcal{P})$–generic.

For this, let $A \in N^*$ be a maximal antichain of $\mathcal{P}$, and let q^1 be a condition in $\mathcal{P}$ extending both q^* and a condition $t \in A$. Note that A is in fact in N by the κ^{++}–chain condition of $\mathcal{P}$. The goal now is of course to see that $t \in N^*$, and for this it will suffice to show that there is a condition in $A \cap N$ compatible with t.

[7]More precisely, it is not of the form "Assume, for an arbitrarily given α, that all $\mathcal{P}_\beta$ (for $\beta < \alpha$) satisfy the relevant form of properness, and then argue that $\mathcal{P}_\alpha$ satisfies it as well."

[8]The construction is certainly direct but it is, perhaps, also the hardest part of the proof of Theorem 1.6.

We may extend q^1 to a condition q^2 such that for every $\gamma \in \mathrm{dom}(F_t) \cap N$,

(i) $q^2|_\gamma$ forces that $\mathsf{sup}(\mathrm{range}(f_{q^2}(\gamma) \restriction \delta_N))$ is a limit point of $\dot{C}^\gamma_{\delta_N}$, and
(ii) there is an ordinal $\eta_\gamma < \delta_N$ such that $q^2|_\gamma$ forces $\dot{C}^\gamma_\delta \cap \delta_N \subseteq \eta_\gamma$ for every $\delta \in \mathrm{range}(f_{q^2}(\gamma))$ of cofinality κ such that $\delta > \delta_N$.

q^2 can be taken to be the canonical greatest lower bound of a decreasing sequence $(q^2_n)_{n<\omega}$ of conditions extending q^1 such that for all n it holds that

(iii) for all $\gamma \in \mathrm{dom}(F_t) \cap N$, $q^2_{n+1}|_\gamma$ forces that the supremum of $\mathrm{range}(f_{q^2_{n+1}}(\gamma) \restriction \delta_N)$ is a limit point of $\dot{C}^\gamma_{\delta_N}$, and
(iv) for all $\gamma \in \mathrm{dom}(F_{q^2_n})$, there is an ordinal $\eta < \delta_N$ for which $q^2_{n+1}|_\gamma$ forces $\dot{C}^\gamma_\delta \cap \delta_N \subseteq \eta$ for every $\delta \in \mathrm{range}(f_{q^2_n}(\gamma))$ of cofinality κ such that $\delta > \delta_N$.

Given q^2_n, q^2_{n+1} can be found by first extending q^2_n to a condition r satisfying (iv) – which exists since every $\mathcal{P}_\gamma$ forces both that $\mathrm{cf}(\delta_N) = \kappa$ (since it is $<\kappa$–closed) and that, for all δ, all limit points of $\dot{C}^\gamma_\delta$ below δ have cofinality less than κ – and then extending r to a condition q^2_{n+1} satisfying (iii) as well. Let $(\gamma_i)_{i<\mu}$, for some limit ordinal $\mu < \kappa$, be an enumeration of $\mathrm{dom}(t) \cap N$ such that every member of $\mathrm{dom}(t) \cap N$ is γ_i for unboundedly many $i < \mu$ (we may assume $\mathrm{dom}(t) \cap N \neq \emptyset$ as otherwise the proof is easier). Then q^2_{n+1} can be taken to be any lower bound of a decreasing sequence $(r_i)_{i<\mu}$ of conditions extending r such that for all i, $r_i|_{\gamma_i}$ forces that $\mathsf{sup}(\mathrm{range}(f_{r_i}(\gamma_i) \restriction \delta_N))$ is a limit point of $\dot{C}^{\gamma_i}_{\delta_N}$. Given i and $(r_{i'})_{i'<i}$, r_i can be found by first finding a lower bound r' of $(r_{i'})_{i'<i}$ and then extending r' in at most ω stages (as in the proof of Lemma 3.4).

Let now $\rho < \kappa$ be such that $\gamma \notin X^{\Phi(\gamma)}_\rho$ for every $\gamma \in \mathrm{dom}(F_t) \cap N$. Since $H(\theta^+) \in N^*$, we can pick $M^* \in N^*$ of size κ such that ${}^{<\kappa}M^* \subseteq M^*$, $\delta_{M^*} \in S_\rho$, $M^* \preccurlyeq H(\theta^+)$, $\mathsf{sup}\{\eta_\gamma \mid \gamma \in \mathrm{dom}(F_{q^2}) \cap N\} < \delta_M$, and such that M^* contains $\mathcal{P}$, Φ, A, $f_{q^2}(\gamma) \restriction \delta_N$ for all $\gamma \in N \cap \mathrm{dom}(F_{q^2})$, and $\mathrm{dom}(\Delta_{q^2}) \cap N$. Let $M = M^* \cap H(\theta)$ and note that ${}^{<\kappa}M \subseteq M$ and that, since $\{H(\theta), \Phi\} \in M^*$, $(M, \in, \Phi) \preccurlyeq (H(\theta), \in, \Phi)$.

Claim 3.13. *There is a condition q^4 stronger than q^2 such that*

(1) $(M, \mathsf{sup}(M \cap \theta)) \in \Delta_{q^4}$ *and*
(2) *there is some $\eta < \delta_M$ such that for every $\gamma \in \mathrm{dom}(F_t) \cap M$, $q^4|_\gamma$ forces $\max(\dot{C}^\gamma_\delta \cap \delta_M) < \eta$ for every $\delta \in \mathrm{range}(f_{q^4}(\gamma))$ of cofinality κ such that $\delta > \delta_M$.*

Proof. By extending q^2 slightly if necessary using Lemma 2.3 (ii) we may assume that $(M, 0) \in \Delta_{q^2}$: Since $\mathrm{dom}(\Delta_{q^2}) \cap N \in M$ is a symmetric system by Lemma 2.3 (i), $(\mathrm{dom}(\Delta_{q^2}) \cap N) \cup \{M\} \in N$ is a symmetric system. By Lemma 2.3 (ii) there is a symmetric Φ–system $\mathcal{M} \supseteq \mathrm{dom}(\Delta_{q^2}) \cup \{M\}$. But now $(F_{q^2}, \Delta_{q^2} \cup \{(M', 0) \mid M' \in \mathcal{M}\})$ is a condition extending q^2 of the specified form.

We may find a condition q^3 stronger than q^2 and such that the pair $(N, \mathsf{sup}(M \cap \theta))$ is in Δ_{q^3}. This condition q^3 can be built by recursion on $\mathrm{dom}(F_{q^2}) \cap M$ using the fact that $\mathcal{P}$ is $<\kappa$–closed. The details are as follows:

q^3 is the result of taking any lower bound r^* of a certain decreasing sequence $(r_i)_{i<\bar{\mu}}$, for $\bar{\mu} < \kappa$, of conditions extending q_2, and adding $(M, \mathsf{sup}(M \cap \theta))$ to its Δ.

Let $(\bar{\gamma}_i)_{i<\bar{\mu}}$ be the strictly increasing enumeration of $\mathrm{dom}(F_{q^2}) \cap M$ (which without loss of generality we may assume nonempty). The sequence $(r_i)_{i<\bar{\mu}}$ is built using $(\bar{\gamma}_i)_{i<\bar{\mu}}$. At any given stage i of the construction, suppose that we are handed a decreasing sequence $(r_{i'})_{i'<i}$ of conditions such that $(M, \bar{\gamma}_{i'} + 1) \in \Delta_{r_{i'}|_{\bar{\gamma}_{i'}+1}}$ for all i'. Let $r = q^2$ if $i = 0$, let $r = r_{i_0}$ if $i = i_0 + 1$, and let r be any lower bound of $(r_{i'})_{i'<i}$ if i is a nonzero limit ordinal. Let $\bar{r} = (F_r, \Delta_r \cup \{(M, \bar{\gamma}_i)\})$. Using the fact that $(M, \bar{\gamma}_{i'} + 1) \in \Delta_{r_{i'}|_{\bar{\gamma}_{i'}+1}}$ for all $i' < i$ and that $\mathrm{dom}(F_r)$ has empty intersection with the interval $[\mathsf{sup}\{\bar{\gamma}_{i'} + 1 \mid i' < i\}, \bar{\gamma}_i)$, it is easy to check that $\bar{r}$ is indeed a condition. By Lemma 3.4 (c) we may extend $\bar{r}|_{\bar{\gamma}_i}$ to a condition $\bar{r}' \in \mathcal{P}_{\bar{\gamma}_i}$ for which there is a function $f \supseteq f_{q_2(\bar{\gamma}_i)}$ in $\mathbf{V}$ such that δ_M is a fixed point of f and $\bar{r}'$ forces $(f, \pi_{q_2(\bar{\gamma}_i)}) \in \mathcal{Q}_{\Phi^*(\bar{\gamma}_i)}$. Let now F be the function with domain $\mathrm{dom}(F_{\bar{r}'}) \cup \mathrm{dom}(F_{q_2})$ such that $F \restriction \bar{\gamma}_i = F_{\bar{r}'}$, $F \restriction (\bar{\gamma}_i, \theta) = F_{q^2} \restriction (\bar{\gamma}_i, \theta)$ and $F(\bar{\gamma}_i) = (f, \pi_{q_2(\bar{\gamma}_i)})$, and let $r_i = (F, \Delta_{\bar{r}'} \cup \{(M, \bar{\gamma}_i + 1)\})$. It is easy to verify that r_i is a $\mathcal{P}$–condition extending $\bar{r}'$, and that the final move of going from a lower bound r^* of $(r_i)_{i<\bar{\mu}}$ to q^3 yields also a $\mathcal{P}$–condition.

Using again that $\mathrm{cf}(\delta_M) = \kappa$ holds in all extensions by all $\mathcal{P}_\gamma$ and that, for every relevant club–sequence $\langle C_\delta \mid \delta \in (\kappa^+ \cap \mathrm{cf}(\kappa))^{\mathbf{V}} \rangle$, every limit point of every C_δ below δ is forced to have cofinality less than κ, we may now extend q^3 to a condition q^4 for which there is an ordinal $\eta < \delta_M$ such that $\eta > \delta_Q$ for every $Q \in \mathrm{dom}(\Delta_{q^4})$ with $\delta_Q < \delta_M$, and such that for every $\gamma \in \mathrm{dom}(F_t) \cap M$ and every $\delta \in \mathrm{range}(f_{q^4}(\gamma))$ of cofinality κ such that $\delta > \delta_M$, $q^4|_\gamma$ forces $\max(\dot{C}^\gamma_\delta \cap \delta_M) < \eta$. This condition q^4 can be obtained as in the above constructions. $\square$

Let now q^4 be given by the above claim, and for any $\gamma \in \mathrm{dom}(F_t) \setminus M$ let

$$\beta_\gamma = \min(((M \cap \theta) \cup \{\theta\}) \setminus \gamma)$$

and

$$\alpha_\gamma = \mathsf{sup}\{\mathsf{sup}(Q \cap \beta_\gamma) \mid Q \in \mathrm{dom}(\Delta_{q^4}) \cap M\}$$

Note that α_γ is in M, and that β_γ is also in M if $\beta_\gamma < \theta$. Note also that $\mathrm{cf}(\beta_\gamma) > \kappa$, and that therefore $\alpha_\gamma < \beta_\gamma$ since every $Q \in \mathrm{dom}(\Delta_{q^4}) \cap M$ has size κ and since $\mathrm{dom}(\Delta_{q^4}) \cap M \in M$ has size less than κ, and that in fact we have that $\alpha_\gamma < \gamma$ by the choice of β_γ.

Claim 3.14. *$Q \cap [\alpha_\gamma, \beta_\gamma] \cap M = \emptyset$ for every $\gamma \in \mathrm{dom}(F_t) \setminus M$ and every $Q \in \mathrm{dom}(\Delta_{q^4})$ such that $\delta_Q < \delta_M$.*

Proof. Let $\gamma \in \mathrm{dom}(F_t) \setminus M$ and suppose $Q \in \mathrm{dom}(\Delta_{q^4})$ is such that $\delta_Q < \delta_M$. Then, by Fact 2.2 there is some $Q' \in \mathrm{dom}(\Delta_{q^4}) \cap M$ such that $\delta_{Q'} = \delta_Q$ and such that $Q \cap M = Q \cap Q'$. Then $\zeta \in Q \cap [\alpha_\gamma, \beta_\gamma] \cap M$ would imply $\zeta \in [\alpha_\gamma, \beta_\gamma] \cap Q'$, but this would contradict the choice of α_γ. $\square$

Working now in M^*, we may find a condition $t^* \in A$ satisfying the following.

- (a) There is a function $\varphi : \mathrm{dom}(F_t) \setminus M \longrightarrow M \cap \theta$ such that
  - (a1) for all $\gamma \in \mathrm{dom}(F_t) \setminus M$, $\varphi(\gamma) \in [\alpha_\gamma, \beta_\gamma] \setminus (\mathrm{dom}(q^4) \cap M)$, and
  - (a2) $\mathrm{dom}(F_{t^*}) = (\mathrm{dom}(F_t) \cap M) \cup \mathrm{range}(\varphi)$.
- (b) For every $\gamma \in \mathrm{dom}(F_{t^*})$,
  - (b1) $\delta_{N'}$ is a fixed point of $f_{t^*}(\gamma)$ whenever $(N', \tau) \in \Delta_{q^4} \cap M$ is such that $\gamma \in N'$ and $\tau \geq \gamma$, and
  - (b2) $f_{t^*}(\gamma) \restriction \eta = f_t(\gamma) \restriction \delta_M$ and $\pi_{t^*}(\gamma) \restriction \eta = \pi_t(\gamma) \restriction \delta_M$ if $\gamma \in \mathrm{dom}(F_t)$.
- (c) There is a symmetric κ–Φ–system $\mathcal{N}$ such that $\mathcal{N} \supseteq \mathrm{dom}(\Delta_{t^*}) \cup \mathrm{dom}(\Delta_{q^4} \cap M)$.

This condition $t^* \in A$ may be found by the correctness of M^* in $H(\theta^+)$ since the existence of a condition $t^* \in A$ satisfying (a)–(c) can be expressed in $H(\theta^+)$ by a sentence σ with a certain parameter $p \in [M^*]^{<\kappa}$ which is in M^* by the closure of M^* under $<\kappa$–sequences, and such that t witnesses the truth of σ.

By the existence of $\mathcal{N}^*$ as in (c), we get that by Lemma 2.3 (ii) there is a condition $\bar{q}^4$ extending q^4 of the form (F_{q^4}, Δ) and such that $\mathcal{N}^* \subseteq \mathrm{dom}(\Delta_{\bar{q}^4})$.[9] It remains to see that $\bar{q}^4$ and t^* are compatible. For this, I am going to build a common extension q^5 of $\bar{q}^4$ and t^* by recursion on $\mathrm{dom}(F_{t^*})$. The construction of q^5 is along the lines of the construction of q^3 from q^2 in the proof of Claim 3.13. Let $(\gamma_i^*)_{i<\mu^*}$, for some $\mu^* < \kappa$, be the strictly increasing enumeration of $\mathrm{dom}(F_{t^*})$, which

[9] We already saw this argument in the proof of Claim 3.13.

we may assume is nonempty. We build a certain decreasing sequence $(q^5_i)_{i<\mu^*}$ of conditions such that for all i,

(i) q^5_i is a condition in $\mathcal{P}$ extending $\bar{q}_4$,
(ii) q^5_i is of the form $\bar{q}^4 \wedge r$ for some $r \in \mathcal{P}_{\gamma^*_i+1}$, and
(iii) $q^5_i|_{\gamma^*_i+1}$ $(= r)$ extends $t^*|_{\gamma^*_i+1}$.

At a given stage i of the construction we are handed a decreasing sequence $(q^5_{i'})_{i'<i}$ of conditions such that each $q^5_{i'}$ satisfies (i)–(iii). We let $r = (F_{\bar{q}^4}, \Delta_{\bar{q}^4} \cup \Delta_{t^*|_{\gamma_0}})$ if $i = 0$, $r = q^5_{i'}$ if $i = i' + 1$, and let r be the greatest lower bound of $\{q^5_{i'} \mid i' < i\}$ if i is a nonzero limit ordinal. It is easily checked that r is indeed a condition in $\mathcal{P}$ in each case. We consider now the following two cases.

Suppose first $\gamma^*_i \in \mathrm{dom}(F_t)$. Then $r|_{\gamma^*_i}$ forces that

$$p = (f_{q^4}(\gamma^*_i) \cup f_{t^*}(\gamma^*_i), \pi_{q^4}(\gamma^*_i) \cup \pi_{t^*}(\gamma^*_i))$$

is a condition in $\mathcal{Q}_{\Phi^*(\gamma^*_i)}$ by (b) in the choice of t^* together with Lemma 3.5 and together with the fact that $\delta_M \notin \mathrm{dom}(\pi_{q^4}(\gamma^*_i))$ by (4) (C) in the definition of $\mathcal{P}_{\gamma^*_i+1}$ for $q^4|_{\gamma^*_i+1}$ and γ_i since $f_{q^4}(\gamma^*_i)(\delta_M) = \delta_M \in S_\rho$ and $\gamma^*_i \notin X^{\Phi(\gamma^*_i)}_\rho$. It follows that, letting F be the function with the same domain as F_r and such that $F \restriction \gamma^*_i = F_r \restriction \gamma^*_i$, $F(\gamma^*_i) = p$, and $F \restriction (\gamma^*_i, \theta) = F_r \restriction (\gamma^*_i, \theta)$, $q^5_i := (F, \Delta_r)$ is a condition in $\mathcal{P}$ extending r and is such that $q^5_i \restriction \gamma^*_i + 1$ extends $t^*|\gamma^*_i + 1$.

The other case is when $\gamma^*_i \notin \mathrm{dom}(F_t)$. Note that also $\gamma^*_i \notin \mathrm{dom}(F_{q^4})$ in this case by the choice of $\varphi(\gamma)$ in (a1). By Lemma 3.4 (c) together with the $<\kappa$–closure of $\mathcal{Q}_{\Phi^*(\gamma^*_i)}$ in $\mathbf{V}^{\mathcal{P}_{\gamma^*_i}}$ (Lemma 3.6 (1)), there is a function $f \supseteq f_{t^*}(\gamma^*_i)$ and an extension s of $r \restriction \gamma^*_i$ such that

(i) δ_Q is a fixed point of f for every Q such that $(Q, \gamma^*_i + 1) \in \Delta_{\bar{q}^4|_{\gamma^*_i+1}}$ and $\delta_Q \geq \delta_M$, and
(ii) r forces $(f, \pi) \in \mathcal{Q}_{\Phi^*(\gamma^*_i)}$.

Since $\gamma^*_i \notin Q$ for any Q such that $(Q, \gamma^*_i + 1) \in \Delta_{\bar{q}^4|_{\gamma^*_i+1}}$ and $\delta_Q < \delta_M$ by Claim 3.14, we have that δ_Q is a fixed point of f for every Q such that $(Q, \gamma^*_i + 1) \in \Delta_{\bar{q}^4|_{\gamma^*_i+1}} \cup \Delta_{t^*|_{\gamma^*_i+1}}$. Now we can amalgamate the relevant objects into a condition q^5_i as in the previous case.

Finally we can take q^5 to be any lower bound of $(q^5_i)_{i<\mu^*}$. □

By Lemmas 3.9, 3.10 and 3.12, $\mathcal{P}$ does not collapse cofinalities. In particular, κ^+ and $\kappa^+ \cap \mathrm{cf}(\kappa)$ have the same meaning in $\mathbf{V}$ as in any generic extension by any $\mathcal{P}_\alpha$, which means that the statements of the following lemmas are not ambiguous.

The proof of the following lemma is similar to the proof of Lemma 3.4 by standard density arguments.

Lemma 3.15. *Suppose $\beta < \theta$, G is a $\mathcal{P}_{\beta+1}$–generic filter over $\mathbf{V}$, $\langle C_\delta \mid \delta \in \kappa^+ \cap \mathrm{cf}(\kappa)\rangle$ is the interpretation of $\Phi^*(\beta)$ by the restriction G_β of G to $\mathcal{P}_\beta$, and D is the union of all sets of the form* $\mathrm{range}(f_q(\beta))$, *where $q \in G$ and $\beta \in \mathrm{dom}(F_q)$. Then the following holds.*

(a) *D is a club of κ^+.*
(b) *If $\langle C_\delta \mid \delta \in \kappa^+ \cap \mathrm{cf}(\kappa)\rangle = \Phi(\beta)_{G_\beta}$ and $\rho < \kappa$ is such that $\beta \in X_\rho^{\Phi(\beta)}$, then for every $\delta \in D \cap S_\rho$,*
  (1) *$\{\alpha < \delta \mid \{C_\delta(\alpha+1), C_\delta(\alpha+2)\} \subseteq D\}$ is bounded in δ, and*
  (2) *if, in addition, $\mathrm{cf}(C_\delta(\alpha+1)) < \kappa$ for all $\delta \in \kappa^+ \cap \mathrm{cf}(\kappa)$ and $\alpha < \kappa$, then $\{\alpha < \delta \mid C_\delta(\alpha+1) \in D\}$ is bounded in δ.*

Lemma 3.16. *$\mathcal{P}$ forces that if $\langle C_\delta \mid \delta \in \kappa^+ \cap \mathrm{cf}(\kappa)\rangle$ is a club–sequence with $\mathrm{ot}(C_\delta) = \kappa$ for all δ, then there is a club $D \subseteq \kappa^+$ such that*

(1) *$\{\alpha < \kappa \mid \{C_\delta(\alpha+1), C_\delta(\alpha+2)\} \subseteq D\}$ is bounded for every δ and such that*
(2) *if, in addition, $\mathrm{cf}(C_\delta(\alpha+1)) < \kappa$ for all $\delta \in \kappa^+ \cap \mathrm{cf}(\kappa)$ and $\alpha < \kappa$, then $\{\alpha < \delta \mid C_\delta(\alpha+1) \in D\}$ is bounded in δ for every δ.*

Proof. Let G be $\mathcal{P}$–generic and, for every $\beta < \theta$, let D_β be the union of all sets of the form $\mathrm{range}(f_q(\beta))$, where $q \in G$ and $\beta \in \mathrm{dom}(F_q)$. Let $\vec{C} = \langle C_\delta \mid \delta \in \kappa^+ \cap \mathrm{cf}(\kappa)\rangle \in \mathbf{V}[G]$ be a club–sequence such that $\mathrm{ot}(C_\delta) = \kappa$ for all δ. By the κ^{++}–c.c. of $\mathcal{P}$ in $\mathbf{V}$ we may find some $\mathcal{P}$–name $\dot{x} \in H(\theta)^{\mathbf{V}}$ such that $\dot{x}_G = \vec{C}$. Then, for every $\beta < \theta$ such that $\Phi(\beta) = \dot{x}$, if $\rho < \kappa$ is such that $\beta \in X_\rho^{\dot{x}}$, then by Lemma 3.15 D_β is a club on κ^+ such that for every $\delta \in D_\beta \cap S_\rho$,

(1) $\{\alpha < \delta \mid \{C_\delta(\alpha+1), C_\delta(\alpha+2)\} \subseteq D_\beta\}$ is bounded in δ, and
(2) if, in addition, $\mathrm{cf}(C_\delta(\alpha+1)) < \kappa$ for all $\delta \in \kappa^+ \cap \mathrm{cf}(\kappa)$ and $\alpha < \kappa$, then $\{\alpha < \delta \mid C_\delta(\alpha+1) \in D_\beta\}$ is bounded in δ.

Now let $(\beta_\rho)_{\rho<\kappa}$ be such that $\beta_\rho \in X_\rho^{\dot{x}}$ for all ρ. Then, if $D = \bigcap_{\rho<\kappa} D_{\beta_\rho}$, D is a club of κ^+ and, since $\{S_\rho : \rho < \kappa\}$ is a partition on $\kappa^+ \cap \mathrm{cf}(\kappa)$, we have that for every $\delta \in D$,

(1) $\{\alpha < \delta \mid \{C_\delta(\alpha+1), C_\delta(\alpha+2)\} \subseteq D\}$ is bounded in δ, and
(2) if, in addition, $\mathrm{cf}(C_\delta(\alpha+1)) < \kappa$ for all $\delta \in \kappa^+ \cap \mathrm{cf}(\kappa)$ and $\alpha < \kappa$, then $\{\alpha < \delta \mid C_\delta(\alpha+1) \in D\}$ is bounded in δ.

□

Lemma 3.16 concludes the proof of Theorem 1.6.

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