Coding into $H(\omega_2)$, together (or not) with forcing axioms. A survey.

David Asperó

Abstract

This paper is mainly a survey of recent results concerning the possibility of building forcing extensions in which there is a simple definition, over the structure $\langle H(\omega_2), \in\rangle$ and without parameters, of a prescribed member of $H(\omega_2)$ or of a well–order of $H(\omega_2)$. Some of these results are in conjunction with strong forcing axioms like PFA^{++} or MM, some are not. I also observe (Corollary 4.4) that the existence of certain objects of size $\aleph_1$ follows outright from the existence of large cardinals. This observation is motivated by an attempt to extend the PFA^{++} result to a result mentioning MM^{++}.

1 Main starting questions and some pieces of notation

The work presented here deals mostly with the problem of finding optimal definitions of well–orders of the reals and other objects. More precisely, it addresses the following two questions.[1]

Question 1: Suppose A is a subset of ω_1. Suppose we are given the task of going over to a *nice* set-forcing extension[2] in which A admits a *simple* definition $\Phi(x)$, without parameters, over the structure $\langle H(\omega_2), \in\rangle$ or over some natural (definable) extension of this structure, like $\langle H(\omega_2), \in, NS_{\omega_1}\rangle$.

[1] As quoted from [As2].

[2] So, if *nice* is to be interpreted as preserving stationary subsets of ω_1 in the ground model and A is a stationary and co-stationary subset of ω_1, then A will remain stationary and co-stationary in the extension.

What is the lowest degree of logical complexity that can be attributed to a definition $\Phi(x)$ for which we can perform the above task?

Question 2: What is the lowest degree of logical complexity of formulas for which there is a formula $\Phi(x, y)$ (again without parameters) with that complexity and with the property that we can go over to a set-forcing extension in which the set of real numbers admits a well–order defined by $\Phi(x, y)$ (again over the structure $\langle H(\omega_2), \in\rangle$ or over some natural extension of it)?

For some background on these problems the reader is referred to [As2]. Logical complexity – for formulas of a language extending the language of set theory – will be measured in this paper by the familiar Levy hierarchy $\bigcup_{n<\omega}\{\Sigma_n, \Pi_n\}$. Recall that a formula is Σ_0 (equivalently, Π_0) if all of its quantifiers are restricted[3] and that, for $n > 0$, a formula is Σ_n (respectively, Π_n) if it is of the form $(\exists x)\varphi$ for a Π_{n-1} formula φ (respectively, if it is of the form $(\forall x)\varphi$ for a Σ_{n-1} formula φ). Note that, in any model M of ZF without the Power Set Axiom, if P is a definable class in M and $\varphi(x_0, \dots x_k)$ is a formula in the language of the structure $\langle M, \in, P\rangle$, then there is some formula $\psi(x_0, \dots x_k) \in \bigcup_{n<\omega}\{\Sigma_n, \Pi_n\}$ (in the same language) such that, in $\langle M, \in, P\rangle$, $\varphi(x_0, \dots x_k)$ is logically equivalent to $\psi(x_0, \dots x_k)$.[4] In other words, the Levy hierarchy provides a classification, up to logical equivalence, of all formulas over structures $\langle M, \in, P\rangle$ as above. Also, note that, for every $n < \omega$, every formula in $\Sigma_n \cup \Pi_n$ is logically equivalent to a formula in $\Sigma_{n+1} \cap \Pi_{n+1}$.

Throughout this paper, $H(\omega_2)$ denotes the set of all sets whose transitive closure has size at most $\aleph_1$, and NS_{ω_1} denotes the nonstationary ideal on ω_1. Given a regular cardinal κ, $cf(\kappa)$ is the class of all ordinals of cofinality κ. $\mathcal{L}$ will denote the first order language of the structure $\langle H(\omega_2), \in, NS_{\omega_1}\rangle$. Given a set X of ordinals, $ot(X)$ will denote the order type of X. Recall that a partial order $\mathcal{P}$ is proper if for every regular cardinal $\theta > |TC(\mathcal{P})|$, every countable $N \preccurlyeq H(\theta)$ containing $\mathcal{P}$ and every $p \in \mathcal{P} \cap N$ there is some $q \in \mathcal{P}$ extending p such that q is $(N, \mathcal{P})$–generic, i.e. such that $q \Vdash_{\mathcal{P}} \tau \in \check{N}$ whenever $\tau \in N$ is a $\mathcal{P}$–name for an ordinal. Also, $\mathcal{P}$ is semiproper in case for every θ, N and p as above there is a condition q extending p such that q is $(N, \mathcal{P})$–semigeneric, i.e. such that $q \Vdash_{\mathcal{P}} \tau \in \check{N}$ for every name $\tau \in N$

[3] In other words, if all its quantifiers occur in a subformula of the form $(\forall x)(x \in y \rightarrow \varphi)$ or $(\exists x)(x \in y \wedge \varphi)$.

[4] That is, $\langle M, \in, P\rangle \models (\forall x_0, \dots x_k)\,(\varphi(x_0, \dots x_k) \leftrightarrow \psi(x_0, \dots x_k))$.

for an ordinal in ω_1^V. Every proper partial order is semiproper and, if $\mathcal{P}$ is semiproper, then every stationary subset of ω_1 remains stationary after forcing with $\mathcal{P}$.

$L(\mathbb{R})$ is the $\subseteq$–minimal transitive inner model of ZF containing all reals and all ordinals. Some arguments in Section 4, and in the proofs of Theorems 2.8 and 2.9 in Section 2, involve $\mathbb{P}_{max}$ forcing. $\mathbb{P}_{max}$ is a poset belonging to $L(\mathbb{R})$ and definable in $L(\mathbb{R})$ (without parameters). If $x^\dagger$ exists for every real x,[5] then $\mathbb{P}_{max}$ is a homogeneous forcing and is σ–closed (in V and in $L(\mathbb{R})$). In particular, forcing with it over $L(\mathbb{R})$ does not add new reals. The standard reference for $\mathbb{P}_{max}$ forcing is [W].

It will be convenient to fix a notion of incompatibility, for pairs of formulas, which is absolute with respect to sufficiently arbitrary models of set theory. We will say that two $\mathcal{L}$–formulas $\Phi_0(x)$ and $\Phi_1(x)$ are *ZFC–provably incompatible* if ZFC proves that for every uncountable regular cardinal κ and every $x \in H(\kappa^+)$, $\langle H(\kappa^+), \in, NS_\kappa\rangle \models \neg(\Phi_0(x) \wedge \Phi_1(x))$. Also, for an $\mathcal{L}$–formula in two free variables $\Phi(x, y)$, we will say that $\Phi(x, y)$ is *ZFC–provably antisymmetric* if ZFC proves that for every uncountable regular cardinal κ and every x, $y \in H(\kappa^+)$, $x \neq y$, $\langle H(\kappa^+), \in, NS_\kappa\rangle \models \neg(\Phi(x, y) \wedge \Phi(y, x))$.[6]

Acknowledgment. Many of the results in this paper were presented in two talks that I gave in Singapore in July 2005. These talks were part of the IMS workshop "Computational Prospects of Infinity". I thank the members of the Organizing Committee for inviting me.

2 Results not mentioning forcing axioms

The following theorems are proved in [As4].

Theorem 2.1 *([As4]) There are Σ_3 $\mathcal{L}$–formulas $\Phi_0(x)$ and $\Phi_1(x)$ and Π_3 $\mathcal{L}$–formulas $\Psi_0(x)$ and $\Psi_1(x)$ with the following two properties.*

---

[5]Which follows from all large cardinal assumptions used in the arguments alluded to here.

[6]The notions of provably incompatible pairs of formulas over $\langle H(\omega_2), \in, NS_{\omega_1}\rangle$ and of provably incompatible formulas in the language of set theory (over $\langle H(\omega_2), \in\rangle$) are of course defined in the natural way. And the same goes for the corresponding notions of provably antisymmetric formulas.

(1) $(\Phi_0(x), \Phi_1(x))$ and $(\Psi_0(x), \Psi_1(x))$ are two pairs of ZFC–provably incompatible formulas over the structure $\langle H(\omega_2), \in, NS_{\omega_1}\rangle$.

(2) Given any $A \subseteq \omega_1$ there is a proper poset forcing that

(a) A is defined, over $\langle H(\omega_2), \in, NS_{\omega_1}\rangle$, by $\Phi_0(x)$ and by $\Psi_0(x)$, and

(b) $\omega_1 \backslash A$ is defined, over $\langle H(\omega_2), \in, NS_{\omega_1}\rangle$, by $\Phi_1(x)$ and by $\Psi_1(x)$.

Theorem 2.2 *([As4]) There is a Σ_3 $\mathcal{L}$–formula $\Phi(x, y)$ and a Π_3 $\mathcal{L}$–formula $\Psi(x, y)$ with the following two properties.*

(1) $\Phi(x, y)$ and $\Psi(x, y)$ are ZFC–provably antisymmetric formulas over the structure $\langle H(\omega_2), \in, NS_{\omega_1}\rangle$.

(2) If there is an inaccessible cardinal, then there is a proper poset $\mathcal{P}$ forcing the existence of a well–order $\leq$ of $H(\omega_2)$ of order type ω_2 such that $\leq$ is defined, over $\langle H(\omega_2), \in, NS_{\omega_1}\rangle$, both by $\Phi(x, y)$ and by $\Psi(x, y)$.

In fact, Theorem 2.1 can be easily derived[7] from the following result.

Theorem 2.3 *([As4]) There is a Σ_2 $\mathcal{L}$–formula $\Phi(x)$ such that for every uncountable regular cardinal κ and every $A \subseteq \kappa$ there is a poset $\mathcal{P}$ with the following properties.*

(1) $\mathcal{P}$ is κ–distributive,[8] proper, and preserves κ. Also, if $2^\mu = \mu^+$ whenever μ is an infinite cardinal with $\mu^+ < \kappa$, then $\mathcal{P}$ preserves all stationary subsets of κ. Finally, if $2^{<\kappa} = \kappa$ and $2^\kappa = \kappa^+$, then $\mathcal{P}$ has the κ^+–chain condition.

(2) $\mathcal{P}$ forces

$$A = \{\xi < \kappa \,:\, \langle H(\kappa^+), \in, NS_\kappa\rangle \models \Phi(\xi)\}$$

Similarly, Theorem 2.2 is a consequence of the following result, also proved in [As4], by taking $\kappa = \omega_1$.

[7] By taking $\kappa = \omega_1$ and by taking $\Phi_0(x)$ and $\Psi_0(x)$ to be $\Phi(x)$ and $\Phi_1(x)$ and $\Psi_1(x)$ to be $\neg\Phi(x)$.

[8] Recall that a forcing notion is κ–distributive if and only if it does not add new sequences of ordinals of length less than κ.

Theorem 2.4 *([As4]) There is a Σ_3 $\mathcal{L}$–formula $\Phi(x,y)$ and a Π_3 $\mathcal{L}$–formula $\Psi(x,y)$ satisfying (1) and (2) below.*

(1) $\Phi(x,y)$ and $\Psi(x,y)$ are ZFC–provably antisymmetric formulas.

(2) Given any uncountable regular cardinal κ there is a poset $\mathcal{P}$ with the following properties.

(a) $\mathcal{P}$ preserves ω_1 and, if $\kappa > \omega_1$, then it satisfies (1) from Theorem 2.3.

(b) $\mathcal{P}$ forces that

$$\{(x,y) \in H(\kappa^+) \times H(\kappa^+) \,:\, \langle H(\kappa^+), \in, NS_\kappa \rangle \models \Phi(x,y)\}$$

is equal to

$$\{(x,y) \in H(\kappa^+) \times H(\kappa^+) \,:\, \langle H(\kappa^+), \in, NS_\kappa \rangle \models \Psi(x,y)\}$$

and is a well–order of $H(\kappa^+)$ of order type κ^+ (assuming there is an inaccessible cardinal in the case $\kappa = \omega_1$).

Note that none of Theorems 2.3 and 2.4 can hold for $\kappa = \omega$: By [Mar-St], Projective Determinacy holds if there are infinitely many Woodin cardinals. In particular, under this large cardinal assumption there can be no well–order of the reals definable over $\langle H(\omega_1), \in \rangle$ (even allowing parameters). And it can be seen that if $\delta < \kappa$ are such that δ is a limit of infinitely many Woodin cardinals and κ is a measurable cardinal, then given any poset $\mathcal{P}$ of size less than δ, any $\mathcal{P}$–generic filter G over V and any real $r \in V[G] \backslash V$, r is not definable over $\langle H(\omega_1)^{V[G]}, \in \rangle$ by any formula with a real number in V as parameter.[9]

Given a regular cardinal $\kappa \geq \omega_1$, the proofs of Theorems 2.3 and 2.4 involve the manipulation, by forcing, of certain weak club–guessing properties for club–sequences defined on stationary subsets of κ, in such a way that the Σ_2 theory of $\langle H(\kappa^+), \in, NS_\kappa \rangle$ with ordinals in κ as parameters codes any prescribed subset of κ.

[9]This follows from a result of Woodin to the effect that the theory of $L(\mathbb{R})$ with real numbers as parameters cannot be changed by forcing with $\mathcal{P}$ whenever $\mathcal{P}$ is a poset with $|\mathcal{P}| < \delta$ and $\delta < \kappa$ are as above (see [L], Theorem 3.1.12 for a proof).

Given an ordinal γ, $Lim(\gamma)$ denotes the set of nonzero limit ordinals in γ. A *club–sequence* will be a sequence of the form $\overline{\alpha} = \langle \alpha_\delta : \delta \in Lim(\gamma)\rangle$ – for some ordinal γ – such that each α_δ is a subset of δ. The set S of $\delta \in Lim(\gamma)$ such that α_δ is a club of δ is called the *domain of* $\overline{\alpha}$. It will also be denoted by $dom(\overline{\alpha})$. We may say that $\overline{\alpha}$ *is defined on* S. If $\overline{\alpha}$ is a club–sequence and γ is such that $sup(dom(\overline{\alpha})) = \gamma$, then we may say that $\overline{\alpha}$ *is a club–sequence on* γ. If τ is an ordinal such that the order type of α_δ is τ for each $\delta \in dom(\overline{\alpha})$, then we say that the *height of* $\overline{\alpha}$ is τ. $ht(\overline{\alpha})$ will denote the height of $\overline{\alpha}$ (if it exists). As in [A-Sh] (for ladder systems, that is, for club–sequences of height ω), if $\overline{\alpha}$ is a club–sequence and $\delta \in dom(\overline{\alpha})$, then α_δ will denote $\overline{\alpha}(\delta)$ (and similarly with other Greek letters). We will say that $\overline{\alpha}$ is a *coherent club–sequence* if there is a club–sequence $\overline{\beta}$ with $dom(\overline{\alpha}) \subseteq dom(\overline{\beta})$ and $\overline{\beta} \restriction dom(\overline{\alpha}) = \overline{\alpha} \restriction dom(\overline{\alpha})$,[10] and such that $\overline{\beta}$ is coherent in the usual sense, that is, such that for every $\delta \in dom(\overline{\beta})$ and every limit point γ of β_δ, $\gamma \in dom(\overline{\beta})$ and $\beta_\delta \cap \gamma = \beta_\gamma$.

The concepts in this paragraph are defined in [A-Sh] for ladder systems.[11] Let $\overline{\alpha}$ be a club–sequence on an ordinal γ of uncountable cofinality. We say that $\overline{\alpha}$ is *guessing* in case for every club $C \subseteq \gamma$ there is some $\delta \in C \cap dom(\overline{\alpha})$ such that $\alpha_\delta \backslash C$ is bounded in δ. Furthermore, we say that $\overline{\alpha}$ is *strongly guessing* if for every club $C \subseteq \gamma$ there is a club $D \subseteq \gamma$ such that $\alpha_\delta \backslash C$ is bounded in δ for every $\delta \in D \cap dom(\overline{\alpha})$.[12] $\overline{\alpha}$ is *avoidable* if there is a club $C \subseteq \gamma$ such that $\alpha_\delta \cap C$ is bounded in δ for each $\delta \in dom(\overline{\alpha}) \cap C$. Given two club–sequences $\overline{\alpha}$ and $\overline{\beta}$ on the same ordinal γ, γ of uncountable cofinality, $\overline{\beta}$ is *disjoint from* $\overline{\alpha}$ if $\beta_\delta \cap \alpha_\delta = \emptyset$ for every $\delta \in dom(\overline{\alpha}) \cap dom(\overline{\beta})$. Given a strongly guessing club–sequence $\overline{\alpha}$ on an ordinal γ and a set $X \subseteq \gamma$ including $dom(\overline{\alpha})$, $\overline{\alpha}$ is said to be *maximal for* X in case every ladder system defined on X and disjoint from $\overline{\alpha}$ is avoidable.

The proofs of Theorems 2.3 and 2.4 make use of a certain weak club–guessing property for club–sequences,[13] which is best defined after introducing the following strong version of intersection of two sets of ordinals: Given two sets of ordinals, X and Y, $X \cap^* Y$ is defined as the set of $\delta \in X \cap Y$ such that δ is not a limit point of X. Now, given a club–sequence $\overline{\alpha}$ on an ordinal

[10] Where, given a club–sequence $\overline{\alpha}$ and a set X, the restriction of $\overline{\alpha}$ to X, to be denoted by $\overline{\alpha} \restriction X$, is that club–sequence which is equal to $\overline{\alpha}$ on X and is $\emptyset$ elsewhere.

[11] They will occur in Definition 2.3.

[12] Note that a strongly guessing club–sequence is guessing if and only if its domain is stationary.

[13] Defined in [As4].

γ of uncountable cofinality, we will say that $\overline{\alpha}$ *is type-guessing* in case for every club $C \subseteq \gamma$ there is some $\delta \in C \cap dom(\overline{\alpha})$ with $ot(\alpha_\delta \cap^* C)$ as high as possible, that is, with $ot(\alpha_\delta \cap^* C) = ot(\alpha_\delta)$. We will say that $\overline{\alpha}$ *is strongly type-guessing* in case for every club $C \subseteq \gamma$ there is a club $D \subseteq \gamma$ such that $ot(\alpha_\delta \cap^* C) = ot(\alpha_\delta)$ for every $\delta \in D \cap dom(\overline{\alpha})$.

Also, for a set X of ordinals and an ordinal δ, the *Cantor–Bendixson rank of δ with respect to X*, $rnk_X(\delta)$, is defined by specifying that $rnk_X(\delta) = 0$ if and only if δ is not a limit point of X, that $rnk_X(\delta) \geq 1$ if and only if δ is a limit point of X and, for each ordinal $\eta \geq 1$, that $rnk_X(\delta) > \eta$ if and only if δ is a limit ordinal and there is a sequence $(\delta_\xi)_{\xi < ot(\delta)}$ converging to δ such that $rnk_X(\delta_\xi) \geq \eta$ for every ξ.[14] An ordinal δ will be said to be *perfect* if $rnk_\delta(\delta) = \delta$.[15] Note that $rnk_\delta(\delta) \leq \delta$ for every ordinal δ and that, given any uncountable regular cardinal κ, the set of perfect ordinals below κ is a club.

Definition 2.1 *([As4]) Given an uncountable regular cardinal κ, $\mathcal{A} = \{\overline{\alpha}^\nu : \nu < \lambda\}$ (for $1 \leq \lambda \leq \kappa$) is an almost specifiable set of club–sequences on κ (asscs on κ, for short) if and only if*

(a) there is a one–to–one sequence $\langle \tau_\nu : \nu < \lambda \rangle$ of perfect ordinals below κ such that, for each ν, τ_ν has countable cofinality and $\overline{\alpha}^\nu$ is a coherent club–sequence on κ of height τ_ν,

(b) $\langle dom(\overline{\alpha}^\nu) : \nu < \lambda \rangle$ is a sequence of pairwise disjoint stationary sets,

(c) each $\overline{\alpha}^\nu$ is strongly type-guessing, and

(d) given any coherent club–sequence $\overline{\beta}$ on κ with stationary domain, if $\overline{\beta}$ has height υ of countable cofinality and $\upsilon \neq ht(\overline{\alpha}^\nu)$ for every $\nu < \lambda$, then $\overline{\beta}$ is not strongly type-guessing.

Lemma 2.5 gives a justification for the use of the phrase 'almost specifiable' in Definition 2.1. It implies that $\{ht(\overline{\alpha}^\nu) : \nu < \lambda_0\} = \{ht(\overline{\beta}^\nu) : \nu < \lambda_1\}$ holds whenever $\{\overline{\alpha}^\nu : \nu < \lambda_0\}$ and $\{\overline{\beta}^\nu : \nu < \lambda_1\}$ are two asscs's on the same κ.

[14] Thus, for example, $rnk_X(\delta) = 1$ if and only if δ is a limit point of ordinals in X but not a limit point of limit points of X.

[15] Thus, with this definition, the first perfect ordinal is 0, the second is $\epsilon_0 = sup\{\omega, \omega^\omega, \omega^{(\omega^\omega)}, \omega^{(\omega^{(\omega^\omega)})}, \ldots\}$, etc.

Lemma 2.5 *([As4]) Suppose $\mathcal{A} = \{\overline{\alpha}^\nu : \nu < \lambda\}$ is an almost specifiable set of club–sequences on an uncountable regular cardinal κ. Then, $\{ht(\overline{\alpha}^\nu) : \nu < \lambda\}$ is equal to the set of perfect ordinals $\tau < \kappa$ of countable cofinality and such that there is a strongly type-guessing coherent club–sequence on κ of height τ and with stationary domain.*

The above lemma follows immediately from the definition of asscs. Theorem 2.3 is an immediate consequence of Lemma 2.5 and of the following result.

Theorem 2.6 *([As4]) Let $\kappa \geq \omega_1$ be a regular cardinal, let A be a subset of κ and let $(\xi_\eta)_{\eta<\kappa}$ be the strictly increasing enumeration of all perfect ordinals ξ less than κ with $cf(\xi) = \omega$. Then there is a poset $\mathcal{P}$ satisfying the niceness properties in (1) from Theorem 2.3 and forcing that there is an asscs $\{\overline{\alpha}^\nu : \nu < ot(A)\}$ on κ such that $A = \{\eta < \kappa : (\exists \nu < ot(A))(ht(\overline{\alpha}^\nu) = \xi_\eta)\}$.*

Theorem 2.3 follows then since, for any given regular $\kappa \geq \omega_1$ and any $A \subseteq \kappa$, the poset given by the above theorem forces that A is the set of $\eta < \kappa$ with the property that there is a coherent strongly type-guessing club–sequence with stationary domain included in κ and of height ξ_η (where $(\xi_\eta)_{\eta<\kappa}$ is as in the above theorem).[16] The strategy for building the poset $\mathcal{P}$ in Theorem 2.6 is quite as one would expect: $\mathcal{P}$ is the limit of a forcing iteration of length κ^+ built with supports of size less than κ. We may assume that $2^{<\kappa} = \kappa$ and $2^\kappa = \kappa^+$ hold in the ground model. In the first step of the iteration one adds, by initial segments, a set $\mathcal{A}$ of coherent club–sequences with the relevant heights. Then one kills, along the iteration, all possible obstacles to $\mathcal{A}$ being an asscs. In fact, it suffices to force, for all clubs $C \subseteq \kappa$ arising during the iteration, with the natural poset for shooting a club $D \subseteq \kappa$ such that $ot(\alpha_\delta \cap^* C) = ot(\alpha_\delta)$ for all $\delta \in D \cap dom(\overline{\alpha})$ (for all $\overline{\alpha} \in \mathcal{A}$). This is enough to ensure that (d) from Definition 2.1 holds for $\mathcal{A}$. The bulk of the proof is of course the verification that this poset $\mathcal{P}$ has the desired properties. Also, several of the technicalities involves in the coding – for example the restriction to perfect ordinals of countable cofinality or the consideration of the operation $\cap^*$ – are there precisely to make (d) hold.

The proof of Theorem 2.4 involves a certain principle which provides a simple way of encoding members of $H(\kappa^+)$ (for some κ) by ordinals in κ^+, quite in the spirit of the principle ψ_{AC} (for $H(\omega_2)$) from [W].

[16] Noting that this property can indeed be expressed by a Σ_2 $\mathcal{L}$–formula over the structure $\langle H(\kappa^+), \in, NS_\kappa\rangle$.

Definition 2.2 *([As4]) Let κ be an uncountable regular cardinal, let F be a function from κ into $\mathcal{P}(\kappa)$, and let $\mathcal{S} = \langle S_i : i < \kappa\rangle$ be a sequence of pairwise disjoint stationary subsets of κ. $\overline{\Psi}^{F,\mathcal{S}}_{AC}$ is then the statement that there is an enumeration $\langle B_\zeta : \kappa \leq \zeta < \kappa^+\rangle$ of all subsets of κ such that, for each ζ, there is a club $E \subseteq [\zeta]^{<\kappa}$ with the property that for every $X \in E$ and every $i < \kappa$,*

(a) $ot(X) \in F(X \cap \kappa)$ if $X \cap \kappa \in S_i$ and $i \in B_\zeta$, and

(b) $ot(X) \notin F(X \cap \kappa)$ if $X \cap \kappa \in S_i$ and $i \notin B_\zeta$.

It is easy to see that there is a Σ_1 formula $\Theta(x, y, z)$ with the property that if κ, F and $\mathcal{S}$ are such that

(1) κ is an uncountable regular cardinal, $F : \kappa \longrightarrow \mathcal{P}(\kappa)$ is a function and $\mathcal{S}$ is a κ–sequence of pairwise disjoint stationary subsets of κ, and

(2) $\overline{\Psi}^{F,\mathcal{S}}_{AC}$ holds,

then $\leq_{F,\mathcal{S}} := \{\langle x, y\rangle \in H(\kappa^+) \times H(\kappa^+) : \langle H(\kappa^+), \in\rangle \models \Theta(x, y, \langle F, \mathcal{S}\rangle)\}$ is a well–order of $H(\kappa^+)$ of order type κ^+. Moreover, this formula can be taken so that ZFC proves that there are no κ, F and $\mathcal{S}$ as in (1) for which there are any distinct $x, y \in H(\kappa^+)$ with $H(\kappa^+) \models \Theta(x, y, \langle F, \mathcal{S}\rangle) \wedge \Theta(y, x, \langle F, \mathcal{S}\rangle)$.

The following result is proved in [As4].

Theorem 2.7 *Let $\kappa \geq \omega_1$ is a regular cardinal, and suppose $2^{<\kappa} = \kappa$ and $2^\kappa = \kappa^+$ hold. Then there is a κ–distributive poset $\mathcal{P}$ adding a function $F : \kappa \longrightarrow [\kappa]^{<\kappa}$ and a κ–sequence $\mathcal{S}$ of pairwise disjoint stationary subsets of κ such that $\overline{\Psi}^{F,\mathcal{S}}_{AC}$ holds. Furthermore, $\mathcal{P}$ has the κ^+–chain condition and, if $2^\mu = \mu^+$ whenever μ is an infinite cardinal with $\mu^+ < \kappa$, then $\mathcal{P}$ preserves all stationary subsets of κ.*

When $\kappa > \omega_1$, Theorem 2.4 is proved by combining the forcing construction for proving the above result with the one for proving Theorem 2.6 with respect to a subset of κ coding the parameter $(F, \mathcal{S})$ for which we force $\overline{\Psi}^{F,\mathcal{S}}_{AC}$: We may assume that $2^{<\kappa} = \kappa$ and $2^\kappa = \kappa^+$ hold in the ground model. We start by adding, by forcing with initial segments, a function $F : \kappa \longrightarrow [\kappa]^{<\kappa}$ and a κ–sequence $\mathcal{S}$ of mutually disjoint stationary subsets of κ. Then we build a forcing iteration of length κ^+ with supports of size less than κ in which we simultaneously perform the tasks of

(1) forcing $\overline{\Psi}^{F,\mathcal{S}}_{AC}$ (as in the proof of Theorem 2.7), and

(2) adding a suitable asscs on κ (as in the proof of Theorem 2.6) coding a fixed subset of κ that encodes (in some Σ_1 way) the pair $(F, \mathcal{S})$.

The desired forcing is then the limit of this iteration. Now, the $\mathcal{L}$–formula $\Phi(x, y)$ witnessing Theorem 2.4 can be taken to express the following property $P_0(x, y)$: "There is a maximal set $\mathcal{T}$ of perfect ordinals $\tau < \kappa$ of countable cofinality with the property that there is a coherent strongly type-guessing club–sequence with stationary domain and with height τ such that $\mathcal{T}$ encodes a pair $(F, \mathcal{S})$ with F a function from κ into $\mathcal{P}(\kappa)$ and $\mathcal{S}$ a κ–sequence of mutually disjoint stationary subsets of κ, and $x \leq_{F,\mathcal{S}} y$[17]".

$P_0(x, y)$ can also be expressed, in a slightly more convoluted way, by saying that there is a set $\mathcal{T}$ such that

(a) for every $\tau \in \mathcal{T}$, τ is a perfect ordinal in κ of countable cofinality and there is a coherent club–sequence $\overline{\alpha}$ of height τ, with $dom(\overline{\alpha}) \subseteq \kappa$ and $dom(\overline{\alpha}) \notin NS_\kappa$, and such that for every club $C \subseteq \kappa$, $\{\delta \in dom(\overline{\alpha}) : ot(\alpha_\delta \cap^* C) \neq \tau\} \in NS_\kappa$,

(b) for every $\tau \in \kappa \cap cf(\omega)$, $\tau \in \mathcal{T}$ or else for every coherent club–sequence $\overline{\alpha}$ of height τ, either $dom(\overline{\alpha})$ is not a stationary subset of κ or there is a club $C \subseteq \omega_1$ such that $\{\delta \in dom(\overline{\alpha}) : ot(\alpha_\delta \cap^* C) \neq \tau\} \notin NS_\kappa$, and such that

(c) $\mathcal{T}$ encodes a pair $(F, \mathcal{S})$ with F a function from κ into $\mathcal{P}(\kappa)$ and $\mathcal{S}$ a κ–sequence of mutually disjoint stationary subsets of κ, and $x \leq_{F,\mathcal{S}} y$.

Clearly, (a) and (b) are, respectively, a Σ_2 property of $\mathcal{T}$ over the structure $\langle H(\kappa^+), \in, NS_\kappa\rangle$ and a Π_2 property of $\mathcal{T}$, also over $\langle H(\kappa^+), \in, NS_\kappa\rangle$. And, since $\leq_{F,\mathcal{S}}$ is as described right after Definition 2.2, (c) is a Σ_1 property, over $\langle H(\kappa^+), \in\rangle$, about $\mathcal{T}$, x and y. Thus, $P_0(x, y)$, being expressible over $\langle H(\kappa^+), \in, NS_\kappa\rangle$ as $(\exists \mathcal{T})[\Phi_a(\mathcal{T}) \wedge \Phi_b(\mathcal{T}) \wedge \Phi_c(\mathcal{T}, x, y)]$, with $\Phi_a(u)$ a Σ_2 $\mathcal{L}$–formula, $\Phi_b(u)$ a Π_2 $\mathcal{L}$–formula, and $\Phi(u, v, w)$ a Σ_1 formula in the language of set theory, can be written as a Σ_3 $\mathcal{L}$–formula over $\langle H(\kappa^+), \in, NS_\kappa\rangle$.

As to $\Psi(x, y)$, it can be taken to express the property – let us call it $P_1(x, y)$ – that every maximal set $\mathcal{T}$ of ordinals $\tau < \kappa$ with the property stated in the description of $\Phi(x, y)$ encodes a pair $(F, \mathcal{S})$ as before, and

[17] Where $\leq_{F,\mathcal{S}}$ is as in the paragraph right after Definition 2.2.

$x \leq_{F,\mathcal{S}} y$. Let us call $Q(\mathcal{T})$ the property of $\mathcal{T}$ expressed by the conjunction of (a) and (b) in the above description. $P_1(x,y)$ can be written as

$$(\forall \mathcal{T})[Q(\mathcal{T}) \rightarrow (\mathcal{T} \text{ encodes a pair } (F,\mathcal{S}) \text{ and } x \leq_{F,\mathcal{S}} y)].$$

$Q(\mathcal{T})$ is a Σ_3 property of $\mathcal{T}$ over $\langle H(\kappa^+), \in, NS_\kappa\rangle$, and the expression to the right of the implication sign is a Σ_1 property of $\mathcal{T}$, x and y. It follows then that $P_1(x,y)$ can be written as a Π_3 formula over $\langle H(\kappa^+), \in, NS_\kappa\rangle$.

The idea behind the proof when $\kappa = \omega_1$ is the same. The proof in this case combines the forcing iteration for Theorem 2.6 with a certain iteration of proper posets, due to Moore [Mo] (see the next section), which codes subsets of ω_1 by ordinals using some fixed parameter p. Now we force this coding while at the same time making p definable. In the extension, the inaccessible cardinal becomes ω_2.

One final word on the proofs of Theorems 2.6 and 2.4: They do not depend on any general forcing iteration lemmas. Instead, they rely on direct constructions depending quite closely on the actual definition of the iteration. By a forcing iteration lemma here I mean a statement typically asserting that if $\langle \mathcal{P}_\xi : \xi \leq \lambda\rangle$ is *any* forcing iteration built with some fixed kind of supports, based on a sequence $\langle \dot{\mathcal{Q}}_\xi : \xi < \lambda\rangle$ of names for posets,[18] and each $\dot{\mathcal{Q}}_\xi$ is forced, by $\mathcal{P}_\xi$, to have a certain property P, then $\mathcal{P}_\lambda$ also has property P. It is well–known[19] that fairly general iteration lemmas can be obtained for countable support iterations (or for some reasonable variation of this type of iterations). On the other hand, there are serious obstacles to proving similar general lemmas for iterations built using uncountable supports, which are precisely the kind of iterations one is typically faced with when forcing some statement about subsets of κ, for some $\kappa > \omega_2$, while at the same time preserving ω_1 and ω_2.

It turns out that Theorems 2.1 and 2.2 are optimal as stated from the point of view of the Levy hierarchy. More specifically, by appealing mainly to results of Woodin one can prove that, in the presence of sufficiently strong large cardinals,[20] 3 cannot be replaced by 2 in the statement of either Theorem 2.1 or Theorem 2.2. In fact, one cannot prove a version of either Theorem 2.1 or Theorem 2.2 in which Σ_3 (equivalently, Π_3) is replaced by Π_2. Theorems 2.8 and 2.9 present more precise formulations of this claim.

[18]That is, for all ξ with $\xi + 1 \leq \lambda$, $\mathcal{P}_{\xi+1}$ is the set of $\xi + 1$–sequences p such that $p \restriction \xi \in \mathcal{P}_\xi$ and such that $p \restriction \xi \Vdash_\xi p(\xi) \in \dot{\mathcal{Q}}_\xi$.

[19]See for example [Sh].

[20]For example a proper class of Woodin cardinals.

Theorem 2.8 *([As2]) Given any stationary and co-stationary $A \subseteq \omega_1$, if AD, the Axiom of Determinacy, holds in $L(\mathbb{R})$ ($AD^{L(\mathbb{R})}$) and if there is a Woodin cardinal below a measurable cardinal, then there is no pair $\Phi_0(x)$, $\Phi_1(x)$ of necessarily incompatible Π_2 formulas over $\langle H(\omega_2), \in, NS_{\omega_1}, r\rangle_{r\in\mathbb{R}}$ such that A is defined, over $\langle H(\omega_2), \in, NS_{\omega_1}, r\rangle_{r\in\mathbb{R}}$, by $\Phi_0(x)$ and $\omega_1\backslash A$ is defined, over $\langle H(\omega_2), \in, NS_{\omega_1}, r\rangle_{r\in\mathbb{R}}$, by $\Phi_1(x)$.*

Theorem 2.9 *([As2]) Assume $AD^{L(\mathbb{R})}$ and suppose there is a Woodin cardinal with a measurable cardinal above. Then there is no necessarily antisymmetric Π_2 formula $\Phi(x, y)$ over the structure $\langle H(\omega_2), \in, NS_{\omega_1}, r\rangle_{r\in\mathbb{R}}$ such that $\Phi(x, y)$ defines, over $\langle H(\omega_2), \in, NS_{\omega_1}, r\rangle_{r\in\mathbb{R}}$, a well–order of $\mathbb{R}$.*

In Theorem 2.8 above, two formulas $\Phi_0(x)$ and $\Phi_1(x)$ in the language of the structure $\langle H(\omega_2), \in, NS_{\omega_1}, r\rangle$ – where r is a real number – are said to be *necessarily incompatible* if, for every generic extension M of $L(\mathbb{R})$ satisfying ZFC,

$$\langle H(\omega_2), \in, NS_{\omega_1}, r\rangle^M \models \neg(\Phi_0(x) \wedge \Phi_1(x))$$

for every $x \in H(\omega_2)^M$. Also, in Theorem 2.9, a formula $\Phi(x, y)$ in the language of the structure $\langle H(\omega_2), \in, NS_{\omega_1}, r\rangle$ – where, again, r is a real number – is *necessarily antisymmetric* in case for every generic extension M of $L(\mathbb{R})$ satisfying ZFC,

$$\langle H(\omega_2), \in, NS_{\omega_1}, r\rangle^M \models (\neg\exists\, x,\, y)(x \neq y \wedge \Phi(x, y) \wedge \Phi(y, x))$$

Theorems 2.8 and 2.9 are easily proved using the theory of $\mathbb{P}_{max}$ forcing by arguments very much contained in the proof of Observation 4.3 in Section 4.[21] From Theorem 2.8 it follows that if there is a proper class of Woodin cardinals and A is a stationary and co-stationary subset of ω_1, then there is no pair $(\Phi_0(x), \Phi_1(x))$ of ZFC–provably incompatible Π_2 $\mathcal{L}$–formulas for which there is a poset $\mathcal{P}$ such that $\mathcal{P}$ preserves the stationarity of both A and $\omega_1\backslash A$ and such that $\mathcal{P}$ forces that A and $\omega_1\backslash A$ are defined over $\langle H(\omega_2), \in, NS_{\omega_1}\rangle$ by, respectively, $\Phi_0(x)$ and $\Phi_1(x)$.[22] Likewise, Theorem 2.9 implies that, under the same large cardinal assumption, there is no poset forcing the existence

[21] Using the fact that, under $AD^{L(\mathbb{R})}$, $\mathbb{P}_{max}$ is homogeneous and σ–closed and using, respectively, the fact that AD prohibits the existence of stationary and co-stationary subsets of ω_1, and the fact that AD prohibits the existence of well–orders of $\mathbb{R}$.

[22] Since, by a result of Woodin, $AD^{L(\mathbb{R})}$ follows from the existence of infinitely many Woodin cardinals with a measurable cardinal above them.

of a well–order of $H(\omega_2)$ definable over the structure $\langle H(\omega_2), \in, NS_{\omega_1}\rangle$ by a ZFC–provably antisymmetric Π_2 $\mathcal{L}$–formula.

One may ask wether it is possible to drop the predicate NS_{ω_1} in the statement of either Theorem 2.1 or 2.2. Concerning this question, there is a version of the above theorems with $\langle H(\omega_2), \in\rangle$ replacing the more expressive $\langle H(\omega_2), \in, NS_{\omega_1}\rangle$. The coding techniques employed in the proof of these theorems are quite different from the ones used in the proofs of Theorems 2.6 and 2.4. These results use ZFC + "There is an inaccessible limit of measurable cardinals" as base theory, rather than just ZFC (+ there is an inaccessible cardinal).

Recall that, if $\alpha < \omega_2$ is an ordinal and $\pi : \omega_1 \longrightarrow \alpha$ is a surjection, the function $g : \omega_1 \longrightarrow \omega_1$ defined by letting $g(\nu) = ot(\pi``\nu)$ for each ν is called a *canonical function for* α. This name is justified by the fact that any two functions thus obtained (for the same α) coincide on a club of ω_1. By a canonical function will be meant a canonical function for some ordinal below ω_2. Given $S \subseteq \omega_1$ and two functions f, $g : S \longrightarrow \omega_1$, we will say that g *dominates* f *on* S *mod. a club* (equivalently, f *is dominated by* g *on* S *mod. a club*) if there is a club $C \subseteq \omega_1$ such that $f(\nu) < g(\nu)$ for every $\nu \in S \cap C$.

The following notion is defined in [As5].

Definition 2.3 *([As5]) Given an ordinal λ, $1 \leq \lambda \leq \omega_1$, $\langle S, \langle S_i : i < \lambda\rangle, f, \overline{\alpha}\rangle$ is a simple decoding object if*

(a) $\{S_i : i < \lambda\} \cup \{S, \omega_1 \backslash (S \cup \bigcup_{i<\lambda} S_i)\}$ *is a collection of pairwise disjoint stationary subsets of* ω_1*,*

(b) every function from S into ω_1 is dominated on S mod. a club by some canonical function,

(c) f is a function from $\omega_1 \backslash (S \cup \bigcup_{i<\lambda} S_i)$ into ω_1 dominating every canonical function on $\omega_1 \backslash (S \cup \bigcup_{i<\lambda} S_i)$ mod. a club,

(d) $\overline{\alpha}$ is a ladder system defined on $\bigcup_{i<\lambda} S_i$ which is strongly guessing and maximal for ω_1, and $\alpha_\delta \cap \bigcup_{i<\lambda} S_i = \emptyset$ for every $\delta \in \bigcup_{i<\lambda} S_i$, and

(e) there is a sequence $\langle r_i : i < \lambda\rangle$ such that for every $i < \lambda$ and every $\delta \in S_i$, r_i is the set of $k < \omega$ for which there are infinitely many $n < \omega$ such that $\{\alpha_\delta(n), \alpha_\delta(n+k+1)\} \subseteq S$ and $\{\alpha_\delta(n+j) : 1 \leq j \leq k\} \cap S = \emptyset$.

If $\langle S, \langle S_i : i < \lambda\rangle, f, \overline{\alpha}\rangle$ is a simple decoding object, then we let code$(S, \langle S_i : i < \lambda\rangle, f, \overline{\alpha}) = \{r_i : i < \lambda\}$, where $\langle r_i : i < \lambda\rangle$ witnesses (e) for $\langle S, \langle S_i : i < \lambda\rangle, f, \overline{\alpha}\rangle$. We will say that $\langle S, \langle S_i : i < \lambda\rangle, f, \overline{\alpha}\rangle$ encodes $\{r_i : i < \lambda\}$.

Lemma 2.10 shows that simple decoding objects are unique in a quite strong sense.

Lemma 2.10 *([As5]) Suppose $\langle S^0, \langle S^0_i : i < \lambda_0\rangle, f_0, \overline{\alpha}^0\rangle$ and $\langle S^1, \langle S^1_i : i < \lambda_1\rangle, f_1, \overline{\alpha}^1\rangle$ are simple decoding objects. Then there are clubs $D \subseteq C$ of ω_1 such that*

(i) $S^0 \cap C = S^1 \cap C$ and $\bigcup_{i<\lambda_0} S^0_i \cap C = \bigcup_{i<\lambda_1} S^1_i \cap C$,

(ii) for every $\delta \in D$, both $\alpha^0_\delta \Delta \alpha^1_\delta$ and $\alpha^0_\delta \backslash C$ are finite.

In particular, code$(S^0, \langle S^0_i : i < \lambda_0\rangle, f_0, \overline{\alpha}^0)$ = code$(S^1, \langle S^1_i : i < \lambda_1\rangle, f_1, \overline{\alpha}^1)$.

The following theorem is proved in [As5].

Theorem 2.11 *([As5]) Suppose κ is an inaccessible limit of measurable cardinals. Let λ be an ordinal, $1 \leq \lambda \leq \omega_1$, and let $\langle r_i : i < \lambda\rangle$ be a sequence of sets of integers. There is a semiproper poset $\mathcal{P} \subseteq V_\kappa$ forcing that there is a simple decoding object $\langle S, \langle S_i : i < \lambda\rangle, f, \overline{\alpha}\rangle$ such that code$(S, \langle S_i : i < \lambda\rangle, f, \overline{\alpha}) = \{r_i : i < \lambda\}$.*

The first of the following two results is a corollary of Theorem 2.11, and the second follows from combining the forcing construction for proving Theorem 2.11 with, for example, the one for proving Theorem 2.7.[23]

Theorem 2.12 *([As5]) There are Σ_3 formulas $\Phi_0(x)$ and $\Phi_1(x)$ in the language of set theory such that*

(1) $(\Phi_0(x), \Phi_1(x))$ are ZFC–provably incompatible formulas over the structure $\langle H(\omega_2), \in\rangle$, and

(2) given any $A \subseteq \omega_1$, if there is an inaccessible limit κ of measurable cardinals, then there is a semiproper poset $\mathcal{P} \subseteq V_\kappa$ forcing that A and $\omega_1 \backslash A$ are defined over $\langle H(\omega_2), \in\rangle$ by, respectively, $\Phi_0(x)$ and $\Phi_1(x)$.

[23]Very much as in the proof of Theorem 2.4.

Theorem 2.13 *([As5]) There is a Σ_3 formula $\Phi(x, y)$ in the language of set theory such that*

(1) $\Phi(x, y)$ is a ZFC–provably antisymmetric formula over $\langle H(\omega_2), \in\rangle$, and

(2) if there is an inaccessible limit κ of measurable cardinals, then there is a semiproper poset $\mathcal{P} \subseteq V_\kappa$ forcing that $\Phi(x, y)$ defines, over $\langle H(\omega_2), \in\rangle$, a well–order of $H(\omega_2)$ of order type ω_2.

For Theorem 2.12, each formula $\Phi_\epsilon(x)$ (for $\epsilon \in \{0, 1\}$) will be a Σ_3 formulas expressing the property $P_\epsilon(x)$, where $P_0(x)$ and $P_1(x)$ are defined by, respectively, "x is a countable ordinal and there is a real r encoding x (in some Σ_1 way), together with a simple decoding object encoding a set of reals to which r belongs" and "x is a countable ordinal and there is a simple decoding object encoding a set of reals X such that $r \notin X$ whenever r is a real encoding x (in the same way as before)".

It is easy to see that both properties can be expressed by Σ_3 formulas over $\langle H(\omega_2), \in\rangle$: For $P_0(x)$, since "$x \in \omega_1$" and "r encodes x" are Σ_1 properties of the relevant objects, the verification will be finished once we see that

$$(\exists S, (S_i)_{i<\lambda}, f, \overline{\alpha})\, Q(S, (S_i)_{i<\lambda}, f, \overline{\alpha}, r)$$

can be written as a Σ_3 sentence over $\langle H(\omega_2), \in\rangle$ with r as parameter, where $Q(S, (S_i)_{i<\lambda}, f, \overline{\alpha}, r)$ expresses that $(S, (S_i)_{i<\lambda}, f, \overline{\alpha})$ is a simple decoding object and that $r \in \mathit{code}(S, (S_i)_{i<\lambda}, f, \overline{\alpha})$. But $Q(S, (S_i)_{i<\lambda}, f, \overline{\alpha}, r)$ can be expressed by saying that (a)–(d) from Definition 2.3 hold for the relevant parameters, and that $r \in \mathit{code}(S, (S_i)_{i<\lambda}, f, \overline{\alpha})$. Since ω_1 is Σ_2 definable over $\langle H(\omega_2), \in\rangle$, (a) from Definition 2.3 is a Σ_2 property of S and $(S_i)_{i<\lambda}$ (over $\langle H(\omega_2), \in\rangle$). (b) from Definition 2.3 is clearly a Π_2 property of S, and (c) is also a Π_2 property (of f, S and $(S_i)_{i<\lambda}$). (d) is expressed by saying that $\overline{\alpha}$ is a ladder system with $\mathit{dom}(\overline{\alpha}) = \bigcup_{i<\lambda} S_i$, that for every club $C \subseteq \omega_1$ there is a club $D \subseteq \omega_1$ such that $\alpha_\delta \backslash C$ is bounded in δ for every $\delta \in D \cap \mathit{dom}(\overline{\alpha})$, and that for every ladder system $\overline{\beta}$ disjoint from $\overline{\alpha}$ there is a club $C \subseteq \omega_1$ such that $\beta_\delta \cap C$ is finite for every $\delta \in C \cap \mathit{dom}(\overline{\beta})$; hence, it can be written as a Π_2 formula over $\langle H(\omega_2), \in\rangle$. Finally, "$r \in \mathit{code}(S, (S_i)_{i<\lambda}, f, \overline{\alpha})$" can be expressed by saying that there is some $i < \lambda$ such that $r = \{k < \omega : (\exists^\infty n < \omega)(\{\alpha_\delta(n),\, \alpha_\delta(n+k+1)\} \subseteq S \wedge \{\alpha_\delta(n+j) : 1 \leq j \leq k\} \cap S = \emptyset)\}$ for every $\delta \in S_i$, and so it is a Σ_0 property of r, S and $(S_i)_{i<\lambda}$. It follows that $P_0(x)$

can be written as a Σ_3 formula over $\langle H(\omega_2), \in\rangle$. The verification for $P_1(x)$ is along the same lines.

The formula $\Phi(x, y)$ witnessing Theorem 2.13 can be taken to say that there is a simple decoding object encoding a set of reals X such that X encodes (in some simple standard way) a pair $(F, \mathcal{S})$ as in Definition 2.2 (for $\kappa = \omega_2$) and $x \leq_{F,\mathcal{S}} y$, for the relation $\leq_{F,\mathcal{S}}$ described after Definition 2.2. It is easy to see, by an analysis as before, that there is indeed a Σ_3 formula expressing the above property of x and y over $\langle H(\omega_2), \in\rangle$.

Question 2.1 *Is it possible to prove versions of either Theorem 2.12 or 2.13 with* Π_3 *replacing* Σ_3*?*

3 Results mentioning strong forcing axioms

The forcing constructions presented in the previous section are flexible enough to accommodate posets with the countable chain condition. Thus, all models built there can be taken to be models of MA_{ω_1}. However, it is not possible to modify those constructions so as to produce models of, for example, $BPFA$.[24] In fact, the techniques presented there for coding a fixed subset of ω_1 are incompatible with $BPFA$. The reason for this is that $BPFA$ implies that every club–sequence is avoidable.[25]

PFA^{++} is the following strong form of PFA:

Given any proper poset $\mathcal{P}$, any sequence $\langle D_i : i < \omega_1\rangle$ of dense subsets of ω_1 and any sequence $\langle \tau_i : i < \omega_1\rangle$ of $\mathcal{P}$–names for stationary subsets of ω_1 there is a filter $G \subseteq \mathcal{P}$ such that, for each $i < \omega_1$, $G \cap D_i \neq \emptyset$ and $\{\nu < \omega_1 : (\exists p \in G)(p \Vdash_{\mathcal{P}} \nu \in \tau_i)\}$ is a stationary subset of ω_1.

The first main result in this section is the following.

Theorem 3.1 *([As3]) Suppose* κ *is a supercompact cardinal and* A *is a subset of* ω_1*. Then there is a semiproper poset* $\mathcal{P} \subseteq V_\kappa$ *such that*

[24] $BPFA$ is the assertion that $\langle H(\omega_2), \in\rangle$ is a Σ_1–elementary substructure of the structure $\langle H(\omega_2), \in\rangle$ as computed in any forcing extension via a proper poset. $BPFA$ is a trivial consequence of the Proper Forcing axiom (PFA).

[25] One can easily verify that the standard poset for introducing, by initial conditions, a club avoiding a fixed club–sequence defined on a subset of ω_1 and whose height exists is always proper.

(1) $\mathcal{P}$ forces PFA^{++},

(2) $\mathcal{P}$ forces that A is definable over the structure $\langle H(\omega_2), \in\rangle$ by a Σ_5 formula without parameters, and

(3) $\mathcal{P}$ forces the existence of a well–order of $H(\omega_2)$ definable over the structure $\langle H(\omega_2), \in\rangle$ by a Σ_5 formula without parameters.

This time the proofs involve the manipulation of certain guessing properties of functions $F : S \longrightarrow \mathcal{P}(\omega_1)$[26] with respect to canonical functions.

Definition 3.1 *([As3]) Let S be a stationary subset of ω_1. Given $I \subseteq \omega_1$, S has guessing density I if for every stationary $S^* \subseteq S$,*

(a) there is a function $F : S^* \longrightarrow \mathcal{P}(\omega_1)$ such that $ot(F(\nu)) \in I$ for all $\nu \in S^*$ and such that $\{\nu \in S^* : g(\nu) \in F(\nu)\}$ is stationary for every $\alpha < \omega_2$ and every canonical function g for α, and

(b) given any function $F' : S^* \longrightarrow \mathcal{P}(\omega_1)$, if $ot(F'(\nu)) < min(I)$ for all $\nu \in S^*$, then there is an ordinal $\alpha < \omega_2$ such that $\{\nu \in S^* : g(\nu) \in F'(\nu)\}$ is nonstationary for every canonical function g for α.

Note that every stationary $S \subseteq \omega_1$ has density ω_1 and that there is no such thing as the unique guessing density of S: if S has guessing density I_0 and $I_1 \subseteq \omega_1$ is such that $min(I_1) \leq min(I_0)$ and $sup(I_0) \leq sup(I_1)$, then S also has guessing density I_1. Also, it is easy to see that for every stationary $S \subseteq \omega_1$, the assumption that $\diamondsuit(S^*)$[27] holds for every stationary $S^* \subseteq S$ implies that S has guessing density $\{1\}$. Finally, BMM[28] implies that no stationary subset of ω_1 has guessing density bounded in ω_1.

The following theorem is proved in [As3].

Theorem 3.2 *([As3]) Suppose κ is an inaccessible cardinal which is a limit of measurable cardinals. Let $\langle S_i : i < \omega_1\rangle$ be a sequence of pairwise disjoint*

[26]Where $S \subseteq \omega_1$ and where, for every $\nu \in S$, $ot(F(\nu))$ is in some prescribed interval of countable ordinals.

[27]Namely the statement that there is a sequence $\langle X_\alpha : \alpha \in S^*\rangle$ with $X_\alpha \subseteq \alpha$ for all α and $\{\alpha \in S^* : X \cap \alpha = X_\alpha\}$ stationary for each $X \subseteq \omega_1$.

[28]Namely the statement that the structure $\langle H(\omega_2), \in\rangle$ is a Σ_1–elementary substructure of $\langle H(\omega_2), \in\rangle^{V^{\mathcal{P}}}$ for every partial order $\mathcal{P}$ preserving stationary subsets of ω_1.

stationary subsets of ω_1 *and let* $\langle \alpha_i : i < \omega_1 \rangle$ *be a sequence of nonzero countable ordinals.*

Then there is a semiproper poset $\mathcal{P} \subseteq V_\kappa$ *forcing, for every* $i < \omega_1$*, that* S_i *has guessing density the interval* $[\alpha_i, \omega^{\alpha_i \cdot \omega})$[29] *if* $\alpha_i > 1$ *and that* S_i *has guessing density* $\{1\}$ *if* $\alpha_i = 1$.

The proof of Theorem 3.2 involves an analysis of iterations of models of set theory relative to sequences of (possibly different) measurable cardinals. It uses a generalization of a theorem of Kunen ([Ku]) saying that for every ordinal ϵ there are only finitely many measurable cardinals γ for which there is a normal measure U on γ such that ϵ is not a fixed point by the elementary embedding of the universe derived from U.

The forcing iteration for proving Theorem 3.2 consists of semiproper posets. In fact, the posets $\mathcal{Q}$ used there are quite close to being proper, in the sense that, although it may not be true that, for an arbitrary structure N containing $\mathcal{Q}$ and an arbitrary $p \in \mathcal{Q} \cap N$, there is an $(N, \mathcal{Q})$–generic condition extending p, it is nevertheless true that for every N and p as above there is a name $\tilde{N}$ for a structure including N and there is a condition extending p which is $(\tilde{N}, \mathcal{Q})$–generic (in a natural way) and, moreover, there is sufficient control in V on what the order type of the interpretation of $\tilde{N} \cap \overline{\kappa}$ (for the relevant $\overline{\kappa}$) is going to be. This ensures that things work as desired.[30] This type of argument also shows that our forcing iteration is robust enough to accommodate arbitrary proper posets (of size less than κ), even on a suitable club with complement unbounded in κ.

By a result of Moore ([Mo]), $BPFA$ implies the existence, given any ladder system $\overline{\epsilon}$ on ω_1 and any sequence $(U_i)_{i<\omega_1}$ of pairwise disjoint stationary subsets of ω_1, of a well–order of $H(\omega_2)$ definable over $\langle H(\omega_2), \in \rangle$ by a Σ_2 formula with $p = (\overline{\epsilon}, (U_i)_{i<\omega_1})$ as parameter. The proof of Theorem 3.1 follows from the above considerations: Suppose we start from a supercompact cardinal instead of just an inaccessible limit of measurable cardinals. Suppose $\langle S_i : i < \omega_1 \rangle$ and $\langle \alpha_i : i < \omega_1 \rangle$ (in the statement of Theorem 3.2) are such that $S_i \cap (i+1) = \emptyset$ for all $i > 0$ and such that $A^* = \{\alpha_i : i < \omega_1\}$ is a sparse enough set of countable ordinals coding in some simple (say, Σ_1) way, both a prescribed $A \subseteq \omega_1$ and a parameter $(\overline{\epsilon}, (U_i)_{i<\omega_1})$ as in Moore's theorem. We

[29] An expression of the form β^γ denotes ordinal exponentiation.

[30] It is worth mentioning that, unlike the forcing construction for proving Theorem 2.11 – which is a revised countable support iteration –, the construction for Theorem 3.2 works with a countable support iteration.

build a forcing iteration in which we perform the usual Baumgartner construction for PFA^{++} on a suitable club C of κ, and in which we force as in Theorem 3.2 on the complement of C. In the end we obtain a model in which PFA^{++} holds and in which A^* is defined as the set of of $\alpha < \omega_1$ for which there is a stationary subset of ω_1 with guessing density equal to the interval $[\alpha, \omega^{\alpha\cdot\omega})$. It follows easily that in this model A^* – and therefore also A – is definable over $\langle H(\omega_2), \in\rangle$ by a Σ_5 formula without parameters. Since A^* also codes a parameter as in Moore's theorem, it follows by that theorem that in the resulting model there is a well–order of $H(\omega_2)$ definable over $\langle H(\omega_2), \in\rangle$ also by a Σ_5 formula without parameters. This proves Theorem 3.1.

The second main result in this section is due to P. Larson and involves a strong form of Martin's Maximum (MM). Recall that MM is the following provably maximal forcing axiom for collections of $\aleph_1$–many dense:[31]

Suppose $\mathcal{P}$ is a poset such that forcing with $\mathcal{P}$ preserves the stationarity of all stationary subsets of ω_1 and suppose $\langle D_i : i < \omega_1\rangle$ is a sequence of dense subsets of $\mathcal{P}$. Then there is a filter $G \subseteq \mathcal{P}$ such that $G \cap D_i \neq \emptyset$ for every $i < \omega_1$.

MM is a maximal forcing axiom in the sense that, on the one hand, if $\mathcal{P}$ is a poset forcing that some stationary subset of ω_1 from the ground model is no longer stationary, then one can easily find a collection $\langle D_i : i < \omega_1\rangle$ of dense subsets of ω_1 such that $D_i \cap G = \emptyset$ for some i whenever $G \subseteq \mathcal{P}$ is a filter; whereas, on the other hand, MM can be forced over any model with a supercompact cardinal. In an older version of this paper I was asking whether the hypothesis that there is a supercompact cardinal (or some other reasonable large cardinal assumption) implies that there is a partial order forcing Martin's Maximum, together with the existence of a well–order of $H(\omega_2)$ definable, over $\langle H(\omega_2), \in\rangle$, by a formula without parameters (or even by a formula with a real number as parameter). Regarding this question, P. Larson has recently proved the following result ([L2]).

Theorem 3.3 *(Larson) Suppose κ is a supercompact limit of supercompact cardinals. Then there is semiproper poset $\mathcal{P} \subseteq V_\kappa$ such that*

(1) $\mathcal{P}$ forces $MM^{+\omega}$,[32] and

[31] Defined and proved consistent in [F-M-Sh].

[32] $MM^{+\omega}$ is the strengthening of MM saying that for every poset $\mathcal{Q}$ preserving station-

(2) $\mathcal{P}$ forces that there is well–order of $H(\omega_2)$ definable, over $\langle H(\omega_2), \in\rangle$, by a formula without parameters.

4 Open questions and some consequences of large cardinal axioms.

In the model of Theorem 4.1, MM^{++} fails necessarily. As far as I know, the following questions remain open.

Questions 4.1 *Assume some reasonable large cardinal hypothesis. Is it possible to force in such a way that MM^{++} holds in the extension, together with the existence of a well–order of $H(\omega_2)$ definable, over $\langle H(\omega_2), \in\rangle$, by a formula without parameters (or even by a formula with a real number as parameter)?*

Does MM^{++} imply that there is a well–order of $H(\omega_2)$ definable, over $\langle H(\omega_2), \in\rangle$, by a formula with at most a real number as parameter?

Let $\mathcal{D}$ denote the class of all subsets of $H(\omega_2)$ which are definable, over $\langle H(\omega_2), \in\rangle$, by a formula with at most a real number as parameter. I will finish this paper with a remark motivated by a failed attempt (so far) to construct a model of MM^{++} in which there is a well–order of $H(\omega_2)$ belonging to $\mathcal{D}$ (thus providing an answer to the first question above).

One first observation – due to Larson – I want to mention is that, in the presence of MM, the existence of a subset $A \subseteq \omega_2 \cap cf(\omega)$ such that A belongs to $\mathcal{D}$ and such that both A and $(\omega_2 \cap cf(\omega))\backslash A$ are stationary suffices to prove the existence of a stationary and co-stationary subset of ω_1 belonging to $\mathcal{D}$. In fact,

Fact 4.1 *(Larson) Suppose NS_{ω_1} is saturated and suppose $(\mathcal{P}(\omega_1))^\sharp$ exists. Suppose as well that for every stationary subset A of $\omega_2 \cap cf(\omega)$ there is some $\delta < \omega_2$ of cofinality ω_1 such that both A and $\omega_2\backslash A$ reflect at δ. If there is some $A \subseteq \omega_2 \cap cf(\omega)$ such that $A \in \mathcal{D}$ and such that both A and $(\omega_2 \cap cf(\omega))\backslash A$ are*

---

ary subsets of ω_1, every sequence $\langle D_i : i < \omega_1\rangle$ of dense subsets of $\mathcal{Q}$ and every sequence $\langle \tau_n : n < \omega\rangle$ of $\mathcal{Q}$–names for stationary subsets of ω_1 there is a filter $G \subseteq \mathcal{Q}$ such that $G \cap D_i \neq \emptyset$ for every $i < \omega_1$ and such that $\{\alpha < \omega_1 : (\exists p \in G)(p \Vdash_{\mathcal{Q}} \alpha \in \tau_n)\}$ is stationary for every $n < \omega$. MM^{++} is the strengthening of $MM^{+\omega}$ incorporating sequences of length ω_1 – instead just of length ω – of names for stationary subsets of ω_1.

stationary subsets of ω_2, then there is a stationary and co-stationary $S \subseteq \omega_1$ such that $S \in \mathcal{D}$.

Proof: Let r_0 be a real such that A is definable over $\langle H(\omega_2), \in\rangle$ by a formula with only r_0 as parameter. Let $\delta < \omega_2$ be an ordinal of uncountable cofinality such that both $A \cap \delta$ and $\delta \backslash A$ are stationary subsets of δ. From the saturation of NS_{ω_1} and the existence of $(\mathcal{P}(\omega_1))^\sharp$ it follows, by [W], Theorem 3.17, that the second uniform indiscernible[33] (u_2) is ω_2. In particular, we may fix a real r such that $|\delta|^{L[r]} = \omega_1^V$ and such that δ is definable, in $L[r]$, from ω_1^V. Notice that $cf(\delta)^{L[r]}$ is then exactly ω_1^V. Let C be the $<_{L[r]}$–least club of δ of order type ω_1^V and let $(\alpha_\nu)_{\nu<\omega_1^V}$ be its strictly increasing enumeration. Then, since $A \cap \delta$ and $\delta \backslash A$ are both stationary, $S := \{\nu < \omega_1 \,:\, \alpha_\nu \in A\}$ is a stationary and co-stationary subset of ω_1 which is definable, over $\langle H(\omega_2), \in\rangle$, by a formula with r_0, δ and r as parameters, and therefore also by a formula with r_0 and r as parameters. $\square$

The hypotheses of Fact 4.1 follow from MM: The saturation of NS_{ω_1} follows from [F-M-Sh], the existence of the sharp of every set follows in fact from BMM by a result of Schindler ([S]), and the simultaneous reflection of pairs of stationary subsets of $\omega_2 \cap cf(\omega)$ follows from [F-M-Sh].

A second observation is that, again in the presence of MM, the existence of a stationary and co-stationary subset belonging to $\mathcal{D}$ suffices to prove that in fact every subset of ω_1 belongs to $\mathcal{D}$:

Fact 4.2 *([As1], Observation 1.1) Suppose that BMM holds and NS_{ω_1} is saturated. Then, given any stationary and co-stationary $S \subseteq \omega_1$ and any $A \subseteq \omega_1$ there is a real r such that $A \in L[r, S]$. In particular, if $S \in \mathcal{D}$, then A is also in $\mathcal{D}$.*

Finally, MM implies the existence, given a sequence $\langle S_\alpha \,:\, \alpha < \omega_1\rangle$ of pairwise disjoint stationary subsets of ω_1, of a well–order of $H(\omega_2)$ which is definable over $\langle H(\omega_2), \in\rangle$ by a formula with $\langle S_\alpha \,:\, \alpha < \omega_1\rangle$ as parameter (for example by [W], Theorem 5.14 and Lemma 5.13).

A consequence of these observations is that, under MM, the existence of a well–order of $H(\omega_2)$ belonging to $\mathcal{D}$ follows from – and in fact is equivalent to – the existence of a stationary $A \subseteq \omega_2 \cap cf(\omega)$ belonging to $\mathcal{D}$ and such

[33] An ordinal is a uniform indiscernible if it is a Silver indiscernible for $L[r]$ for every real r.

that $(\omega_2 \cap cf(\omega))\backslash A$ is also stationary. Now, suppose we are in a context in which MM holds and we want to argue that there is a well–order of $H(\omega_2)$ in $\mathcal{D}$. One possibility would be to prove that there is some formula $\Phi(x)$, perhaps with a real parameter, such that $A_{\Phi(x)}$ and $(\omega_2 \cap cf(\omega))\backslash A_{\Phi(x)}$ are both stationary for $A_{\Phi(x)} := \{\alpha \in \omega_2 \cap cf(\omega) : \langle H(\omega_2), \in\rangle \models \Phi(\alpha)\}$.

This strategy asks for the stationarity of sets of ordinals of countable cofinality. On the other hand, Namba forcing (Nm) preserves stationary subsets of ω_1 and forces $cf(\omega_2^V) = \omega$. It certainly forces the following property about $x = \omega_2^V$, for any collection Δ of axioms of set theory of bounded Levy complexity:[34] "$cf(x) = \omega$, there is a transitive set $M \models \Delta$ such that $\omega_1^M = \omega_1$ and such that x is ω_2^M, and there is a Nm^M–generic filter over M". By general arguments involving forcing axioms it easily follows then that, by MM, there are stationarily many $\alpha < \omega_2$ of countable cofinality such that this statement holds with $x = \alpha$. Thus, it seemed to be a good idea to choose as $\Phi(x)$ a formula expressing something like the above property. In other words, it seemed a good idea to try to argue that the complement, relative to $\omega_2 \cap cf(\omega)$, of the set defined by the above property – or some other related property – must be stationary. For example, one could have expected to start with a model with a supercompact cardinal κ, perform the usual MM^{++}–forcing construction by a forcing with the κ–chain condition, as in [F-M-Sh], and argue that, in the extension, the above property fails for (say) stationarily many $\alpha < \omega_2 = \kappa$ of countable cofinality in the ground model. However, the observations I am about to present show that this hope is sterile.

It is well–known that certain properties for definable sets of reals follow from the existence of large cardinals. The first observation I want to make extends this type of results to the level of $H(\omega_2)$. In other words, it shows that the existence, in the universe, of certain objects of size $\aleph_1$ *follows* outright from large cardinal axioms.

Observation 4.3 *Suppose there are cardinals $\delta < \kappa$ such that δ is a limit of infinitely many Woodin cardinals and κ is measurable. Let $\Phi(x)$ be either a Σ_1 formula or a Π_1 formula (in the language $\mathcal{L}^*$ for the structure $\langle H(\omega_2), \in, NS_{\omega_1}, r\rangle_{r\in\mathbb{R}}$), let $\lambda \in \{\omega, \omega_1\}$ and let $\Phi^*(x)$ be a Σ_1 $\mathcal{L}^*$–formula or a Π_1 $\mathcal{L}^*$–formula expressing $\Phi(x) \wedge cf(x) = \lambda$. Suppose there is a poset $\mathcal{P} \in V_\delta$ forcing that there are stationarily many $\alpha < u_2$ such that $\langle H(\omega_2), \in, NS_{\omega_1}, r\rangle_{r\in\mathbb{R}} \models$*

[34] Or even for $\Delta = ZFC$ if, for example, some rank–initial segment of the universe satisfies ZFC.

$\Phi^*(\alpha)$. *Then, in V, there is a club $C \subseteq u_2$ in $L(\mathbb{R})$ such that $\langle H(\omega_2), \in, NS_{\omega_1}, r\rangle_{r\in\mathbb{R}} \models \Phi^*(\alpha)$ holds, in V, for every $\alpha \in C$.*

Proof: Let $\Phi(x)$, λ and $\mathcal{P}$ provide a counterexample. Suppose $\Phi^*(x)$ is a Σ_1 formula (the argument when $\Phi^*(x)$ is Π_1 is the same). Let A be, in $V^{\mathcal{P}}$, the set of $\alpha < u_2$ such that $\langle H(\omega_2), \in, NS_{\omega_1}, r\rangle_{r\in\mathbb{R}} \models \Phi^*(\alpha)$. In $V^{\mathcal{P}}$, there are infinitely many Woodin cardinals with a measurable above them. In particular, $AD^{L(\mathbb{R})}$ holds and there is a Woodin cardinal with a measurable above. Hence, by the proof of [W], Theorem 4.65, in $V^{\mathcal{P}}$, $\mathbb{P}_{max}$ forces $\langle H(\omega_2), \in, NS_{\omega_1}, r\rangle_{r\in\mathbb{R}} \models \Phi^*(\alpha)$ over $L(\mathbb{R})^{V^{\mathcal{P}}}$ whenever $\alpha \in A$.[35] In particular, by the definability of the forcing relation over ZF–models, $A \in L(\mathbb{R})^{V^{\mathcal{P}}}$. If $u_2 \backslash A$ were stationary in $V^{\mathcal{P}}$, then A and $(\omega_2 \cap cf(\lambda))\backslash A$ would be stationary subsets of ω_2 $(= u_2)$ in $L(\mathbb{R})^{V^{\mathcal{P}}}$.[36] But this contradicts a result, of Martin and Paris, saying that $\{C \cap cf(\lambda) : C \text{ a club of } \omega_2\}$ generates an ultrafilter of $\omega_2 \cap cf(\lambda)$ under AD (see [K], p. 395). Hence, in $L(\mathbb{R})^{V^{\mathcal{P}}}$ it holds that there is a club $C \subseteq u_2$ of ordinals of cofinality λ such that $\mathbb{P}_{max}$ forces $\langle H(\omega_2), \in, NS_{\omega_1}, r\rangle_{r\in\mathbb{R}} \models \Phi(\alpha)$ for every $\alpha \in C$ of cofinality λ. By another result of Woodin referred to already in Section 2, forcing with $\mathcal{P}$ does not change the theory of $L(\mathbb{R})$ with real numbers as parameters. Hence, in $L(\mathbb{R})^V$ there is also a club $C \subseteq u_2$ with the above property. Take any $\alpha \in C$ of cofinality λ and suppose, towards a final contradiction, that $\langle H(\omega_2), \in, NS_{\omega_1}, r\rangle_{r\in\mathbb{R}} \models \neg\Phi(\alpha)$ holds in V. Again by the proof of [W], Theorem 4.65, this time applied in V, $\langle H(\omega_2), \in, NS_{\omega_1}, r\rangle_{r\in\mathbb{R}} \models \neg\Phi(\alpha)$ also holds in $L(\mathbb{R})^{\mathbb{P}_{max}}$. Contradiction. $\square$

The following consequence of Observation 4.3 is relevant to the possible scenario presented in this section for building a model of MM^{++} with a well–order of $H(\omega_2)$ belonging to $\mathcal{D}$.

Corollary 4.4 *Suppose there is a supercompact cardinal. Then there is a club $C \subseteq u_2$, $C \in L(\mathbb{R})$, with the property, in V, that for every $\lambda \in \{\omega, \omega_1\}$ and every formula $\varphi(x)$, if ZFC proves that $\varphi(x)$ defines a poset – call it $\mathcal{P}_{\varphi(x)}$ – preserving stationary subsets of ω_1 and forcing $cf(\omega_2^V) = \lambda$, then for every $\alpha \in C$ of cofinality λ there is a transitive model M of ZFC computing stationary subsets of ω_1 correctly and with $\omega_2^M = \alpha$ and there is a $(\mathcal{P}_{\varphi(x)})^M$–generic filter over M.*

[35] Since every $\alpha < u_2$ is Σ_1 definable from ω_1 together with a real.

[36] As $L(\mathbb{R})^{V^{\mathcal{P}}}$ computes u_2 and cofinalities below u_2 correctly and as $AD \models u_2 = \omega_2$.

Proof: Fix $\lambda \in \{\omega, \omega_1\}$ and a formula $\varphi(x)$ a above. Since $cf(u_2) \geq \omega_1$, it suffices to show that there is a club $C \subseteq u_2$ such that every $\alpha \in C \cap cf(\lambda)$ has the following property (in V):

$P(\alpha)$: There is a transitive model M of ZFC computing stationary subsets of ω_1 correctly and such that $\omega_2^M = \alpha$ and there is a $(\mathcal{P}_{\varphi(x)})^M$–generic filter over M.

By the construction in [F-M-Sh] there is an iteration $\langle \mathcal{P}_\alpha : \alpha \leq \kappa \rangle$, $\mathcal{P}_\kappa \subseteq V_\kappa$, such that each $\mathcal{P}_\kappa / \dot{G}_\alpha$ is semiproper in $V^{\mathcal{P}_\alpha}$, and forcing both $u_2 = \omega_2$ and that there are stationarily many $\alpha < \omega_2 = \overline{\kappa}$ such that $\alpha = \omega_2^{V^{\mathcal{P}_\alpha}}$ and such that there is a $(\mathcal{P}_{\varphi(x)})^{V^{\mathcal{P}_\alpha}}$–generic filter over $V^{\mathcal{P}_\alpha}$. Hence, by reflection, we may fix $\overline{\kappa} < \kappa$ and a semiproper poset $\mathcal{P} \subseteq V_{\overline{\kappa}}$ forcing $u_2 = \omega_2$ and, since κ is inaccessible in $V^{\mathcal{P}_{\overline{\kappa}}}$, forcing that there are stationarily many $\alpha < \omega_2 = \overline{\kappa}$ such that $P(\alpha)$. Now we can apply Observation 4.3 since $P(\alpha)$ can be written as a Σ_1 sentence, over $\langle H(\omega_2), \in, NS_{\omega_1} \rangle$, with α as parameter. $\square$

Remember the approach pointed out before for producing, in the presence of MM and using Namba forcing, a stationary $A \subseteq \omega_2 \cap cf(\omega)$ in $\mathcal{D}$ with $(\omega_2 \cap cof(\omega)) \backslash A$ also stationary. Corollary 4.4 shows that this approach cannot work (just take $\varphi(x)$ to be a definition of Namba forcing).

References

[A-Sh] U. Abraham, S. Shelah, *Coding with ladders a well ordering of the reals.* J. of Symbolic Logic, 67 (2002), 579–597.

[As1] D. Asperó, *The nonexistence of robust codes for subsets of* ω_1. Fundamenta Mathematicae, vol. 186 (2005), 215–231.

[As2] D. Asperó, *Coding by club–sequences*, Ann. Pure and Applied Logic, vol. 142 (2006), 98–114.

[As3] D. Asperó, *Guessing and non-guessing of canonical functions*, Ann. Pure and Applied Logic, vol. 146 (2007), 150-179.

[As4] D. Asperó, *Coding into* $H(\kappa^+)$ *without parameters.* Submitted.

[As5] D. Asperó, *Forcing a simply definable well–order of* $H(\omega_2)$. Preprint.

[K] A. Kanamori, *The Higher Infinite: Large Cardinals in Set Theory from their Beginnings*. Springer-Verlag, Berlin (1994).

[Ku] K. Kunen, *A model for the negation of the Axiom of Choice*, in Cambridge Summer School in Mathematical Logic (A. Mathias and H. Rogers Jr., eds.), Lecture Notes in Math., vol. 337, Springer, Berlin (1973), 489–494.

[F-M-Sh] M. Foreman, M. Magidor, S. Shelah, *Martin's Maximum, saturated ideals and non-regular ultrafilters, I.* Ann. of Mathematics, 127 (1988), 1–47.

[L1] P. Larson, *The Stationary Tower. Notes on a Course by W. Hugh Woodin*, AMS University Lecture Series, 32. Providence, RI (2004).

[L2] P. Larson, *Martin's Maximum and definability in* $H(\aleph_2)$. Preprint.

[Mar-St] D. Martin, J. Steel, *A proof of projective determinacy*, J. American Math. Soc., 2 (1989), 71–125.

[Mo] J. Moore, *Set mapping reflection.* J. Math. Logic, 5 (2005), 87–98.

[S] R. Schindler, *Semi-proper forcing, remarkable cardinals, and Bounded Martin's Maximum*, Math. Logic Q., 50 (2004), 527–532.

[Sh] S. Shelah, *Proper and Improper Forcing*, 2nd. ed., Perspectives in Mathematical Logic, Springer, Berlin (1998).

[W] H. Woodin, *The axiom of Determinacy, Forcing Axioms, and the Nonstationary ideal.* De Gruyter Series in Logic and its Applications, Number 1. Berlin, New York (1999).

Institució Catalana de Recerca i Estudis Avançats (ICREA)
and
Dept. de Lògica, Història i Filosofia de la Ciència,
Universitat de Barcelona,
Montalegre 6, 08001 Barcelona

http://www.icrea.es/pag.asp?id=David.Aspero
david.aspero@icrea.es