THE SPECIAL ARONSZAJN TREE PROPERTY AT $\aleph_2$ AND GCH

DAVID ASPERÓ AND MOHAMMAD GOLSHANI

Abstract. Starting from the existence of a weakly compact cardinal, we build a generic extension of the universe in which GCH holds and all $\aleph_2$-Aronszajn trees are special and hence there are no $\aleph_2$-Souslin trees. This result answers a longstanding open question from the 1970's.

1. introduction

Let κ be an uncountable regular cardinal. Let us recall that a κ-tree is a tree T of height κ all of whose levels are smaller than κ, and that a κ-tree is called a κ-Aronszajn tree if it has no κ-branches. Also, T is called a κ-Souslin tree if it has no κ-branches and no antichains of size κ. When $\kappa = \lambda^+$ is a successor cardinal, a κ-Aronszajn tree is said to be *special* if and only if it is a union of λ antichains.[1] Let us make the following definition:

Definition 1.1. (1) *Souslin's Hypothesis at κ,* SH_κ*, is the statement "there are no κ-Souslin trees".*
(2) *The special Aronszajn tree property at $\kappa = \lambda^+$,* SATP_κ*, is the statement "there exist κ-Aronszajn trees and all such trees are special" (see* [5]*).*

Aronszajn trees were introduced by Aronszajn (see [9]), who proved the existence, in ZFC, of a special $\aleph_1$-Aronszajn tree. Later, Specker ([17]) showed that $2^{<\lambda} = \lambda$ implies the existence of special λ^+-Aronszajn trees for λ regular, and Jensen ([7]) produced special λ^+-Aronszajn trees for singular λ in L.

In [16], Solovay and Tennenbaum proved the consistency of Martin's Axiom + $2^{\aleph_0} > \aleph_1$ and showed that this implies $\mathrm{SH}_{\aleph_1}$. This was later extended by Baumgartner, Malitz and Reinhardt [3], who showed that

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[1] $A \subseteq T$ is an antichain of T iff A consists of pairwise incomparable nodes.

Martin's Axiom + $2^{\aleph_0} > \aleph_1$ implies $\mathrm{SATP}_{\aleph_1}$. Later, Jensen (see [4] and [13]) produced a model of GCH in which $\mathrm{SATP}_{\aleph_1}$ holds.

The situation at $\aleph_2$ turned out to be more complicated. In [7], Jensen proved that the existence of an $\aleph_2$-Souslin tree follows from each of the hypotheses CH + $\diamondsuit(S^2_1)$ and $\square_{\omega_1} + \diamondsuit(S^2_0)$ (where, given $m < n < \omega$, $S^n_m = \{\alpha < \aleph_n \mid \mathrm{cf}(\alpha) = \aleph_m\}$). The second result was improved by Gregory in [6], where he proved that GCH together with the existence of a non-reflecting stationary subset of S^2_0 yields the existence of an $\aleph_2$-Souslin tree. In [10], Laver and Shelah produced, relative to the existence of a weakly compact cardinal, a model of ZFC + CH in which the special Aronszajn tree property at $\aleph_2$ holds. But in their model $2^{\aleph_1} > \aleph_2$, and the task of finding a model of ZFC+GCH+$\mathrm{SATP}_{\aleph_2}$, or even of ZFC+GCH+$\mathrm{SH}_{\aleph_2}$, remained a major open problem. The earliest published mention of this problem seems to appear in [8] (see also [10], [18], [15], [14], or [11]).

In this paper we solve the above problem by proving the following theorem.

Theorem 1.2. *Suppose κ is a weakly compact cardinal. Then there exists a set-generic extension of the universe in which GCH holds, $\kappa = \aleph_2$, and the special Aronszajn tree property at $\aleph_2$ (and hence Souslin's Hypothesis at $\aleph_2$) holds.*

Remark 1.3.
(1) Our argument can be easily extended to deal with the successor of any regular cardinal.
(2) By results of Shelah and Stanley ([15]) and of Rinot ([12]), our large cardinal assumption is optimal. Specifically:
   (a) It is proved in [15] that if ω_2 is not weakly compact in L, then either $\square_{\omega_1}$ holds or there is a non-special $\aleph_2$-Aronszajn tree; in particular, GCH+$\mathrm{SATP}_{\aleph_2}$ implies that ω_2 is weakly compact in L by one of Jensen's results mentioned above.
   (b) Rinot proved in [12] that if GCH holds, $\lambda \geq \omega_1$ is a cardinal, and $\square(\lambda^+)$ holds, then there is a λ-closed λ^+-Souslin tree; on the other hand, Todorčević ([19]) proved that if $\kappa \geq \omega_2$ is a regular cardinal and $\square(\kappa)$ fails, then κ is weakly compact in L.

The rest of the paper is devoted to the proof of Theorem 1.2. We will next give an (inevitably) vague and incomplete description of the forcing witnessing the conclusion of this theorem.

The construction of the forcing witnessing Theorem 1.2 combines a natural iteration for specializing $\aleph_2$-Aronszajn trees, due to Laver and Shelah ([10]), with ideas from [2]. More specifically, we build a certain countable support forcing iteration $\langle \mathbb{Q}_\beta \mid \beta \leq \kappa^+\rangle$ with side conditions.

The first step of the construction is essentially the Lévy collapse of the weakly compact cardinal κ to become ω_2. At subsequent stages, we consider forcings for specializing $\aleph_2$-Aronszajn trees by countable approximations. Conditions in a given $\mathbb{Q}_\beta$, for $\beta > 0$, will consist of a working part f_q, together with a certain side condition. The working part f_q will be a countable function with domain contained in β such that for all $\alpha \in \mathrm{dom}(f_q)$,

- $f_q(\alpha)$ is a condition in the Lévy collapse if $\alpha = 0$, and
- if $\alpha > 0$, $f_q(\alpha)$ is a countable subset of $\kappa \times \omega_1$.

Letting $\alpha = \alpha_0 + \nu$, where α_0 is a multiple of ω_1 and $\nu < \omega_1$, any two distinct members of $f_q(\alpha)$, when $\alpha > 0$, will be forced to be incomparable nodes in a certain κ-Aronszajn tree $\utilde{T}_{\alpha_0}$ on $\kappa \times \omega_1$ chosen via a given bookkeeping function $\Phi : \mathcal{X} \longrightarrow H(\kappa^+)$, where $\mathcal{X}$ denotes the set of multiples of ω_1 below κ^+.

The side condition will be a countable directed graph τ_q whose vertices are ordered pairs of the form (N, γ), where N is an elementary submodel of $H(\kappa^+)$ such that $|N| = |N \cap \kappa|$ and ${}^{<|N|}N \subseteq N$, and where γ is an ordinal in the closure of $N \cap (\beta+1)$ in the order topology. Given any such (N, γ), γ is to be seen as a *marker for* N in q, telling us up to which stage is N 'active' as a model. We will tend to call such pairs (N, γ) *models with markers*. Whenever $\langle (N_0, \gamma_0), (N_1, \gamma_1)\rangle$ is an edge in τ_q, for a condition q, $(N_0, \in)$ and $(N_1, \in)$ are $\in$-isomorphic via a (unique) isomorphism Ψ_{N_0,N_1} such that $\Psi_{N_0,N_1}(\xi) \leq \xi$ for every ordinal $\xi \in N_0$ and such that Ψ_{N_0,N_1} is in fact an isomorphism between the structures $(N_0, \in, \Phi_\alpha)$ and $(N_1, \in, \Phi_{\Psi_{N_0,N_1}(\alpha)})$ whenever $\alpha \in N_0 \cap \gamma_0$ is such that $\Psi_{N_0,N_1}(\alpha) < \gamma_1$, for a certain sequence $(\Phi_\alpha)_{\alpha<\kappa^+}$ of increasingly expressive predicates contained in $H(\kappa^+)$.

In the above situation, N_0 and N_1 are to be seen as 'twin models', relative to q, with respect to all stages α and $\Psi_{N_0,N_1}(\alpha)$ such that $\alpha \in N_0 \cap \gamma_0$ and $\Psi_{N_0,N_1}(\alpha) < \gamma_1$. This means that the natural restriction of $f_q(\alpha)$ to N_0, i.e., $f_q(\alpha) \cap N_0$, is to be copied over, via Ψ_{N_0,N_1}, into the restriction of $f_q(\Psi_{N_0,N_1}(\alpha))$ to N_1, i.e., we require that

$$\Psi_{N_0,N_1}(f_q(\alpha) \cap N_0) = f_q(\alpha) \cap N_0 \subseteq f_q(\Psi_{N_0,N_1}(\alpha)),$$

and similarly for the restriction of $\tau_q \restriction \alpha+1$ to N_0 (with the restriction $\tau_q \restriction \alpha + 1$ being defined naturally).

We can describe our copying procedure by saying that we are *copying into the past information coming from the future via the edges in* τ_q. Given an edge $\langle (N_0, \gamma_0), (N_1, \gamma_1)\rangle$ as above and some $\alpha \in N_0 \cap \gamma_0$ such that $\bar{\alpha} = \Psi_{N_0,N_1}(\alpha) < \gamma_1$, the intersection of $f_q(\bar{\alpha})$ with $\delta_{N_0} \times \omega_1$ may certainly contain more information than the intersection of $f_q(\alpha)$ with

$\delta_{N_0} \times \omega_1$. Thanks to the way we are setting up the copying procedure—namely, only copying from the future into the past via edges as we have described—it is straightforward to see that our construction is in fact a forcing iteration, in the sense that $\mathbb{Q}_\alpha$ is a complete suborder of $\mathbb{Q}_\beta$ for all $\alpha < \beta$. This need not be true in general, in forcing constructions of this sort, if we allow also to copy 'from the past into the future'.[2]

For technical reasons, given an edge $\langle (N_0, \gamma_0), (N_1, \gamma_1) \rangle$ in τ_q and a stage $\alpha \in N_0 \cap \gamma_0$, we would like to require $\mathbb{Q}_{\alpha+1} \cap N_0$ to be a complete suborder of $\mathbb{Q}_{\alpha+1}$; indeed, having this would be useful in the proof that our construction has the κ-chain condition.[3] This cannot be accomplished, while defining $\mathbb{Q}_{\alpha+1}$, on pain of circularity. However, a certain approximation to the above situation can be meaningfully stipulated, which we do,[4] and this suffices for our purposes.

Our construction is σ-closed for all $\beta \leq \kappa^+$. In particular, forcing with $\mathbb{Q}_{\kappa^+}$ preserves ω_1 and CH. The preservation of all higher cardinals proceeds by showing that the construction has the κ-chain condition. For this, we use the weak compactness of κ in an essential way. The proof of the κ-c.c. of $\mathbb{Q}_\beta$, for each $\beta < \kappa^+$, is modelled after the corresponding proof in [10]; in fact it is a natural adaptation, to the current setting, of the proof in [10] of the κ-c.c. of the main forcing in that paper. The fact that the length of our iteration is not greater than κ^+ seems to be needed in this proof. Finally, the copying, for a given condition q, of all information coming from q via the edges occurring in τ_q is crucially used in the proof that our forcing preserves $2^{\aleph_1} = \aleph_2$ (s. the proof of Lemma 5.1).

Side conditions are often employed in forcing constructions with the purpose of guaranteeing that certain cardinals are preserved. In the present construction, on the other hand, they are used to ensure that the relevant level of GCH[5] is preserved. This use of side conditions is taken from [2], where they are crucially used in the proof of CH-preservation. It is worth observing that, while in the construction from [2] a certain amount of structure is needed among the models occurring in the side condition,[6] no structure whatsoever (for the underlying set

[2]It turns out that, in our specific construction, and thanks to clause (7) in the definition of condition, we could in fact have required to copy information in both directions, i.e. that, in the above situation, full symmetry obtains, below $\delta_{N_0} \times \omega_1$, between stages α and $\Psi_{N_0,N_1}(\alpha)$. However, the current presentation, only deriving full symmetry for a dense set of conditions, seems to be cleaner.

[3]We elaborate on this point at the end of Section 3.

[4]This is clause (7) in the definition of condition.

[5]$2^{\aleph_1} = \aleph_2$.

[6]Using the terminology of [1], they need to come from a symmetric system.

of models) is needed in the present construction. We should point out that even if it preserves $2^{\aleph_1} = \aleph_2$, our construction does add new subsets of ω_1 after collapsing κ to become ω_2, although only $\aleph_2$-many of them (cf. the construction in [2], where CH is preserved but $\aleph_1$-many new reals are added).

The paper is organized as follows. In Section 2 we define the notions of model with marker and edge, which we will be using throughout the paper, and prove some of their basic properties. In Section 3 we define our forcing construction and prove some if its basic properties. In Section 4 we show that the forcing has the κ-chain condition. This is the most elaborate proof in the paper.[7] Finally, in Section 5 we complete the proof of Theorem 1.2. The main argument in this section is to show that our forcing preserves $2^{\aleph_1} = \aleph_2$.

2. Models with markers and edges

In this section we set up the side condition part of our main forcing construction and discuss some of its properties. As we will see, our side condition forcing (i.e., the collection of our side conditions, with the natural extension relation) is a trivial forcing notion in the sense that any two conditions are compatible.

Let us fix, for the remainder of this paper, a weakly compact cardinal κ, and let us assume, without loss of generality, that $2^\mu = \mu^+$ for every cardinal $\mu \geq \kappa$.[8]

Given functions $f_0, \ldots, f_n$, for $n < \omega$, we let

$$f_n \circ \ldots \circ f_0$$

denote the function f with domain the set of $x \in \mathrm{dom}(f_0)$ such that for every $i < n$, $(f_i \circ \ldots \circ f_0)(x) \in \mathrm{dom}(f_{i+1})$, and such that for every $x \in \mathrm{dom}(f)$, $f(x) = f_n((f_{n-1} \circ \ldots \circ f_0)(x))$. For a function f and a set x we let $f(x)$ denote the empty set whenever $x \notin \mathrm{dom}(f)$.

Throughout the paper, if N is a set such that $N \cap \kappa$ is an ordinal, we denote this ordinal by δ_N and call it *the height of* N. If X is a set, we set

$$\mathrm{cl}(X) = X \cup \{\alpha \in \mathrm{Ord} \mid \alpha = \sup(X \cap \alpha)\}$$

If, in addition, γ is an ordinal, we let γ_X be the highest ordinal $\xi \in \mathrm{cl}(X)$ such that $\xi \leq \gamma$.

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[7]Cf. the proof in [10], where the hardest part is to prove that the forcing is κ-c.c., or the proof in [2], where the hardest part is to prove that the forcing is proper.

[8]In fact, if κ is weakly compact, then GCH at every cardinal $\mu \geq \kappa$ can be easily arranged by collapsing cardinals with conditions of size $\leq \kappa$, which will preserve the weak compactness of κ.

Let $\mathcal{X} = \{\omega_1 \cdot \alpha \mid \alpha < \kappa^+\}$.[9] Given an ordinal $\alpha < \kappa^+$, there is a unique representation $\alpha = \alpha_0 + \nu$, where $\alpha_0 \in \mathcal{X}$ and $\nu < \omega_1$. We will denote the above ordinal α_0 by $\mu(\alpha)$.

Let

$$\Phi : \mathcal{X} \longrightarrow H(\kappa^+)$$

be such that for each $x \in H(\kappa^+)$, $\Phi^{-1}(x)$ is a stationary subset of $\mathcal{X}$.

This function Φ exists by $2^\kappa = \kappa^+$. Also, let $F : \kappa^+ \longrightarrow H(\kappa^+)$ be a bijection which is definable over the structure $\langle H(\kappa^+), \in, \Phi\rangle$. (We may for example let $\langle W_\alpha \mid \alpha < \kappa^+\rangle$ be the $<_\Phi$-increasing enumeration of $\{\Phi^{-1}(x) \mid, x \in H(\kappa^+)\}$, where $<_\Phi$ is defined by setting $\Phi^{-1}(x) <_\Phi \Phi^{-1}(y)$ iff $\min(\Phi^{-1}(x)) < \min(\Phi^{-1}(y))$, and then we may define F so that $F^{-1}(x) = \alpha$ if and only $\Phi^{-1}(x) = W_\alpha$.) Let also Φ_0 be the satisfaction predicate for the structure $\langle H(\kappa^+), \in, \Phi\rangle$.

Definition 2.1 (Models with markers)**.** *An ordered pair (N, γ) is called a* model with marker *if and only if:*

1. $(N, \in, \Phi_0) \prec (H(\kappa^+), \in, \Phi_0)$. [10]
2. $N \cap \kappa \in \kappa$, $|N| = |N \cap \kappa|$, *and* ${}^{<|N|}N \subseteq N$.
3. $\gamma \in \mathrm{cl}(N) \cap \kappa^+$.

We will often use, without mention, the fact that $(N, \gamma) \in N'$ whenever (N, γ) and (N', γ') are models with markers and $N \in N'$.[11]

Notation 2.2. *Given models N_0 and N_1 such that $(N_0, \in) \cong (N_1, \in)$, we will denote the unique $\in$-isomorphism $\Psi : (N_0, \in) \to (N_1, \in)$ by Ψ_{N_0,N_1}.*

Given any nonzero ordinal $\eta < \kappa^+$, let e_η be the F-least surjection from κ onto η. Let $\vec{e} = \langle e_\eta \mid 0 < \eta < \kappa^+\rangle$. We will say that a model $N \subseteq H(\kappa^+)$ is *closed under $\vec{e}$* if $e_\eta(\xi) \in N$ for every nonzero $\eta \in N \cap \kappa^+$ and every $\xi \in \kappa \cap N$.

Lemma 2.3. *Suppose N_0 and N_1 are models closed under $\vec{e}$ of the same height. Then $N_0 \cap N_1 \cap \kappa^+$ is an initial segment of both $N_0 \cap \kappa^+$ and $N_1 \cap \kappa^+$. In particular, if (N_0^i, γ_0^i) and (N_1^i, γ_1^i) (for $i \leq n$) are models with markers such that $(N_0^i, \in, \Phi_0) \cong (N_1^i, \in, \Phi_0)$ for all $i \leq n$, then*

$$(\Psi_{N_0^n,N_1^n} \circ \ldots \circ \Psi_{N_0^0,N_1^0})(x) = x$$

for every $x \in \mathrm{dom}(\Psi_{N_0^n,N_1^n} \circ \ldots \circ \Psi_{N_0^0,N_1^0}) \cap N_1^n$.

[9] Where, here and elsewhere, the dot in $\omega_1 \cdot \alpha$ denotes ordinal multiplication.

[10] Given $N \subseteq H(\kappa^+)$ and predicates $P_0, \ldots, P_n \subseteq H(\kappa^+)$, we will tend to write $(N, \in, P_0, \ldots, P_n)$ as short-hand for $(N, \in, P_0 \cap N, \ldots, P_n \cap N)$.

[11] Note that $N \in N'$ implies $\gamma \in N'$ as well. This is because, $\mathrm{cl}(N) \cap \kappa^+ \in N'$ and $\mathrm{cl}(N) \cap \kappa^+$ has size less than κ, so $\mathrm{cl}(N) \cap \kappa^+ \subseteq N'$.

Proof. Let us first prove the first assertion. Given any nonzero $\eta \in N_0 \cap N_1 \cap \kappa^+$ and any $\alpha \in N_0 \cap \eta$ there is some $\xi \in N_0 \cap \kappa$ such that $e_\eta(\xi) = \alpha$. But since η and ξ are both members of N_1, we also have that $\alpha = e_\eta(\xi) \in N_1$.

As to the second assertion, let us first consider the case in which $\delta_{N_0^i} = \delta_{N_0^{i'}}$ for all i, i'. By the choice of $\vec{e}$, each of the models N_ϵ^i, for $i \leq n$ and $\epsilon \in \{0, 1\}$, is closed under $\vec{e}$.

Let $x \in \mathrm{dom}(\Psi_{N_0^n, N_1^n} \circ \ldots \circ \Psi_{N_0^0, N_1^0}) \cap N_1^n$ and $\alpha = F^{-1}(x)$. We first prove by induction on $i \leq n$ that

$$\mathrm{ot}(N_0^0 \cap \alpha) = \mathrm{ot}(N_1^i \cap (\Psi_{N_0^i, N_1^i} \circ \ldots \circ \Psi_{N_0^0, N_1^0})(\alpha))$$

and

$$(\Psi_{N_0^i, N_1^i} \circ \ldots \circ \Psi_{N_0^0, N_1^0})(x) = F((\Psi_{N_0^i, N_1^i} \circ \ldots \circ \Psi_{N_0^0, N_1^0})(\alpha)).$$

For $i = 0$ this is true since $\Psi_{N_0^0, N_1^0}$ is an isomorphism between the structures $(N_0^0, \in, \Phi)$ and $(N_1^0, \in, \Phi)$. For $i > 0$, assuming the above equalities hold for $i-1$, we have that $(\Psi_{N_0^{i-1}, N_1^{i-1}} \circ \ldots \circ \Psi_{N_0^0, N_1^0})(x) \in N_0^i$ and therefore

$$(\Psi_{N_0^{i-1}, N_1^{i-1}} \circ \ldots \circ \Psi_{N_0^0, N_1^0})(\alpha) \in N_0^i$$

since

$$(\Psi_{N_0^{i-1}, N_1^{i-1}} \circ \ldots \circ \Psi_{N_0^0, N_1^0})(x) = F((\Psi_{N_0^{i-1}, N_1^{i-1}} \circ \ldots \circ \Psi_{N_0^0, N_1^0})(\alpha))$$

and $F : \kappa^+ \longrightarrow H(\kappa^+)$ is a bijection. Then we have that

$$\mathrm{ot}(N_0^i \cap (\Psi_{N_0^{i-1}, N_1^{i-1}} \circ \ldots \circ \Psi_{N_0^0, N_1^0})(\alpha))$$

and

$$\mathrm{ot}(N_1^i \cap (\Psi_{N_i^0, N_1^i} \circ \Psi_{N_0^{i-1}, N_1^{i-1}} \circ \ldots \circ \Psi_{N_0^0, N_1^0})(\alpha))$$

are equal (since $\Psi_{N_0^i, N_1^i}$ is an $\in$-isomorphims between N_0^i and N_1^i) and

$$\mathrm{ot}(N_1^{i-1} \cap (\Psi_{N_0^{i-1}, N_1^{i-1}} \circ \ldots \circ \Psi_{N_0^0, N_1^0})(\alpha))$$

and

$$\mathrm{ot}(N_0^i \cap (\Psi_{N_0^{i-1}, N_1^{i-1}} \circ \ldots \circ \Psi_{N_0^0, N_1^0})(\alpha))$$

are equal (by the first assertion as $(\Psi_{N_0^{i-1}, N_1^{i-1}} \circ \ldots \circ \Psi_{N_0^0, N_1^0})(\alpha) \in N_1^{i+1} \cap N_0^i)$. Hence,

$$\mathrm{ot}(N_0^0 \cap \alpha) = \mathrm{ot}(N_1^i \cap (\Psi_{N_0^i, N_1^i} \circ \ldots \circ \Psi_{N_0^0, N_1^0})(\alpha))$$

For the second conclusion, we note that

$$(\Psi_{N_0^i, N_1^i} \circ \ldots \circ \Psi_{N_0^0, N_1^0})(x) = (\Psi_{N_0^i, N_1^i} \circ F \circ \Psi_{N_0^{i-1}, N_1^{i-1}} \circ \ldots \circ \Psi_{N_0^0, N_1^0})(\alpha),$$

which is equal to $F(\Psi_{N_0^i, N_1^i} \circ \ldots \Psi_{N_0^0, N_1^0})(\alpha))$ since $\Psi_{N_0^i N_1^i}$ is an isomorphism between $(N_0^i, \in, \Phi)$ and $(N_1^i, \in, \Phi)$.

Finally, we note that $\mathrm{ot}(N_1^n \cap \alpha) = \mathrm{ot}(N_0^0 \cap \alpha)$ by the first assertion as $\alpha = F^{-1}(x) \in N_0^0 \cap N_1^n$. Hence,

$$\mathrm{ot}(N_1^n \cap (\Psi_{N_0^n, N_1^n} \circ \ldots \circ \Psi_{N_0^0, N_1^0})(\alpha)) = \mathrm{ot}(N_1^n \cap \alpha)$$

and therefore $(\Psi_{N_0^n, N_1^n} \circ \ldots \circ \Psi_{N_0^0, N_1^0})(\alpha) = \alpha$. Then also

$$(\Psi_{N_0^n, N_1^n} \circ \ldots \circ \Psi_{N_0^0, N_1^0})(x) = F(\Psi_{N_0^n, N_1^n} \circ \ldots \circ \Psi_{N_0^0, N_1^0})(\alpha) = F(\alpha) = x$$

Suppose now that $\delta_{N_0^i} \neq \delta_{N_0^{i'}}$ for some $i \neq i'$ and let i^* be such that $\delta_{N_0^{i^*}} = \min\{\delta_{N_0^i} \mid i \leq n\}$. Given any i and ϵ, we know that $(N_\epsilon^i, \in, \Phi_0) \prec (H(\kappa^+), \in, \Phi_0)$ and that N_ϵ^i is closed under sequences of length less than $|N_\epsilon^i|$. Also, $|N_\epsilon^{i^{**}}| = |\delta_{N_0^{i^*}}|$ for every $\epsilon \in \{0, 1\}$ and every i^{**} such that $\delta_{N_0^{i^{**}}} = \delta_{N_0^{i^*}}$. But now it easily follows from the above facts that there is a sequence $(\langle \bar{N}_0^i, \bar{N}_1^i \rangle)_{i \leq n}$ of pairs of models with the following properties.

- For all $i \leq n$, $(\bar{N}_0^i, \in, \Phi)$ and $(\bar{N}_1^i, \in, \Phi)$ are isomorphic elementary submodels of $(H(\kappa^+), \in, \Phi)$ and $\delta_{\bar{N}_0^i} = \delta_{N_0^{i^*}}$.
- For every $i \leq n$ and every $\epsilon \in \{0, 1\}$, $\bar{N}_\epsilon^i = N_\epsilon^i$ or $\bar{N}_\epsilon^i \in N_\epsilon^i$.
- $x \in \mathrm{dom}(\Psi_{\bar{N}_0^n, \bar{N}_1^n} \circ \ldots \circ \Psi_{\bar{N}_0^0, \bar{N}_1^1}) \cap \bar{N}_1^n$
- $(\Psi_{N_0^n, N_1^n} \circ \ldots \circ \Psi_{N_0^0, N_1^0})(x) = (\Psi_{\bar{N}_0^n, \bar{N}_1^n} \circ \ldots \circ \Psi_{\bar{N}_0^0, \bar{N}_1^1})(x)$

Indeed, we can define $\bar{N}_0^{i^*+j}$, $\bar{N}_1^{i^*+j}$, $\bar{N}_0^{i^*-k}$ and $\bar{N}_1^{i^*-k}$, for $j \leq n - i^*$ and $k \leq i^*$, by recursion as follows.

- If $\delta_{N_0^{i^*+j}} = \delta_{N_0^{i^*}}$, then $\bar{N}_0^{i^*+j} = N_0^{i^*+j}$ and $\bar{N}_1^{i^*+j} = N_1^{i^*+j}$, and if $\delta_{N_0^{i^*-k}} = \delta_{N_0^{i^*}}$, then $\bar{N}_0^{i^*-k} = N_0^{i^*-k}$ and $\bar{N}_1^{i^*-k} = N_1^{i^*-k}$.
- If $\delta_{N_0^{i^*}} < \delta_{N_0^{i^*+j}}$ (in which case $j > 0$), then $\bar{N}_0^{i^*+j} \in N_0^{i^*+j}$ is such that
  - $(N_0^{i^*+j}, \in, \Phi) \prec (H(\kappa^+), \in, \Phi)$,
  - $(\bar{N}_1^{i^*+j-1}, \in, \Phi) \cong (\bar{N}_0^{i^*+j}, \in, \Phi)$, and
  - $\bar{N}_1^{i^*+j-1} \cap N_0^{i^*+j} \subseteq \bar{N}_0^{i^*+j}$,

  and $\bar{N}_1^{i^*+j} = \Psi_{N_0^{i^*+j}, N_1^{i^*+j}}(\bar{N}_0^{i^*+j})$.
- If $\delta_{N_0^{i^*}} < \delta_{N_0^{i^*-k}}$ (in which case $k > 0$), then $\bar{N}_1^{i^*-k} \in N_1^{i^*-k}$ is such that
  - $(N_1^{i^*-k}, \in, \Phi) \prec (H(\kappa^+), \in, \Phi)$,
  - $(\bar{N}_0^{i^*-k+1}, \in, \Phi) \cong (\bar{N}_1^{i^*-k}, \in, \Phi)$, and
  - $\bar{N}_0^{i^*-k+1} \cap N_1^{i^*-k} \subseteq \bar{N}_1^{i^*-k}$,

  and $\bar{N}_0^{i^*-k} = \Psi_{N_1^{i^*-k}, N_0^{i^*-k}}(\bar{N}_1^{i^*-k})$.

But now we are done by the previous case. □

It will be convenient to use the following pieces of terminology: given models with markers (N_0, γ_0), (N_1, γ_1), we will say that (N_0, γ_0) *and*

(N_1, γ_1) *are twin models (with markers)* if and only if $(N_0, \in, \Phi_0) \cong (N_1, \in, \Phi_0)$. If $\Psi_{N_0,N_1}(\alpha) \leq \alpha$ for every ordinal $\alpha \in N_0$, then we say that N_1 *is a projection of* N_0.

Definition 2.4 (Edge)**.** *Suppose $\vec{\Phi} = (\Phi_\alpha)_\alpha$ is a sequence of predicates of $H(\kappa^+)$ of length less than κ^+. An ordered pair*

$$\langle (N_0, \gamma_0), (N_1, \gamma_1) \rangle$$

of models with markers is called a $\vec{\Phi}$-edge *if and only if the following are satisfied:*

1. *(N_0, γ_0) and (N_1, γ_1) are twin models with markers;*
2. *for every $\epsilon \in \{0, 1\}$ and every $\alpha \in N_\epsilon \cap \gamma_\epsilon$, $(N_\epsilon, \in, \Phi_\alpha) \prec (H(\kappa^+), \in, \Phi_\alpha)$;*
3. *N_1 is a projection of N_0;*
4. *Ψ_{N_0,N_1} is an isomorphism between the structures $(N_0, \in, \Phi_\alpha)$ and $(N_1, \in, \Phi_{\bar{\alpha}})$ for every $\alpha \in N_0 \cap \gamma_0$ such that $\bar{\alpha} := \Psi_{N_0,N_1}(\alpha) < \gamma_1$.*

Moreover, if $\gamma_0 \leq \beta$ and $\gamma_1 \leq \beta$, then we call $\langle (N_0, \gamma_0), (N_1, \gamma_1) \rangle$ a $\vec{\Phi}$-edge below β.

Definition 2.5 (Generalized edge)**.** *An ordered pair*

$$e = \langle (N_0, \gamma_0), (N_1, \gamma_1) \rangle$$

of models with markers is called a $\vec{\Phi}$-anti-edge *if*

$$e^{-1} := \langle (N_1, \gamma_1), (N_0, \gamma_0) \rangle$$

is a $\vec{\Phi}$-edge. We say that an ordered pair e is a generalized $\vec{\Phi}$-edge *if it is a $\vec{\Phi}$-edge or a $\vec{\Phi}$-anti-edge.*

Convention 2.6. *If τ is a set of generalized $\vec{\Phi}$-edges, we say that a generalized $\vec{\Phi}$-edge e* comes from τ *in case $e \in \tau$ or $e^{-1} \in \tau$. We also set $\tau^{-1} = \{e^{-1} : e \in \tau\}$.*

Given a generalized $\vec{\Phi}$-edge $e = \langle (N_0, \gamma_0), (N_1, \gamma_1) \rangle$ and an ordinal α, we let $e \restriction \alpha$ denote the generalized $(\vec{\Phi} \restriction \alpha)$-edge

$$\langle (N_0, \min\{\alpha, \gamma_0\}_{N_0}), (N_1, \min\{\alpha, \gamma_1\}_{N_1}) \rangle.$$[12]

Given a collection τ of $\vec{\Phi}$-edges and given an ordinal α, we denote by $\tau \restriction \alpha$ the set $\{e \restriction \alpha \mid e \in \tau\}$. Note that $\tau \restriction \alpha$ is a collection of $(\vec{\Phi} \restriction \alpha)$-edges below α.

[12] Recall that $\min\{\alpha, \gamma_\epsilon\}_{N_\epsilon}$, for $\epsilon \in \{0, 1\}$, is the highest ordinal $\xi \in \mathrm{cl}(N_\epsilon)$ such that $\xi \leq \min\{\alpha, \gamma_\epsilon\}$.

Given a sequence $\vec{\Phi} = (\Phi_\alpha)_\alpha$ of predicates of $H(\kappa^+)$, we say that *for each α, Φ_α codes $\langle \Phi_\beta \mid \beta < \alpha \rangle$ in a uniform way* in case there is a formula $\varphi(x, y)$ in the language for the structures $(H(\kappa^+), \in, \Phi_\alpha)$ such that for all $\beta < \alpha$ less than the length of $\vec{\Phi}$, and for each $a \in H(\kappa^+)$, $a \in \Phi_\beta$ if and only if $(H(\kappa^+), \in, \Phi_\alpha) \models \varphi(\beta, a)$.

Given models with markers (N, γ), (N_0, γ_0) and (N_1, γ_1), if $N \in N_0$ and $(N_0, \in) \cong (N_1, \in)$, then we let $\pi^{\gamma, N}_{N_0, \gamma_0, N_1, \gamma_1}$ denote the supremum of the set of ordinals of the form $\Psi_{N_0, N_1}(\xi)$, where

- $\xi \in N \cap (\gamma + 1)$,
- $\xi < \gamma_0$, and
- $\Psi_{N_0, N_1}(\xi) < \gamma_1$.

We let also $\pi^{\gamma, N}_e$ denote $\pi^{\gamma, N}_{N_0, \gamma_0, N_1, \gamma_1}$ whenever $e = \langle (N_0, \gamma_0), (N_1, \gamma_1) \rangle$ is a $\vec{\Phi}$-edge. Given generalized $\vec{\Phi}$-edges $e = \langle (N_0, \gamma_0), (N_1, \gamma_1) \rangle$ and $e' = \langle (N'_0, \gamma'_0), (N'_1, \gamma'_1) \rangle$ such that $e' \in N_0$, we denote

$$\langle (\Psi_{N_0, N_1}(N'_0), \pi^{\gamma'_0, N'_0}_{N_0, \gamma_0, N_1, \gamma_1}), (\Psi_{N_0, N_1}(N'_1), \pi^{\gamma'_1, N'_1}_{N_0, \gamma_0, N_1, \gamma_1}) \rangle$$

by $\Psi_e(e')$. Note that if $\vec{\Phi}$ is such that for each α, Φ_α codes $\langle \Phi_\beta \mid \beta < \alpha \rangle$ in a uniform way, then $\Psi_e(e')$ is a generalized $\vec{\Phi}$-edge.

Definition 2.7 (Closedness under copying)**.** *Suppose $\vec{\Phi}$ is such that for each α, Φ_α codes $\langle \Phi_\beta \mid \beta < \alpha \rangle$ in a uniform way. A set τ of $\vec{\Phi}$-edges is* closed under copying *in case for all $\vec{\Phi}$-edges $e = \langle (N_0, \gamma_0), (N_1, \gamma_1) \rangle$ and $e' = \langle (N'_0, \gamma'_0), (N'_1, \gamma'_1) \rangle$ in τ such that $e' \in N_0$ there are ordinals $\gamma^*_0 \geq \pi^{\gamma'_0, N'_0}_e$ and $\gamma^*_1 \geq \pi^{\gamma'_1, N'_1}_e$ such that*

$$\langle (\Psi_{N_0, N_1}(N'_0), \gamma^*_0), (\Psi_{N_0, N_1}(N'_1), \gamma^*_1) \rangle \in \tau.$$

Given a sequence $\vec{\mathcal{E}} = (\langle (N^i_0, \gamma^i_0), (N^i_1, \gamma^i_1) \rangle \mid i < n)$ of generalized $\vec{\Phi}$-edges, we will tend to denote the expression

$$\Psi_{N^{n-1}_0, N^{n-1}_1} \circ \ldots \circ \Psi_{N^0_0, N^0_1}$$

by $\Psi_{\vec{\mathcal{E}}}$. If $\vec{\mathcal{E}}$ is the empty sequence, we let $\Psi_{\vec{\mathcal{E}}}$ be the identity function.

Definition 2.8 (Pure side conditions forcing)**.** *Suppose $\vec{\Phi} = (\Phi_\alpha)_\alpha$ is a sequence of predicates of $H(\kappa^+)$ such that for each α, Φ_α codes $\langle \Phi_\beta \mid \beta < \alpha \rangle$ in a uniform way. Let $\beta < \kappa^+$. Let $\mathbb{P}^{\mathsf{e}}_{\vec{\Phi}, \beta}$ be the set of all countable sets τ of $\vec{\Phi}$-edges below β which are closed under copying and $\vec{e}$. Given conditions τ_0 and τ_1 in $\mathbb{P}^{\mathsf{e}}_{\vec{\Phi}, \beta}$, let $\tau_1 \leq \tau_0$ if for every $\langle (N_0, \gamma_0), (N_1, \gamma_1) \rangle \in \tau_0$ there are $\gamma'_0 \geq \gamma_0$ and $\gamma'_1 \geq \gamma_1$ such that $\langle (N_0, \gamma'_0), (N_1, \gamma'_1) \rangle \in \tau_1$.*

The next lemma shows that $\mathbb{P}^{\mathsf{e}}_{\vec{\Phi},\beta}$ is the trivial forcing notion.

Lemma 2.9. *Given sets τ^0 and τ^1 of $\vec{\Phi}$-edges, there exists a smallest set $\tau = \tau^0 \oplus \tau^1$ of $\vec{\Phi}$-edges which contains both τ_0 and τ_1 and is closed under copying. Furthermore, if both τ^0 and τ^1 are from $\mathbb{P}^{\mathsf{e}}_{\vec{\Phi},\beta}$, then so is τ.*

Proof. Let $\tau^0 \oplus \tau^1$ be the natural amalgamation of τ^0 and τ^1 obtained by taking copies of $\vec{\Phi}$-edges as dictated by suitable functions $\Psi_{\vec{\mathcal{E}}}$, so that $\tau^0 \oplus \tau^1$ is closed under copying (where the $\vec{\Phi}$-edges generated by this copying procedure have minimal marker so that $\tau^0 \oplus \tau^1$ is closed under copying). To be more specific, $\tau^0 \oplus \tau^1 = \bigcup_{n<\omega} \tau_n$, where $(\tau_n)_n$ is the following sequence.

(1) $\tau_0 = \tau^0 \cup \tau^1$
(2) For each $n < \omega$, $\tau_{n+1} = \tau_n \cup \tau_n'$, where τ_n' is the set of $\vec{\Phi}$-edges of the form $\Psi_e(e')$, for $\vec{\Phi}$-edges $e = \langle (N_0, \gamma_0), (N_1, \gamma_1) \rangle$ and e' in τ_n such that $e' \in N_0$.

Then $\tau = \tau^0 \oplus \tau^1$ is as required. □

Given τ^0 and τ^1, two sets of $\vec{\Phi}$-edges, the construction in the proof of Lemma 2.9 of $\tau^0 \oplus \tau^1$ as $\bigcup_{n<\omega} \tau_n$ gives rise to a natural notion of rank on the set of generalized $\vec{\Phi}$-edges coming from $\tau^0 \oplus \tau^1$. Specifically, given $n < \omega$, a generalized $\vec{\Phi}$-edge $e = \langle (N_0, \gamma_0), (N_1, \gamma_1) \rangle$ coming from $\tau^0 \oplus \tau^1$ has (τ^0, τ^1)-*rank* n iff n is least such that e comes from τ_n. Alternatively, we may define the (τ^0, τ^1)-*rank of* e as follows.

- e has (τ^0, τ^1)-rank 0 if e comes from $\tau^0 \cup \tau^1$.
- For every $n < \omega$, a generalized $\vec{\Phi}$-edge e coming from $\tau^0 \oplus \tau^1$ has (τ^0, τ^1)-rank $n + 1$ iff e does not have (τ^0, τ^1)-rank m for any $m \leq n$ and there are $\vec{\Phi}$-edges $e_0 = \langle (N_0, \gamma_0), (N_1, \gamma_1) \rangle$ and e_1 coming from $\tau^0 \oplus \tau^1$ and such that
  - the maximum of the (τ^0, τ^1)-ranks of e_0 and e_1 is n,
  - $e_1 \in N_0$, and
  - $e = \Psi_{e_0}(e_1)$.

Definition 2.10 (τ-thread)**.** *Given a set τ of $\vec{\Phi}$-edges, a sequence $\vec{\mathcal{E}} = (\langle (N_0^i, \gamma_0^i), (N_1^i, \gamma_1^i) \rangle \mid i < n)$ of generalized $\vec{\Phi}$-edges coming from τ, and $x \in N_0^0$, we will call $\langle \vec{\mathcal{E}}, x \rangle$ a* τ-thread *in case $x \in \mathrm{dom}(\Psi_{\vec{\mathcal{E}}})$. In the above situation, if $x = (y, \alpha)$, where $y \in H(\kappa^+)$ and $\alpha < \kappa^+$, we call $\langle \vec{\mathcal{E}}, x \rangle$ a* correct τ-thread *if and only if*

(1) $\alpha < \gamma_0^0$,
(2) $\Psi_{\vec{\mathcal{E}}}(\alpha) \in \gamma_1^{n-1}$, *and*

(3) $\Psi_{\vec{\mathcal{E}}}$ *is a (partially defined) elementary embedding from the structure* $(N_0^0, \in, \Phi_\alpha)$ *into the structure* $(N_1^{n-1}, \in, \Phi_{\Psi_{\vec{\mathcal{E}}}(\alpha)})$.

We will sometimes just say *thread* when τ is not relevant. It will be useful to consider the following strengthening of the notion of correct thread:

Definition 2.11. *Given a set* τ *of* $\vec{\Phi}$*-edges, a sequence*

$$\vec{\mathcal{E}} = (\langle (N_0^i, \gamma_0^i), (N_1^i, \gamma_1^i) \rangle \mid i < n)$$

of generalized $\vec{\Phi}$*-edges coming from* τ*, and* $x = (y, \alpha) \in H(\kappa^+) \times \kappa^+$*,* $\langle \vec{\mathcal{E}}, x \rangle$ *is a* connected τ-thread *in case*

(1) $\langle \vec{\mathcal{E}}, x \rangle$ *is a* τ*-thread,*
(2) $\alpha < \gamma_0^0$*, and*
(3) *for each* $i < n$*,*
   (a) $(\Psi_{N_i^0, N_i^1} \circ \ldots \circ \Psi_{N_0^0, N_1^0})(\alpha) < \gamma_1^i$*, and*
   (b) $(\Psi_{N_i^0, N_i^1} \circ \ldots \circ \Psi_{N_0^0, N_1^0})(\alpha) < \gamma_0^{i+1}$ *if* $i + 1 < n$*.*

Remark 2.12. Given a set τ of $\vec{\Phi}$-edges, every connected τ-thread is correct.

The following lemma can be easily proved by induction on the (τ^0, τ^1)-rank of the members of $\tau^0 \oplus \tau^1$.

Lemma 2.13. *Suppose for each* α*,* Φ_α *codes* $\langle \Phi_\beta \mid \beta < \alpha \rangle$ *in a uniform way. Let* τ^0 *and* τ^1 *be sets of* $\vec{\Phi}$*-edges, and let* $\lambda < \kappa$ *be an ordinal such that all members of* τ^0 *involve models of height less than* λ*. Suppose* τ^1 *is closed under copying. Then all members of* $\tau^0 \oplus \tau^1$ *involving models of height at least* λ *are in* τ^1*.*

As we will see, the following lemma will enable us to ease our path through the proof of Claim 4.7, in Section 4, in a significant way.

Lemma 2.14. *Suppose for each* α*,* Φ_α *codes* $\langle \Phi_\beta \mid \beta < \alpha \rangle$ *in a uniform way. For all sets* τ^0 *and* τ^1 *of* $\vec{\Phi}$*-edges, every set* x*, and every* $\tau^0 \oplus \tau^1$*-thread* $\langle \vec{\mathcal{E}}, x \rangle$ *there is a* $\tau^0 \cup \tau^1$*-thread* $\langle \vec{\mathcal{E}}_*, x \rangle$ *such that*

$$\Psi_{\vec{\mathcal{E}}}(x) = \Psi_{\vec{\mathcal{E}}_*}(x)$$

Furthermore, if $x = (y, \alpha) \in H(\kappa^+) \times \kappa^+$ *and* $\langle \vec{\mathcal{E}}, x \rangle$ *is connected, then* $\vec{\mathcal{E}}_*$ *may be chosen to be connected as well.*

Proof. Let $\vec{\mathcal{E}} = (e_i \mid i \leq n)$, where $e_i = \langle (N_0^i, \gamma_0^i), (N_1^i, \gamma_1^i) \rangle$ for each i. We aim to prove that there is a $\tau^0 \oplus \tau^1$-thread $\langle \vec{\mathcal{E}}_*, x \rangle$ with the following properties.

(1) $\Psi_{\vec{\mathcal{E}}}(x) = \Psi_{\vec{\mathcal{E}}_*}(x)$
(2) If every e_i has (τ^0, τ^1)-rank 0, then $\vec{\mathcal{E}}_* = \vec{\mathcal{E}}$.
(3) If some e_i has positive (τ^0, τ^1)-rank, then the maximum (τ^0, τ^1)-rank of the members of $\vec{\mathcal{E}}_*$ is strictly less than the maximum (τ^0, τ^1)-rank of the members of $\vec{\mathcal{E}}$.
(4) The following holds, where $\vec{\mathcal{E}}_* = (e^*_i \mid i \leq n^*)$.
 (a) $e^*_0 = e_0$ if e_0 comes from $\tau^0 \cup \tau^1$.
 (b) If there are generalized $\vec{\Phi}$-edges $e = \langle (N_0, \gamma_0), (N_1, \gamma_1)\rangle$ and e' coming from $\tau^0 \oplus \tau^1$, both of rank less than the rank of e_0, such that $e' \in N_0$ and $e_0 = \Psi_e(e')$, then

$$e^*_0 = \langle (N_1, \gamma_1), (N_0, \gamma_0)\rangle,$$

 where $\langle (N_0, \gamma_0), (N_1, \gamma_1)\rangle$ is some generalized $\vec{\Phi}$-edge as above.
 (c) $e^*_{n^*} = e_n$ if e_n comes from $\tau^0 \cup \tau^1$.
 (d) If there are generalized $\vec{\Phi}$-edges $e = \langle (N_0, \gamma_0), (N_1, \gamma_1)\rangle$ and e' coming from $\tau^0 \oplus \tau^1$, both of rank less than the rank of e_n, such that $e' \in N_0$ and $e_n = \Psi_e(e')$, then

$$e^*_{n^*} = \langle (N_0, \gamma_0), (N_1, \gamma_1)\rangle,$$

 where $\langle (N_0, \gamma_0), (N_1, \gamma_1)\rangle$ is some generalized $\vec{\Phi}$-edge as above.
(5) If $x = (y, \alpha) \in H(\kappa^+) \times \kappa^+$ and $\langle \vec{\mathcal{E}}, x\rangle$ is connected, then $\vec{\mathcal{E}}_*$ is connected.

The proof of (1)–(5) will be by induction on n. We may obviously assume that there is some $i < n$ such that e_i does not come from $\tau^0 \cup \tau^1$. Then there are generalized $\vec{\Phi}$-edges $e = \langle (N_0, \gamma_0), (N_1, \gamma_1)\rangle$ and $e' \in N_0$ coming from $\tau^0 \oplus \tau^1$, both of rank less than e_i, and such that $e_i = \Psi_e(e')$.

By induction hypothesis there is a $\tau^0 \oplus \tau^1$-thread $\langle \vec{\mathcal{E}}_0, x\rangle$, together with a $\tau^0 \oplus \tau^1$-thread of the form $\langle \vec{\mathcal{E}}_2, \Psi_{\vec{\mathcal{E}}\restriction i+1}(x)\rangle$, such that

$$\Psi_{\vec{\mathcal{E}}_0}(x) = \Psi_{\vec{\mathcal{E}}\restriction i}(x)$$

and

$$\Psi_{\vec{\mathcal{E}}_2}(\Psi_{\vec{\mathcal{E}}\restriction i+1}(x)) = \Psi_{\vec{\mathcal{E}}\restriction[i+1,n)}(\Psi_{\vec{\mathcal{E}}\restriction i+1}(x)) = \Psi_{\vec{\mathcal{E}}}(x),$$

and such that the relevant instances of (1)–(5) hold for $\langle \vec{\mathcal{E}}_0, x\rangle$ and $\langle \vec{\mathcal{E}}_2, \Psi_{\vec{\mathcal{E}}\restriction i+1}(x)\rangle$. Also, by the choice of e_i, the thread $\langle \vec{\mathcal{E}}_1, \Psi_{\vec{\mathcal{E}}\restriction i}(x)\rangle$ satisfies the instances of (1) and (3)–(5) corresponding to $\langle (e_i), \Psi_{\vec{\mathcal{E}}\restriction i}(x)\rangle$, where (e_i) is the sequence whose only member is e_i, and where $\vec{\mathcal{E}}_1 =$

(e^{-1}, e', e). But now we may take $\vec{\mathcal{E}}_*$ to be the concatenation of $\vec{\mathcal{E}}_0$, $\vec{\mathcal{E}}_1$, and $\vec{\mathcal{E}}_2$.

Finally, it follows from clause (3) that after iterating the above construction some finite number of times we obtain a $\tau^0 \cup \tau^1$-thread $\langle \vec{\mathcal{E}}_*, x\rangle$ as desired. □

3. Definition of the forcing and its basic properties

We shall now define our sequence $\langle \mathbb{Q}_\beta \mid \beta \leq \kappa^+\rangle$ of forcing notions and our sequence $\langle \Phi_\beta \mid 0 < \beta < \kappa^+\rangle$ of predicates.[13] We recall that Φ_0 has already been defined.

For each $\alpha \in \mathcal{X}$, assuming $\mathbb{Q}_\alpha$ has been defined and there is a $\mathbb{Q}_\alpha$-name $\underset{\sim}{T} \in H(\kappa^+)$ for a κ-Arosnzajn tree, we let $\underset{\sim}{T}_\alpha$ be such a $\mathbb{Q}_\alpha$-name. Further, if $\Phi(\alpha)$ is a $\mathbb{Q}_\alpha$-name for a κ-Aronszajn tree, then we let $\underset{\sim}{T}_\alpha = \Phi(\alpha)$. For simplicity of exposition we will assume that the universe of $\underset{\sim}{T}_\alpha$ is forced to be $\kappa \times \omega_1$ and that for each $\rho < \kappa$, its ρ-th level is $\{\rho\} \times \omega_1$. We will often refer to members of $\kappa \times \omega_1$ as *nodes*.

As we will see, each forcing notion $\mathbb{Q}_\beta$ in our construction will consist of ordered pairs of the form $q = (f_q, \tau_q)$, where f_q is a function and τ_q is a set of edges below β. Given a nonzero ordinal $\alpha < \kappa^+$ and an ordinal $\delta < \kappa$, we will write $\mathbb{Q}^\delta_{\alpha+1}$ to denote the suborder of $\mathbb{Q}_{\alpha+1}$ consisting of those conditions q such that $\delta_{N_0} < \delta$ for every edge $\langle (N_0, \gamma_0), (N_1, \gamma_1)\rangle$ coming from τ_q such that at least one of γ_0, γ_1 is $\alpha + 1$.[14]

Now suppose that $\beta \leq \kappa^+$ and that $\mathbb{Q}_\alpha$ and Φ_α have been defined for all $\alpha < \beta$. Given an ordered pair $q = (f_q, \tau_q)$, where f_q is a function and τ_q is a set of edges, and given an ordinal α, we denote by $q \restriction \alpha$ the ordered pair $(f_q \restriction \alpha, \tau_q \restriction \alpha)$.

We are now ready to define $\mathbb{Q}_\beta$ and Φ_β.

We start with the definition of $\mathbb{Q}_\beta$. A condition in $\mathbb{Q}_\beta$ is an ordered pair of the form $q = (f_q, \tau_q)$ with the following properties.

(1) f_q is a countable function such that

$$\mathrm{dom}(f_q) \subseteq \beta$$

and such that the following holds for every $\alpha \in \mathrm{dom}(f_q)$.

(a) If $\alpha = 0$, then $f_q(\alpha)$ is a condition in $\mathrm{Col}(\omega_1, {<}\kappa)$, the Lévy collapse turning κ into $\aleph_2$, i.e., $f_q(0)$ is a countable function

[13] The reader should keep in mind the overview of the construction we gave in the introduction.

[14] We note that there is no requirement on the heights of the nodes occurring in $f_q(\alpha)$. Also, despite possible first impressions, there is no circularity in the definition of $\mathbb{Q}^\delta_{\alpha+1}$ (s. Remark 3.3).

with domain included in $\kappa \times \omega_1$ such that $(f_q(0))(\rho, \xi) < \rho$ for all $(\rho, \xi) \in \mathrm{dom}(f_q(0))$.

(b) If $\alpha > 0$, then $f_q(\alpha) \in [\kappa \times \omega_1]^{\leq \aleph_0}$.

(2) τ_q is a countable set of $(\vec{\Phi} \restriction \beta)$-edges below β.

(3) $q \restriction \alpha \in \mathbb{Q}_\alpha$ for all $\alpha < \beta$.

(4) For every nonzero $\alpha < \beta$ such that $\underset{\sim}{T}_{\mu(\alpha)}$ is defined,[15] if $x_0 \neq x_1$ are nodes in $f_q(\alpha)$, then $q \restriction \mu(\alpha)$ forces x_0 and x_1 to be incomparable in $\underset{\sim}{T}_{\mu(\alpha)}$.

(5) τ_q is closed under copying.

(6) For every edge $\langle (N_0, \gamma_0), (N_1, \gamma_1) \rangle \in \tau_q$ and every $\alpha \in N_0 \cap \gamma_0$ such that $\bar{\alpha} := \Psi_{N_0,N_1}(\alpha) < \gamma_1$, if $\alpha \neq 0$, $\alpha \in \mathrm{dom}(f_q)$, and $x \in f_q(\alpha) \cap N_0$, then

(a) $\bar{\alpha} \in \mathrm{dom}(f_q)$, and

(b) $x \in f_q(\bar{\alpha})$.[16]

(7) Suppose $\alpha < \beta$, $e = \langle (N_0, \alpha + 1), (N_1, \gamma_1) \rangle$ is a generalized $(\vec{\Phi} \restriction \alpha + 1)$-edge coming from $\tau_q \restriction \alpha + 1$, $r \in \mathbb{Q}_{\alpha+1}^{\delta_{N_0}}$ is such that $e \restriction \alpha$ comes from τ_r, $\vec{\mathcal{E}} = (\langle (N_0^i, \gamma_0^i), (N_1^i, \gamma_1^i) \rangle \mid i < n)$ is a sequence of generalized $(\vec{\Phi} \restriction \alpha + 1)$-edges coming from $\tau_r \cup \{e\}$ such that $\langle (N_0^0, \gamma_0^0), (N_1^0, \gamma_1^0) \rangle = e$ and $\langle \vec{\mathcal{E}}, (\emptyset, \alpha) \rangle$ is a correct thread. Let $\delta = \min\{\delta_{N_0^i} \mid i < n\}$ and $\bar{\alpha} = \Psi_{\vec{\mathcal{E}}}(\alpha)$. Suppose $r \restriction \mu(\alpha)$ forces every two distinct nodes in $f_r(\bar{\alpha}) \cap (\delta \times \omega_1)$ to be incomparable in $\underset{\sim}{T}_{\mu(\alpha)}$. Then there is an extension $r^* \in \mathbb{Q}_{\alpha+1}^{\delta_{N_0}}$ of r such that

(a) $f_r(\alpha) \cap (\delta \times \omega_1) \subseteq f_{r^*}(\bar{\alpha}) \cap (\delta \times \omega_1)$, and

(b) $r^* \restriction \mu(\alpha)$ forces every two distinct nodes in

$$(f_r(\bar{\alpha}) \cap (\delta \times \omega_1)) \cup f_r(\alpha)$$

to be incomparable in $\underset{\sim}{T}_{\mu(\alpha)}$.

The extension relation on $\mathbb{Q}_\beta$ is defined in the following way:

Given $q_1, q_0 \in \mathbb{Q}_\beta$, $q_1 \leq_{\mathbb{Q}_\beta} q_0$ (q_1 is an extension of q_0) if and only if the following holds.

(1) $\mathrm{dom}(f_{q_0}) \subseteq \mathrm{dom}(f_{q_1})$

(2) For every $\alpha \in \mathrm{dom}(f_{q_0})$, $f_{q_0}(\alpha) \subseteq f_{q_1}(\alpha)$.

(3) For every $\langle (N_0, \gamma_0), (N_1, \gamma_1) \rangle \in \tau_{q_0}$ there are $\gamma_0' \geq \gamma_0$ and $\gamma_1' \geq \gamma_1$ such that $\langle (N_0, \gamma_0'), (N_1, \gamma_1') \rangle \in \tau_{q_1}$.

Finally, if $\beta > 0$, then Φ_β is a subset of $H(\kappa^+)$ canonically coding $\langle \Phi_\alpha \mid \alpha < \beta \rangle$, $\langle \mathbb{Q}_\alpha \mid \alpha \leq \beta \rangle$ and $\langle \Vdash_{\mathbb{Q}_\alpha}^{H(\kappa^+)} \mid \alpha \leq \beta \rangle$, where for each

[15] We will see that in fact each $\underset{\sim}{T}_{\mu(\alpha)}$ is defined.

[16] Note that $\Psi_{N_0,N_1}(x) = x$.

$\alpha \leq \beta$, $\Vdash^{H(\kappa^+)}_{\mathbb{Q}_\alpha}$ denotes the forcing relation restricted to formulas with $\mathbb{Q}_\alpha$-names in $H(\kappa^+)$ as parameters.

We may, and will, assume that the definition of $\langle \Phi_\beta \mid 0 < \beta < \kappa^+ \rangle$ is uniform in β.

Remark 3.1. Having fixed the sequence $\vec{\Phi} = \langle \Phi_\beta \mid 0 < \beta < \kappa^+ \rangle$ of predicates as above, by an edge we always mean a $\vec{\Phi}$-edge, and similarly for other concepts.

Remark 3.2. Given any $\alpha < \kappa^+$, there is a natural map

$$\pi_\alpha : \mathbb{Q}_\alpha \to \mathbb{P}^{\mathsf{e}}_{\vec{\Phi} \restriction \alpha, \alpha},$$

defined by $\pi_\alpha(q) = \tau_q$. However, π_α is not necessarily a projection of forcing notions, as given a condition $q \in \mathbb{Q}_\alpha$ there might exist $\tau_q \subseteq \tau \in \mathbb{P}^{\mathsf{e}}_{\vec{\Phi} \restriction \alpha, \alpha}$ such that $\tau_{q'} \not\leq_{\mathbb{P}^{\mathsf{e}}_{\vec{\Phi} \restriction \alpha, \alpha}} \tau$ for all $q' \leq q$.

Remark 3.3. Despite possible first impressions due to the presence of clause (7), our definition of $\mathbb{Q}_{\alpha+1}$-condition, for a given $\alpha < \kappa^+$, is not circular. Rather, the definition of 'q is a $\mathbb{Q}_{\alpha+1}$-condition' is to be seen, because of that clause, as being by recursion on the supremum of the set of heights of models N_0 occurring in edges $\langle (N_0, \gamma_0), (N_1, \gamma_1) \rangle$ in τ_q. Indeed, given any q satisfying clauses (1)–(6), in order to verify whether or not q satisfies also (7) we check whether for each generalized $(\vec{\Phi} \restriction \alpha + 1)$-edge $e = \langle (N_0, \alpha + 1), (N_1, \gamma_1) \rangle$ coming from $\tau_q \restriction \alpha + 1$ it is the case that some condition holds depending only on $\langle \mathbb{Q}_\beta \mid \beta \leq \alpha \rangle$, e, and $\mathbb{Q}^{\delta_{N_0}}_{\alpha+1}$, which consists of conditions q' with $\delta_{N'_0} < \delta_{N_0}$ for every $\langle (N'_0, \gamma'_0), (N'_1, \gamma'_1) \rangle \in \tau_{q'}$.

Before moving on to the next subsection, we will briefly address the need for, and nature of, clause (7) in our definition of condition. As already mentioned in the introduction, the proof that our forcing satisfies the κ-chain condition is an adaptation, in our present context, of the Laver-Shelah proof that their forcing in [10] has the κ-chain condition. The only potential obstacles to making such an adaptation work may come from our present requirements that a condition q be closed under copying of all the relevant information, as dictated by the presence of edges $\langle (N_0, \gamma_0), (N_1, \gamma_1) \rangle$ in its side condition τ_q, and where this includes the information coming from the working part f_q.

In the proof of the κ-c.c. of $\mathbb{Q}_\beta$, given $A \subseteq \mathbb{Q}_\beta$ such that $|A| = \kappa$, we need to find two distinct conditions in A which are compatible. As we said, we would like to do that following the ideas in the κ-c.c. proof from [10] as closely as possible. Now, due to technical reasons coming from the present copying requirements, in order to do this we seem to need to work under the assumption that all conditions in A have an additional property, namely that they are what we call *adequate conditions* (s. Definition 3.16). One of the requirements for q to be an adequate condition is that it be suitably closed under copying not only via edges from τ_q, but also via the corresponding anti-edges. In particular, if $\langle (N_0, \gamma_0), (N_1, \gamma_1)\rangle$ is an edge in τ_q, then not only are we to copy the information from the working part sitting in N_0 into N_1 (into the past) but also to copy the information sitting in N_1 into N_0 (into the future); and similarly for the edges in N_1 with markers at most α for $\alpha \in \mathrm{dom}(f_q) \cap N_1 \cap \gamma_1$ such that $f_q(\alpha) \cap N_1 \neq \emptyset$ and $\Psi_{N_1,N_0}(\alpha) < \gamma_0$.

Now, the main obstacle for proving that the set of adequate conditions is dense—and this is the motivation behind clause (7)—is the following: take the situation in which there is an edge $\langle (N_0, \gamma_0), (N_1, \gamma_1)\rangle \in \tau_q$ with $\alpha \in N_0 \cap \gamma_0$, $\bar{\alpha} := \Psi_{N_0,N_1}(\alpha) < \gamma_1$, some $x \in f_q(\bar{\alpha})$ of height less than δ_{N_1} $(= \delta_{N_0})$, and some $y \in f_q(\alpha)$ of height at least δ_{N_0}. If q' were to be any adequate condition extending q, it would have to be the case that $x \in f_{q'}(\alpha)$. However, unless we have an extra clause preventing it, it could for example be that y is forced to be above x in $\underset{\sim}{T}_{\mu(\alpha)}$, which would make it impossible for such a q' to exist.[17]

Our way around this difficulty is to incorporate, in our definition, a clause which stipulates that the above operation can be carried out. This is in essence what clause (7) says.[18] Fortunately, the intended content can indeed be expressed (cf. the previous footnote); our device for doing so is to phrase this content by reference to a well-defined suborder $\mathbb{Q}^{\delta}_{\alpha+1}$ of $\mathbb{Q}_{\alpha+1}$—namely the set of conditions $q \in \mathbb{Q}_{\alpha+1}$ all of whose edges of form $\langle (N_0, \alpha + 1), (N_1, \gamma_1)\rangle$ are such that $\delta_{N_0} < \delta$ (but allowing all nodes in $f_q(\alpha)$ to be of any height below κ). Hence, due to the presence of this clause (7), the definition of q being a $\mathbb{Q}_{\alpha+1}$-condition is ultimately to be seen as being given by recursion on the supremum of the collection of heights of models occurring in edges of the form $\langle (N_0, \alpha + 1), (N_1, \gamma_1)\rangle$ (i.e., those edges not coming from the restriction of q to α).[19]

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[17]The problematic configuration can of course be described in slightly more general terms.

[18]A more naive (and simpler-looking) approach would be to require that if $\langle (N_0, \gamma_0), (N_1, \gamma_1)\rangle$ is an edge from τ_q, then $\mathbb{Q}_{\alpha+1} \cap N_0$ is a complete suborder of $\mathbb{Q}_{\alpha+1}$. This would have the intended effect. However, such a condition cannot be expressed without circularity.

[19]Let us reconsider for a second the situation described a few lines earlier. Suppose $q \in \mathbb{Q}_{\alpha+1}$, $\langle (N_0, \gamma_0), (N_1, \gamma_1)\rangle \in \tau_q$, and α and $\bar{\alpha}$ are as in that description. Suppose $x \in f_q(\bar{\alpha})$ is of height less than δ_{N_1} and y is a node of height at least δ_{N_0} such that, say, $q \restriction \mu(\alpha)$ happens to force y to be above x in $\underset{\sim}{T}_{\mu(\alpha)}$. It is then of course

3.1. **Basic properties of $\langle \mathbb{Q}_\beta \mid \beta \leq \kappa^+\rangle$.** Our first lemma follows immediately from the choice of the predicates Φ_α.

Lemma 3.4. *For every nonzero $\alpha < \kappa^+$, $\mathbb{Q}_\alpha$ and $\Vdash^{H(\kappa^+)}_{\mathbb{Q}_\alpha}$ are definable over the structure*

$$(H(\kappa^+), \in, \Phi_\alpha)$$

without parameters. Moreover, this definition is uniform in α.

Our next lemma follows from the fact that $\mathbb{Q}_1$ is essentially the Lévy collapse turning κ into ω_2.

Lemma 3.5. *$\mathbb{Q}_1$ forces $\kappa = \aleph_2$.*

The following lemma is also an easy consequence of the definition of condition.

Lemma 3.6. *For every $\beta \leq \kappa^+$, $\mathbb{Q}_\alpha \subseteq \mathbb{Q}_\beta$ for all $\alpha < \beta$.*

Lemma 3.7 follows easily from the definition of $\langle \mathbb{Q}_\alpha \mid \alpha \leq \kappa^+\rangle$.

Lemma 3.7. *For all $\alpha < \beta \leq \kappa^+$, $q \in \mathbb{Q}_\beta$, and $r \in \mathbb{Q}_\alpha$, if $r \leq_{\mathbb{Q}_\alpha} q \restriction \alpha$, then*

$$(f_r \cup f_q \restriction [\alpha,\, \beta), \tau_q \cup \tau_r)$$

is a common extension of q and r in $\mathbb{Q}_\beta$.

Proof. Let $p = (f_r \cup f_q \restriction [\alpha,\, \beta), \tau_q \cup \tau_r)$. We show that p satisfies items (1)–(7) of the definition of a $\mathbb{Q}_\beta$-condition. It suffices to consider (5) and (6), as all other clauses can be proved easily.

We first show that p satisfies clause (5). Thus let

$$e = \langle (N_0, \gamma_0), (N_1, \gamma_1)\rangle \in \tau_q \cup \tau_r$$

and

$$e' = \langle (N_0', \gamma_0'), (N_1', \gamma_1')\rangle \in (\tau_q \cup \tau_r) \cap N_0.$$

We have to show that there are ordinals $\gamma_0^* \geq \pi^{\gamma_0', N_0'}_{N_0,\gamma_0,N_1,\gamma_1}$ and $\gamma_1^* \geq \pi^{\gamma_1', N_1'}_{N_0,\gamma_0,N_1,\gamma_1}$ such that

$$\langle (\Psi_{N_0,N_1}(N_0'), \gamma_0^*), (\Psi_{N_0,N_1}(N_1'), \gamma_1^*)\rangle \in \tau_q \cup \tau_r.$$

We divide the proof into three cases:

(1) Both e and e' belong to τ_q (resp. τ_r). Then the conclusion is immediate as q (resp. r) is a condition in $\mathbb{Q}_\beta$.

impossible to extend q to a condition q' such that $x \in f_{q'}(\alpha)$. However, we can certainly pick α' such that $\mu(\alpha') = \mu(\alpha)$ and such that q can be extended (trivially) by making $f_{q'}(\alpha') = \{x\}$. This will ensure that the generic specializing function for $\underset{\sim}{T}_{\mu(\alpha)}$ will be defined everywhere (cf. the proof of Lemma 5.3).

(2) $e \in \tau_q$ and $e' \in \tau_r$. Then $e \restriction \alpha \in \tau_{q\restriction\alpha}$, so as $r \leq_{\mathbb{Q}_\alpha} q \restriction \alpha$, we can find $\gamma_0^{''}, \gamma_1^{''} \leq \alpha$ such that $\gamma_0^{''} \geq \min\{\gamma_0, \alpha\}_{N_0}$, $\gamma_1^{''} \geq \min\{\gamma_1, \alpha\}_{N_1}$ and $\langle (N_0, \gamma_0^{''}), (N_1, \gamma_1^{''}) \rangle \in \tau_r$. Hence, as r is a condition, for some

$$\gamma_0^* \geq \pi^{\gamma_0', N_0'}_{N_0, \gamma_0^{''}, N_1, \gamma_1^{''}}$$

and

$$\gamma_1^* \geq \pi^{\gamma_1', N_1'}_{N_0, \gamma_0^{''}, N_1, \gamma_1^{''}}$$

we have

$$\langle (\Psi_{N_0,N_1}(N_0'), \gamma_0^*), (\Psi_{N_0,N_1}(N_1'), \gamma_1^*) \rangle \in \tau_r.$$

As $r \in \mathbb{Q}_\alpha$, $e' \in \tau_r$, and $\gamma_0', \gamma_1' \leq \alpha$, and by the choice of γ_0'', γ_1'', one can easily show that

$$\pi^{\gamma_0', N_0'}_{N_0, \gamma_0^{''}, N_1, \gamma_1^{''}} \geq \pi^{\gamma_0', N_0'}_{N_0, \gamma_0, N_1, \gamma_1},$$

and

$$\pi^{\gamma_1', N_1'}_{N_0, \gamma_0^{''}, N_1, \gamma_1^{''}} \geq \pi^{\gamma_1', N_1'}_{N_0, \gamma_0, N_1, \gamma_1}.$$

Thus $\gamma_0^* \geq \pi^{\gamma_0', N_0'}_{N_0, \gamma_0, N_1, \gamma_1}$ and $\gamma_1^* \geq \pi^{\gamma_1', N_1'}_{N_0, \gamma_0, N_1, \gamma_1}$, from which the result follows.

(3) $e \in \tau_r$ and $e' \in \tau_q$. Then $e' \restriction \alpha \in \tau_{q\restriction\alpha}$ and so we can find $\gamma_0^{''}, \gamma_1^{''} \leq \alpha$ such that $\gamma_0^{''} \geq \min\{\gamma_0', \alpha\}_{N_0'}$, $\gamma_1^{''} \geq \min\{\gamma_1', \alpha\}_{N_1'}$ and $\langle (N_0', \gamma_0^{''}), (N_1', \gamma_1^{''}) \rangle \in \tau_r$. By the discussion after Definition 2.1, $\langle (N_0', \gamma_0^{''}), (N_1', \gamma_1^{''}) \rangle \in N_0$. Hence, as r is a condition, we can find

$$\gamma_0^* \geq \pi^{\gamma_0^{''}, N_0'}_{N_0, \gamma_0, N_1, \gamma_1}$$

and

$$\gamma_1^* \geq \pi^{\gamma_1^{''}, N_1'}_{N_0, \gamma_0, N_1, \gamma_1}$$

such that

$$\langle (\Psi_{N_0,N_1}(N_0'), \gamma_0^*), (\Psi_{N_0,N_1}(N_1'), \gamma_1^*) \rangle \in \tau_r.$$

As $r \in \mathbb{Q}_\alpha$, $e \in \tau_r$, and $\gamma_0, \gamma_1 \leq \alpha$, and by the choice of $\gamma_0^{''}, \gamma_0^{''}$, one can again easily show that

$$\pi^{\gamma_0^{''}, N_0'}_{N_0, \gamma_0, N_1, \gamma_1} \geq \pi^{\gamma_0', N_0'}_{N_0, \gamma_0, N_1, \gamma_1}$$

and

$$\pi^{\gamma_1^{''}, N_1'}_{N_0, \gamma_0, N_1, \gamma_1} \geq \pi^{\gamma_1', N_1'}_{N_0, \gamma_0, N_1, \gamma_1},$$

from which we get that $\gamma_0^* \geq \pi^{\gamma_0', N_0'}_{N_0, \gamma_0, N_1, \gamma_1}$ and $\gamma_1^* \geq \pi^{\gamma_1', N_1'}_{N_0, \gamma_0, N_1, \gamma_1}$, and the result follows.

To show that p satisfies clause (6), let $e = \langle (N_0, \gamma_0), (N_1, \gamma_1)\rangle \in \tau_q \cup \tau_r$, $\eta \in (\mathrm{dom}(f_r) \cup \mathrm{dom}(f_q)) \cap N_0 \cap \gamma_0$, $\eta \neq 0$, and $x \in (f_r(\eta) \cup f_q(\eta)) \cap N_0$. We have to show that $\bar{\eta} \in \mathrm{dom}(f_r) \cup \mathrm{dom}(f_q)$ and $x \in f_r(\bar{\eta}) \cup f_q(\bar{\eta})$, where $\bar{\eta} = \Psi_{N_0,N_1}(\eta)$.

If $\eta < \alpha$, then $\eta \in \mathrm{dom}(f_r)$ and $x \in f_r(\eta)$. As $r \leq_\alpha q \restriction \alpha$, for some γ_0', γ_1' we have $\langle (N_0, \gamma_0'), (N_1, \gamma_1')\rangle \in \tau_r$ (if $e \in \tau_r$, we can take $\gamma_0' = \gamma_0$ and $\gamma_1' = \gamma_1$, otherwise, we can take $\gamma_0' \geq \min\{\gamma_0, \alpha\}_{N_0}$ and $\gamma_1' \geq \min\{\gamma_1, \alpha\}_{N_1}$). But then $\bar{\eta} \in \mathrm{dom}(f_r)$ and $x \in f_r(\bar{\eta})$.

Next suppose that $\eta \geq \alpha$. In this case we must have $e \in \tau_q$ and $\eta \in \mathrm{dom}(f_q) \setminus \mathrm{dom}(f_r)$. But then $\bar{\eta} \in \mathrm{dom}(f_q)$ and $x \in f_q(\bar{\eta})$. □

Throughout the paper, we write $\mathbb{P} \lessdot \mathbb{Q}$ to denote that $\mathbb{P}$ is a complete suborder of $\mathbb{Q}$ (i.e., $\mathbb{P}$ is a suborder of $\mathbb{Q}$ and maximal antichains in $\mathbb{P}$ are also maximal antichains in $\mathbb{Q}$).

The following corollary is a trivial consequence of Lemma 3.7.

Corollary 3.8. *$\langle \mathbb{Q}_\alpha \mid \alpha \leq \kappa^+\rangle$ is a forcing iteration, in the sense that $\mathbb{Q}_\alpha \lessdot \mathbb{Q}_\beta$ for all $\alpha < \beta \leq \kappa^+$.*

We say that a partial order $\mathcal{P}$ is σ-closed if every descending sequence $(p_n)_{n<\omega}$ of $\mathcal{P}$-conditions has a lower bound in $\mathcal{P}$.

Remark 3.9. Suppose $\beta \leq \kappa^+$ and $\langle \tau_n : n < \omega\rangle$ is a $\leq_{\mathbb{P}^{\mathsf{e}}_{\vec{\Phi},\beta}}$-decreasing sequence of conditions in $\mathbb{P}^{\mathsf{e}}_{\vec{\Phi},\beta}$ which are closed under copying. Then $\bigcup_{n<\omega} \tau_n$ is also a condition in $\mathbb{P}^{\mathsf{e}}_{\vec{\Phi},\beta}$ and is closed under copying. The reason that $\bigcup_{n<\omega} \tau_n$ is closed under copying is that if $n < m$ and we have

$$e = \langle (N_0, \gamma_0), (N_1, \gamma_1)\rangle \in \tau_m$$

and

$$e' = \langle (N_0', \gamma_0'), (N_1', \gamma_1')\rangle \in \tau_n \cap N_0,$$

then for some γ_0'', γ_1'' we have $e'' = \langle (N_0', \gamma_0''), (N_1', \gamma_1'')\rangle \in \tau_m$, and by the discussion after Definition 2.1, $e'' \in N_0$, so as in the proof of Lemma 3.7, for some $\gamma_0^* \geq \pi^{\gamma_0', N_0'}_{N_0,\gamma_0,N_1,\gamma_1}$ and $\gamma_1^* \geq \pi^{\gamma_1', N_1'}_{N_0,\gamma_0,N_1,\gamma_1}$ we have

$$\langle (\Psi_{N_0,N_1}(N_0'), \gamma_0^*), (\Psi_{N_0,N_1}(N_1'), \gamma_1^*)\rangle \in \tau_m.$$

Lemma 3.10. *$\mathbb{Q}_\beta$ is σ-closed for every $\beta \leq \kappa^+$. In fact, every decreasing ω-sequence of $\mathbb{Q}_\beta$-conditions has a greatest lower bound in $\mathbb{Q}_\beta$. In particular, forcing with $\mathbb{Q}_\beta$ does not add new ω-sequences of ordinals, and therefore this forcing preserves both ω_1 and CH.*

Proof. Given a decreasing sequence $(q_n)_{n<\omega}$ of $\mathbb{Q}_\beta$-conditions, it is immediate to check that $q = (f, \bigcup_{n<\omega} \tau_{q_n})$ is the greatest lower bound of the set $\{q_n \mid n < \omega\}$, where $\mathrm{dom}(f) = \bigcup_{n<\omega} \mathrm{dom}(f_{q_n})$ and, for each

$n < \omega$ and $\alpha \in \mathrm{dom}(f_{q_n})$, $f(\alpha) = \bigcup\{f_{q_m}(\alpha) \mid m \geq n\}$. For this one proves, by induction on α, that $q \restriction \alpha \in \mathbb{Q}_\alpha$ for every $\alpha \leq \beta$. $\square$

Remark 3.11. Lemma 3.10, or rather its proof, will be used, often without mention, in several places in which we run some construction, in ω steps, along which we build some decreasing sequence $(q_n)_{n<\omega}$ of conditions. At the end of such a construction we will have that the ordered pair $q = (f, \bigcup_{n<\omega} \tau_{q_n})$, where f is given as in the above proof, is the greatest lower bound of $(q_n)_{n<\omega}$.

Given $\alpha \in \mathcal{X}$, a node $x = (\rho, \zeta)$ in $\kappa \times \omega_1$, and an ordinal $\bar{\rho} \leq \rho$, if $\mathbb{Q}_\alpha$ has the κ-c.c., we denote by $A^\alpha_{x,\bar{\rho}}$ the F-first maximal antichain of $\mathbb{Q}_\alpha$ in $H(\kappa^+)$ consisting of conditions deciding, for some ordinal $\bar{\zeta} < \omega_1$, that $(\bar{\rho}, \bar{\zeta})$ is $\underset{\sim}{T}_\alpha$-below x.[20] If $x_0 = (\rho_0, \zeta_0)$ and $x_1 = (\rho_1, \zeta_1)$ are nodes, $\bar{\rho} \leq \rho_0$, ρ_1, $r_0 \in A^\alpha_{x_0,\bar{\rho}}$, $r_1 \in A^\alpha_{x_1,\bar{\rho}}$, and there are ordinals $\bar{\zeta}_0 \neq \bar{\zeta}_1$ in ω_1 such that r_0 forces that $(\bar{\rho}, \bar{\zeta}_0)$ is $\underset{\sim}{T}_\alpha$-below x_0 and r_1 forces that $(\bar{\rho}, \bar{\zeta}_1)$ is $\underset{\sim}{T}_\alpha$-below x_1, then we say that *r_0 and r_1 force x_0 and x_1 to be incomparable in $\underset{\sim}{T}_\alpha$*.[21]

The following lemma will be often used.

Lemma 3.12. *Suppose q is a $\mathbb{Q}_{\kappa^+}$-condition, $\alpha \in \mathrm{dom}(f_q)$, $\alpha \neq 0$, and $\underset{\sim}{T}_{\mu(\alpha)}$ is defined. Suppose $\mathbb{Q}_{\mu(\alpha)}$ has the κ-c.c. Suppose $\langle \vec{\mathcal{E}}, (\rho, \alpha)\rangle$ is a correct τ_q-thread, where $\rho < \kappa$, and $x_0 = (\rho_0, \zeta_0)$ and $x_1 = (\rho_1, \zeta_1)$ are two nodes such that*

- *ρ_0, $\rho_1 \leq \rho$, and*
- *there is some $\bar{\rho} \leq \rho_0$, ρ_1 such that $q \restriction \mu(\alpha)$ extends conditions $r_0 \in A^{\mu(\alpha)}_{x_0,\bar{\rho}}$ and $r_1 \in A^{\mu(\alpha)}_{x_1,\bar{\rho}}$ forcing x_0 and x_1 to be incomparable in $\underset{\sim}{T}_{\mu(\alpha)}$.*

Let $\bar{\alpha} = \Psi_{\vec{\mathcal{E}}}(\alpha)$. Then

1. *r_0 and r_1 are in $\mathrm{dom}(\Psi_{\vec{\mathcal{E}}})$, and*
2. *$\Psi_{\vec{\mathcal{E}}}(r_0)$ and $\Psi_{\vec{\mathcal{E}}}(r_1)$ force x_0 and x_1 to be incomparable in $\underset{\sim}{T}_{\mu(\bar{\alpha})}$.*

Proof. Let

$$\vec{\mathcal{E}} = (\langle (N^i_0, \gamma^i_0), (N^i_1, \gamma^i_1)\rangle \mid i \leq n)$$

Since $\omega_1 \cup \rho + 1 \subseteq \mathrm{dom}(\Psi_{\vec{\mathcal{E}}})$ and $\Psi_{\vec{\mathcal{E}}}$ is a partially defined elementary embedding from $(N^0_0, \in, \Phi_\alpha)$ into $(N^n_1, \in, \Phi_{\bar{\alpha}})$ we have, by definability of $A^{\mu(\alpha)}_{x_0,\bar{\rho}}$ and $A^{\mu(\alpha)}_{x_1,\bar{\rho}}$ over the structure $(H(\kappa^+), \in, \Phi_\alpha)$ by formulas φ_0

[20]In Lemma 4.3 we will prove that each $\mathbb{Q}_\alpha$ has the κ-c.c. Hence, $A^\alpha_{x,\bar{\rho}}$ will be defined for all x and $\bar{\rho}$.

[21]This terminology is apt: since for each $\rho < \kappa$, $\{\rho\} \times \omega_1$ is forced to be the ρ-th level of $\underset{\sim}{T}_\alpha$, we have that every condition in $\mathbb{Q}_\alpha$ extending both of r_0 and r_1 must force that x_0 and x_1 are incomparable nodes in $\underset{\sim}{T}_\alpha$.

and φ_1, respectively, with x_0, x_1 and $\bar{\rho}$ as parameters, and definability of $A^{\mu(\bar{\alpha})}_{x_0,\bar{\rho}}$ and $A^{\mu(\bar{\alpha})}_{x_1,\bar{\rho}}$ over $(H(\kappa^+), \in, \Phi_{\bar{\alpha}})$, also by φ_0 and φ_1, respectively, that

- $A^{\mu(\alpha)}_{x_0,\bar{\rho}}$, $A^{\mu(\alpha)}_{x_1,\bar{\rho}} \in \mathrm{dom}(\Psi_{\vec{\mathcal{E}}})$,
- $A^{\mu(\bar{\alpha})}_{x_0,\bar{\rho}} = \Psi_{\vec{\mathcal{E}}}(A^{\mu(\alpha)}_{x_0,\bar{\rho}})$, and
- $A^{\mu(\bar{\alpha})}_{x_1,\bar{\rho}} = \Psi_{\vec{\mathcal{E}}}(A^{\mu(\alpha)}_{x_1,\bar{\rho}})$.

Again by a definability argument, since $|A^{\mu(\alpha)}_{x_0,\bar{\rho}}|$, $|A^{\mu(\alpha)}_{x_1,\bar{\rho}}| < \kappa$, we also have that $A^{\mu(\alpha)}_{x_0,\bar{\rho}}$ and $A^{\mu(\alpha)}_{x_1,\bar{\rho}}$ are both subsets of $\mathrm{dom}(\Psi_{\vec{\mathcal{E}}})$. Finally, we have $\bar{\zeta}_0 \neq \bar{\zeta}_1$ in ω_1 such that

- r_0 forces in $\mathbb{Q}_{\mu(\alpha)}$ that $(\bar{\rho}, \bar{\zeta}_0)$ is below x_0 in $\underset{\sim}{T}_{\mu(\alpha)}$ and
- r_1 forces in $\mathbb{Q}_{\mu(\alpha)}$ that $(\bar{\rho}, \bar{\zeta}_1)$ is below x_1 in $\underset{\sim}{T}_{\mu(\alpha)}$.

But since $\Psi_{\vec{\mathcal{E}}}$ is a partial elementary embedding from $(N^0_0, \in, \Phi_\alpha)$ into $(N^n_1, \in, \Phi_{\bar{\alpha}})$, by Lemma 3.4 we have that

- $\Psi_{\vec{\mathcal{E}}}(r_0)$ forces in $\mathbb{Q}_{\mu(\bar{\alpha})}$ that $(\bar{\rho}, \bar{\zeta}_0)$ is below x_0 in $\underset{\sim}{T}_{\mu(\bar{\alpha})}$ and that
- $\Psi_{\vec{\mathcal{E}}}(r_1)$ forces in $\mathbb{Q}_{\mu(\bar{\alpha})}$ that $(\bar{\rho}, \bar{\zeta}_1)$ is below x_1 in $\underset{\sim}{T}_{\mu(\bar{\alpha})}$.

$\square$

Given functions f and g, let us momentarily denote by $f + g$ the function with $\mathrm{dom}(f + g) = \mathrm{dom}(f) \cup \mathrm{dom}(g)$ defined by letting

$$(f + g)(x) = f(x) \cup g(x)$$

for all $x \in \mathrm{dom}(f) \cup \mathrm{dom}(g)$.[22]

Given $\mathbb{Q}_{\kappa^+}$-conditions q_0 and q_1, let $q_0 \oplus q_1$ denote the natural amalgamation of q_0 and q_1; to be more specific, $q_0 \oplus q_1$ is the ordered pair $(f, \tau_{q_0} \oplus \tau_{q_1})$, where f is the closure of $f_{q_0} + f_{q_1}$ with respect to relevant (restrictions of) functions of the form Ψ_{N_0,N_1}, for edges $\langle (N_0, \gamma_0), (N_1, \gamma_1)\rangle$ in $\tau_{q_0} \oplus \tau_{q_1}$, so that clause (6) in the definition of condition holds for $q_0 \oplus q_1$. Even more precisely, we define

$$q_0 \oplus q_1 = ((f_{q_0} + f_{q_1}) + f, \tau_{q_0} \oplus \tau_{q_1}),$$

where f is the function with domain X—for X being the collection of all ordinals of the form $\Psi_{\vec{\mathcal{E}}}(\alpha)$, for a connected $\tau_{q_0} \oplus \tau_{q_1}$-thread $\langle \vec{\mathcal{E}}, (\rho, \alpha)\rangle$ such that $\vec{\mathcal{E}}$ consists of edges, and such that $(\rho, \zeta) \in f_{q_0}(\alpha) \cup f_{q_1}(\alpha)$ for some $\zeta < \omega_1$—and such that for every $\bar{\alpha} \in X$, $f(\bar{\alpha})$ is the collection of all nodes (ρ, ζ), for connected $\tau_{q_0} \oplus \tau_{q_1}$-threads $\langle \vec{\mathcal{E}}, (\rho, \alpha)\rangle$ such that

(1) $\vec{\mathcal{E}}$ consists of edges,
(2) $(\rho, \zeta) \in f_{q_0}(\alpha) \cup f_{q_1}(\alpha)$, and
(3) $\Psi_{\vec{\mathcal{E}}}(\alpha) = \bar{\alpha}$.

[22]Where, we recall, if h is a function and $x \notin \mathrm{dom}(h)$, we are setting $h(x)$ to be $\emptyset$.

Lemma 3.13 holds by the construction of $q_0 \oplus q_1$.

Lemma 3.13. *Let q_0 and q_1 be $\mathbb{Q}_{\kappa^+}$-conditions and let $q = q_0 \oplus q_1$. Then the following holds.*

(1) *τ_q is closed under copying.*
(2) *For every edge $\langle (N_0, \gamma_0), (N_1, \gamma_1)\rangle \in \tau_q$ and every $\alpha \in N_0 \cap \gamma_0$ such that $\bar{\alpha} := \Psi_{N_0,N_1}(\alpha) < \gamma_1$, if $\alpha \neq 0$, $\alpha \in \mathrm{dom}(f_q)$, and $x \in f_q(\alpha) \cap N_0$, then*
   (a) *$\bar{\alpha} \in \mathrm{dom}(f_q)$, and*
   (b) *$x \in f_q(\bar{\alpha})$.*

The following lemma is a trivial consequence of Lemmas 2.13 and 2.14.

Lemma 3.14. *For every two $\mathbb{Q}_{\kappa^+}$-conditions q_0 and q_1, if $q_0 \oplus q_1 = (f, \tau)$, then for every $\alpha \in \mathrm{dom}(f)$ and every $x = (\rho, \zeta) \in f(\alpha)$ such that $x \notin f_{q_0}(\alpha) \cup f_{q_1}(\alpha)$ there is some $\alpha^* \in \mathrm{dom}(f_{q_0}) \cup \mathrm{dom}(f_{q_1})$ such that $x \in f_{q_0}(\alpha^*) \cup f_{q_1}(\alpha^*)$ and some connected $\tau_{q_0} \cup \tau_{q_1}$-thread $\langle \vec{\mathcal{E}}, (\rho, \alpha^*)\rangle$ such that $\Psi_{\vec{\mathcal{E}}}(\alpha^*) = \alpha$. Furthermore, if $\lambda < \kappa$ is such that all edges in τ_{q_0} involve models of height less than λ, then all members of $\vec{\mathcal{E}}$ involving models of height at least λ are edges in τ_{q_1}.*

Extending our notation $f + g$ for functions f, g, if $\mathcal{F}$ is a set of functions, we denote by $\bigoplus \mathcal{F}$ the function g with domain

$$\bigcup \{\mathrm{dom}(f) \mid f \in \mathcal{F}\}$$

given by

$$g(x) = \bigcup \{f(x) \mid x \in \mathrm{dom}(f)\}.$$

The following lemma will be used in the proof of Lemma 4.1.

Lemma 3.15. *Let $\beta \leq \kappa^+$, and suppose q_0, $q_1 \in \mathbb{Q}_\beta$ are such that for every $\alpha < \beta$, if*

$$(q_0 \restriction \alpha) \oplus (q_1 \restriction \alpha) \in \mathbb{Q}_\alpha,$$

then

$$(q_0 \restriction \alpha + 1) \oplus (q_1 \restriction \alpha + 1) \in \mathbb{Q}_{\alpha+1}.$$

Then $q_0 \oplus q_1 \in \mathbb{Q}_\beta$.

Proof. The proof is by induction on β. We only need to argue for the conclusion in the case that β is a nonzero limit ordinal. In that case the conclusion follows easily from the induction hypothesis and the fact that for every $\alpha < \beta$,

$$f_{q_0 \oplus q_1} \restriction \alpha = \bigoplus \{f_{(q_0 \restriction \alpha') \oplus (q_1 \restriction \alpha')} \restriction \alpha \mid \alpha \leq \alpha' < \beta\}.$$

To see this equality it suffices to note that for every $\bar{\alpha} \in \mathrm{dom}(f_{q_0 \oplus q_1})$, any given (ρ, ζ) in

$$f_{q_0 \oplus q_1}(\bar{\alpha}) \setminus (f_{q_0}(\bar{\alpha}) \cup f_{q_1}(\bar{\alpha}))$$

has arrived there, thanks to Lemma 3.14, by virtue of some connected $\tau_{q_0 \restriction \alpha'} \cup \tau_{q_1 \restriction \alpha'}$-thread $\langle \vec{\mathcal{E}}, (\rho, \alpha^*) \rangle$ for some high enough $\alpha' < \beta$.

If $q_0 \oplus q_1$ were not a $\mathbb{Q}_\beta$-condition, there would be some $\alpha < \beta$ such that some finite piece of information contained in $f_{q_0 \oplus q_1} \restriction \alpha$ fails to satisfy clause (4), (6) or (7) in the definition of $\mathbb{Q}_\alpha$-condition. But that piece of information would occur in $f_{(q_0 \restriction \alpha') \oplus (q_1 \restriction \alpha')} \restriction \alpha$ for a high enough $\alpha' < \beta$ above α. Hence, by taking α' high enough we may guarantee that the fact that the piece of information violates some clause in the definition of $\mathbb{Q}_\alpha$-condition entails that

$$(f_{(q_0 \restriction \alpha') \oplus (q_1 \restriction \alpha')} \restriction \alpha, \tau_{(q_0 \restriction \alpha') \oplus (q_1 \restriction \alpha')} \restriction \alpha)$$

is not a $\mathbb{Q}_\alpha$-condition. But that contradicts $(q_0 \restriction \alpha') \oplus (q_1 \restriction \alpha') \in \mathbb{Q}_{\alpha'}$, which we know is true by induction hypothesis. □

We will now introduce the notion of adequate condition, which we already alluded to at the beginning of this section.

Definition 3.16. *Given $\beta < \kappa^+$ and $q \in \mathbb{Q}_\beta$, we will say that q is* adequate *in case (1) and (2) below hold.*

1. *For every $\alpha \in \mathrm{dom}(f_q)$ such that $\mathbb{Q}_{\mu(\alpha)}$ has the κ-c.c. and $\underset{\sim}{T}_{\mu(\alpha)}$ is defined and for all distinct $x_0 = (\rho_0, \zeta_0)$, $x_1 = (\rho_1, \zeta_1) \in f_q(\alpha)$ there is some $\bar{\rho} \leq \rho_0$, ρ_1, together with conditions $r_0 \in A^{\mu(\alpha)}_{x_0, \bar{\rho}}$ and $r_1 \in A^{\mu(\alpha)}_{x_1, \bar{\rho}}$ weaker than $q \restriction \mu(\alpha)$ and such that r_0 and r_1 force x_0 and x_1 to be incomparable in $\underset{\sim}{T}_{\mu(\alpha)}$.*
2. *The following holds for every correct τ_q-thread $\langle \vec{\mathcal{E}}, (\rho, \bar{\alpha}) \rangle$, where $\alpha = \Psi_{\vec{\mathcal{E}}}(\bar{\alpha})$, $\rho < \kappa$, $\zeta < \omega_1$, and $x = (\rho, \zeta) \in f_q(\bar{\alpha})$.*
   (a) *$\alpha \in \mathrm{dom}(f_q)$ and $x \in f_q(\alpha)$.*
   (b) *For every $\langle (N_0, \gamma_0), (N_1, \gamma_1) \rangle \in \tau_q \cap \mathrm{dom}(\vec{\mathcal{E}})$ with $\gamma_0, \gamma_1 \leq \bar{\alpha}$ there are $\gamma_0' \geq \Psi_{\vec{\mathcal{E}}}(\gamma_0)$ and $\gamma_1' \geq \Psi_{\vec{\mathcal{E}}}(\gamma_1)$ such that*
   $$\langle (\Psi_{\vec{\mathcal{E}}}(N_0), \gamma_0'), (\Psi_{\vec{\mathcal{E}}}(N_1), \gamma_1') \rangle \in \tau_q$$

We call a condition *weakly adequate* if it satisfies clause (1) from Definition 3.16.

Lemma 3.17. *The set of weakly adequate conditions is dense in $\mathbb{Q}_\beta$ for each $\beta \leq \kappa^+$.*

Proof. For each condition $q \in \mathbb{Q}_\beta$ let $\langle (\alpha^q_n, x^q_{0,n}, x^q_{1,n}) : n < \omega \rangle$ be the F-least enumeration of all triples (α, x_0, x_1) such that $\alpha \in \mathrm{dom}(f_q)$, $\alpha \neq 0$, is such that $\mathbb{Q}_{\mu(\alpha)}$ has the κ-c.c. and $\underset{\sim}{T}_{\mu(\alpha)}$ is defined, and

$x_0 = (\rho_0, \zeta_0)$, $x_1 = (\rho_1, \zeta_1) \in f_{q_i}(\alpha)$ are distinct. Let also $\varphi : \omega \times \omega \to \omega$ be a bijection such that $\varphi(m, n) \geq m$ for all $m, n < \omega$.

By induction on $i < \omega$ we define a decreasing sequence $\langle q_i : i < \omega \rangle$ of $\mathbb{Q}_\beta$-conditions as follows. To start, set $q_0 = q$. Now suppose that $i < \omega$ and that q_i is defined. Let m, n be such that $\varphi(m, n) = i$ and set $(\alpha, x_0, x_1) = (\alpha_n^{q_m}, x_{0,n}^{q_m}, x_{1,n}^{q_m})$. Let q_{i+1} be an extension of q_i such that there are $\bar{\rho} \leq \rho_0, \rho_1$, together with conditions $r_0 \in A^{\mu(\alpha)}_{x_0,\bar{\rho}}$ and $r_1 \in A^{\mu(\alpha)}_{x_1,\bar{\rho}}$ weaker than $q_{i+1} \restriction \mu(\alpha)$ and such that r_0 and r_1 force x_0 and x_1 to be incomparable in $\underset{\sim}{T}_{\mu(\alpha)}$. Then the greatest lower bound of $\{q_i \mid i < \omega\}$ is weakly adequate. □

In fact, the set of adequate conditions is dense in $\mathbb{Q}_\beta$ for each $\beta \leq \kappa^+$, as shown in Lemma 3.19. To show this, we need the following lemma.

Lemma 3.18. *Let $\alpha < \kappa^+$, $q \in \mathbb{Q}_{\alpha+1}$, $e = \langle (N_0, \alpha+1), (N_1, \gamma_1) \rangle$ a generalized edge coming from τ_q, and $\vec{\mathcal{E}} = (\langle (N_0^i, \gamma_0^i), (N_1^i, \gamma_1^i) \rangle \mid i \leq n)$ a sequence of generalized edges coming from τ_q with $\langle (N_0^0, \gamma_0^0), (N_1^0, \gamma_1^0) \rangle = e$ and such that $\langle \vec{\mathcal{E}}, (\emptyset, \alpha) \rangle$ is a correct thread. Let $\bar{\alpha}$ be such that $\Psi_{\vec{\mathcal{E}}}(\alpha) = \bar{\alpha}$. Let also $\delta = \min\{\delta_{N_0^i} \mid i \leq n\}$. Suppose that*

1. *$q \restriction \alpha$ is adequate, and that*
2. *$q \restriction \mu(\alpha)$ forces every two distinct nodes in $f_q(\bar{\alpha}) \cap (\delta \times \omega_1)$ to be incomparable in $\underset{\sim}{T}_{\mu(\alpha)}$.*

Then there is an extension $q^* \in \mathbb{Q}_{\alpha+1}$ of q such that $f_q(\bar{\alpha}) \cap (\delta \times \omega_1) \subseteq f_{q^*}(\alpha)$.

Proof. We may obviously assume $\bar{\alpha} \neq \alpha$ as otherwise there is nothing to prove. Let $r = (f_q, (\tau_q \setminus \tau) \cup (\tau_q \restriction \alpha) \cup (\tau \restriction \alpha))$, where

$$\tau = \{\langle (N_0', \gamma_0'), (N_1', \gamma_1') \rangle \in \tau_q \mid \max\{\gamma_0', \gamma_1'\} = \alpha + 1,\ \delta_{N_0'} \geq \delta_{N_0}\}.$$

Then $r \in \mathbb{Q}_{\alpha+1}^{\delta_{N_0}}$ and $e \restriction \alpha$ comes from τ_r. To see the latter claim, note that e clearly comes from τ, and hence $e \restriction \alpha$ comes from $\tau \restriction \alpha \subseteq \tau_r$. Also note that $r \restriction \mu(\alpha) = q \restriction \mu(\alpha)$, hence by clause (2), $r \restriction \mu(\alpha)$ forces every two distinct nodes in $f_q(\bar{\alpha}) \cap (\delta \times \omega_1)$ to be incomparable in $\underset{\sim}{T}_{\mu(\alpha)}$.

Set

$$\vec{\mathcal{E}}^* = \langle e \rangle^\frown (\langle (N_0^i, \gamma_0^i), (N_1^i, \gamma_1^i) \rangle \restriction \alpha \mid 0 < i \leq n).$$

Then $(\vec{\mathcal{E}}^*, (\emptyset, \alpha))$ is a correct thread (since $\bar{\alpha} < \alpha$). Since clearly all members of $\vec{\mathcal{E}}^*$ come from $\tau_r \cup \{e\}$, by clause (7) of the definition of $\mathbb{Q}_{\alpha+1}$-condition for q there is an extension $r^* \in \mathbb{Q}_{\alpha+1}^{\delta_{N_0}}$ of r such that

$$f_{r^*}(\bar{\alpha}) \supseteq f_r(\alpha) \cap (\delta \times \omega_1) = f_q(\alpha) \cap (\delta \times \omega_1)$$

and such that $r^* \restriction \mu(\alpha)$ forces every two distinct nodes in

$$(f_q(\bar{\alpha}) \cap (\delta \times \omega_1)) \cup f_q(\alpha)$$

to be incomparable in $\underset{\sim}{T}_{\mu(\alpha)}$. Let

$$q^* = (f_{r^*} \restriction \alpha \cup \{(\alpha, f_q(\alpha) \cup (f_q(\bar{\alpha}) \cap (\delta \times \omega_1)))\}, \tau_q \cup \tau_{r^*} \restriction \alpha).$$

It suffices to show that q^* is a $\mathbb{Q}_{\alpha+1}$-condition, as then it is an extension of q in $\mathbb{Q}_{\alpha+1}$ as desired.

To see that $q^* \in \mathbb{Q}_{\alpha+1}$, we only need to show that q^* satisfies clause (6) of the definition of $\mathbb{Q}_{\alpha+1}$-condition, as all other clauses can be checked easily. Thus let $e' = \langle (N'_0, \alpha+1), (N'_1, \gamma'_1) \rangle \in \tau_q \cup \tau_{r^*} \restriction \alpha$ be such that $\alpha' := \Psi_{N'_0, N'_1}(\alpha) < \gamma'_1$ and let us note that in fact $e' \in \tau_q$. We have to show that

$$\big(f_q(\alpha) \cup (f_q(\bar{\alpha})\big) \cap (\delta_{N_0} \times \omega_1) \cap N'_0 \subseteq f_{q^*}(\alpha').$$

We may assume that $\alpha' < \alpha$. Since $e' \in \tau_q$, we have that $f_q(\alpha) \cap N'_0 \subseteq f_q(\alpha') \subseteq f_{q^*}(\alpha')$. To show that $f_q(\bar{\alpha}) \cap (\delta_{N_0} \times \omega_1) \cap N'_0 \subseteq f_{q^*}(\alpha')$, let $x = (\rho, \zeta) \in f_q(\bar{\alpha}) \cap (\delta_{N_0} \times \omega_1)$ with $\rho < \delta_{N'_0}$. Let

$$\vec{\mathcal{E}}^{-1} = (\langle (N_1^{n-i}, \gamma_1^{n-i}), (N_0^{n-i}, \gamma_0^{n-i}) \rangle \mid i \leq n \rangle$$

and let us consider the correct $\tau_q \restriction \alpha$-thread $\langle \vec{\mathcal{F}}, (\rho, \bar{\alpha}) \rangle$, where

$$\vec{\mathcal{F}} = (\vec{\mathcal{E}}^{-1} \restriction \alpha)^\frown \langle e' \restriction \alpha \rangle.$$

Then $\Psi_{\vec{\mathcal{F}}}(\bar{\alpha}) = \alpha'$, and since $q \restriction \alpha$ is adequate, we have $x \in f_{q^*}(\alpha')$. Thus, the conclusion follows. □

Lemma 3.19. *For every $\beta \leq \kappa^+$, the set of adequate $\mathbb{Q}_\beta$-conditions is dense in $\mathbb{Q}_\beta$.*

Proof. Let $q \in \mathbb{Q}_\beta$. We will find an adequate $\mathbb{Q}_\beta$-condition q^* stronger than q. We prove this by induction on β.

First, suppose that β is a limit ordinal of countable cofinality and let $(\beta_i)_{i<\omega}$ be an increasing sequence of ordinals cofinal in β. We define, by induction on $i < \omega$, two sequences $(q_i)_{i<\omega}$ and $(r_i)_{i<\omega}$ of conditions such that $q_0 = q$ and such that for all $i < \omega$,

(1) $q_i \in \mathbb{Q}_\beta$,
(2) r_i is an adequate $\mathbb{Q}_{\beta_i}$-condition,
(3) $r_i \leq_{\mathbb{Q}_{\beta_i}} q_i \restriction \beta_i$, and
(4) $q_{i+1} = (f_{r_i} \cup f_{q_i} \restriction [\beta_i, \beta), \tau_{r_i} \cup \tau_{q_i})$.[23]

The construction can be carried out using the induction hypothesis.

Let q^* be the greatest lower bound of the sequence $(q_i)_{i<\omega}$, which exists by Lemma 3.10. Then q^* is an adequate $\mathbb{Q}_\beta$-condition extending q. The point is that every instance of adequacy depends on ordinals

[23]Note that by Lemma 3.7 (and Lemma 3.6), each q_{i+1} is indeed a $\mathbb{Q}_\beta$-condition.

α, $\bar{\alpha} < \beta$ and is in fact verified at some high enough stage i of the construction.

If β is a limit ordinal of uncountable cofinality, then we fix some $\bar{\beta} < \beta$ such that $\mathrm{dom}(f_q) \subseteq \bar{\beta}$, find an adequate extension q' of $q \restriction \bar{\beta}$ in $\mathbb{Q}_{\bar{\beta}}$ (which exists by the induction hypothesis), and note that $q^* = (f_{q'}, \tau_{q'} \cup \tau_q)$ is an extension of q in $\mathbb{Q}_\beta$ by Lemma 3.7. But then we are done since q^* is adequate by the choice of q'.

Finally, suppose that $\beta = \alpha + 1$ is a successor ordinal. By induction hypothesis together with Lemma 3.7, we may assume that $q \restriction \alpha$ is adequate. We may also assume that there is some generalized edge coming from τ_q of the form $\langle (N_0, \beta), (N_1, \gamma_1) \rangle$, as otherwise we are done. We build q^* as the greatest lower bound of a suitably constructed descending sequence $(q_n)_{n<\omega}$ of conditions extending $q_0 = q$ and such that q_n is weakly adequate, and $q_n \restriction \alpha$ is adequate for each n. For every n, and assuming q_n has been found, we construct q_{n+1} in the following way.

Let us pick some correct τ_{q_n}-thread $\langle \vec{\mathcal{E}}, (\rho, \bar{\alpha}) \rangle$. Let $\rho < \kappa$, $\zeta < \omega_1$, and suppose $x = (\rho, \zeta) \in f_{q_n}(\bar{\alpha})$. Let $\delta = \min\{\delta_{N_0^i} : i \leq m\}$, where $\vec{\mathcal{E}} = (\langle (N_0^i, \gamma_0^i), (N_1^i, \gamma_1^i) \rangle : i \leq m)$. Let $\alpha = \Psi_{\vec{\mathcal{E}}}(\bar{\alpha})$. By the adequacy of $q_n \restriction \alpha$ and using Lemma 3.12, we have that $q_n \restriction \mu(\alpha)$ forces any two distinct nodes in $f_{q_n}(\bar{\alpha}) \cap (\delta \times \omega_1)$ to be incomparable in $\underset{\sim}{T}_{\mu(\alpha)}$. This is true since for any two distinct nodes x, y in $f_{q_n}(\bar{\alpha}) \cap (\delta \times \omega_1)$, by weak adequacy of q_n, some condition $r \in \mathbb{Q}_{\mu(\bar{\alpha})}$ weaker than $q_n \restriction \mu(\bar{\alpha})$ and belonging to the final model of $\vec{\mathcal{E}}$ forces x and y to be incomparable in $T_{\mu(\bar{\alpha})}$. By adequacy of $q_n \restriction \mu(\bar{\alpha})$, $\Psi_{\vec{\mathcal{E}}}(r)$ is also weaker than $q_n \restriction \mu(\bar{\alpha})$. And by Lemma 3.12, $\Psi_{\vec{\mathcal{E}}}(r)$ is a condition in $\mathbb{Q}_{\mu(\alpha)}$ forcing x and y, which of course are fixed by $\Psi_{\vec{\mathcal{E}}}$, to be incomparable in $T_{\mu(\alpha)}$.

By Lemma 3.18 we may find an extension $q^0_{n+1} \in \mathbb{Q}_\beta$ of q_n such that

$$f_{q_n}(\bar{\alpha}) \cap (\delta \times \omega_1) \subseteq f_{q^0_{n+1}}(\alpha) \cap (\delta \times \omega_1).$$[24]

Let then $q^1_{n+1} = (f_{q^0_{n+1}}, \tau_{q^1_{n+1}})$, where $\tau_{q^1_{n+1}}$ is the union of $\tau_{q^0_{n+1}}$ and the set of edges of the form

$$\Psi_{\vec{\mathcal{E}}}(\langle (N_0', \gamma_0'), (N_1', \gamma_1') \rangle)$$

with $\langle (N_0', \gamma_0'), (N_1', \gamma_1') \rangle \in \tau_{q_n} \cap N_0^0$ and γ_0', $\gamma_1' \leq \bar{\alpha}$. We note that $\Psi_{\vec{\mathcal{E}}}(\langle (N_0', \gamma_0'), (N_1', \gamma_1') \rangle)$ is obtained from some already present edge $\langle \Psi_{\vec{\mathcal{E}}}(N_0', \tilde{\gamma}_0'), \Psi_{\vec{\mathcal{E}}}(N_1', \tilde{\gamma}_1') \rangle$ in q_n by, at most, increasing some of the markers $\tilde{\gamma}_\epsilon'$ to α, and that doing so does not force us to add working parts that were not already present in q^0_{n+1}.

[24]In fact, by the proof of Lemma 3.18 it suffices for this to simply add to $f_{q_n}(\alpha)$ all nodes in $f_{q_n}(\bar{\alpha}) \cap (\delta \times \omega_1)$.

Let now q_{n+1} be an extension of q^1_{n+1}, obtained by first extending $q^1_{n+1} \restriction \alpha$ to an adequate condition using the induction hypothesis and then applying clause (7) in the definition of condition, such that

$$f_{q^1_{n+1}}(\alpha) \cap (\delta \times \omega_1) \subseteq f_{q_{n+1}}(\bar{\alpha}).$$

By further extending q_{n+1} using Lemma 3.17 and the induction hypothesis (and Lemma 3.7), we may assume in addition that q_{n+1} is weakly adequate and $q_{n+1} \restriction \alpha$ is adequate.

Using some suitable book-keeping, we can make sure that $(q_n)_{n<\omega}$ is built in such a way that every relevant $\langle \vec{\mathcal{E}}, (\rho, \bar{\alpha})\rangle$ for which there is some $x = (\rho, \zeta) \in f_{q_n}(\bar{\alpha})$, occurring at any stage m in the construction, is taken care of at infinitely many stages $n > m$. Let q^* be the greatest lower bound of $\{q_n \mid n < \omega\}$. We then have that q^* is an adequate condition extending q. □

Given a $\mathbb{Q}_{\kappa^+}$-condition q and a model N, we denote by $q \restriction N$ the ordered pair $(f, \tau_q \cap N)$, where f is the function with domain $\mathrm{dom}(f_q) \cap N$ such that $f(x) = f_q(x) \cap N$ for every $x \in \mathrm{dom}(f)$.

It will be necessary, in the proof of Lemma 5.1, to adjoin a certain edge to some given condition. This will be accomplished by means of the following lemma.

Lemma 3.20. *Suppose for every $\beta < \kappa^+$, $\mathbb{Q}_\beta$ has the κ-c.c. Let $\alpha < \kappa^+$ and let*

$$e = \langle (N_0, \alpha + 1), (N_1, \gamma_1)\rangle$$

be a generalized edge. Suppose $(N_0, \in, \Phi_{\alpha+1}) \prec (H(\kappa^+), \in, \Phi_{\alpha+1})$. Suppose $\mathbb{Q}_\xi \cap N_0$ is a complete suborder of $\mathbb{Q}_\xi$ for every $\xi \in (\alpha + 2) \cap N_0$. Let $r \in \mathbb{Q}^{\delta_{N_0}}_{\alpha+1}$ and suppose $e \restriction \alpha$ comes from τ_r. Let

$$\vec{\mathcal{E}} = (\langle (N^i_0, \gamma^i_0), (N^i_1, \gamma^i_1)\rangle \mid i < n)$$

be a sequence of generalized edges coming from $\tau_r \cup \{e\}$ such that $\langle (N^0_0, \gamma^0_0), (N^1_0, \gamma^1_0)\rangle = e$ and $\langle \vec{\mathcal{E}}, (\emptyset, \alpha)\rangle$ is a correct thread. Let $\delta = \min\{\delta_{N^i_0} \mid i < n\}$ and $\bar{\alpha} = \Psi_{\vec{\mathcal{E}}}(\alpha)$. Suppose $\bar{\alpha} < \alpha$ and suppose $r \restriction \mu(\alpha)$ forces every two distinct nodes in $f_r(\bar{\alpha}) \cap (\delta \times \omega_1)$ to be incomparable in $\underset{\sim}{T}_{\mu(\alpha)}$.

Then there is an extension $r^* \in \mathbb{Q}^{\delta_{N_0}}_{\alpha+1}$ of r such that

1. *$f_r(\alpha) \cap (\delta \times \omega_1) \subseteq f_{r^*}(\bar{\alpha})$ and*
2. *$r^* \restriction \mu(\alpha)$ forces every two distinct nodes in*

$$(f_r(\bar{\alpha}) \cap (\delta \times \omega_1)) \cup f_r(\alpha)$$

to be incomparable in $\underset{\sim}{T}_{\mu(\alpha)}$.

Proof. The proof is by induction on α. To start with, since $\mathbb{Q}_{\alpha+1} \cap N_0 \lessdot \mathbb{Q}_{\alpha+1}$, $r \restriction N_0$ may be extended to a condition $q_0 \in \mathbb{Q}_{\alpha+1} \cap N_0$ forcing that every condition in $\dot{G}_{\mathbb{Q}_{\alpha+1} \cap N_0}$ is compatible with r.

Claim 3.21. *We may extend q_0 to a condition $q_1 \in \mathbb{Q}_{\alpha+1} \cap N_0$ for which there is a generalized edge*

$$e' = \langle (N_0', \alpha), (N_1', \gamma_1') \rangle \in \tau_{q_1}$$

such that $\mathbb{Q}_\xi \cap N_0'$ is a complete suborder of $\mathbb{Q}_\xi$ for every $\xi \in (\alpha+1) \cap N_0'$, together with some $r' \in \mathbb{Q}_{\alpha+1}^{\delta_{N_0'}}$ with $e' \restriction \alpha$ coming from $\tau_{r'}$, and together with a sequence

$$\vec{\mathcal{E}}' = (\langle (N_0'^i, \gamma_0'^i), (N_1'^i, \gamma_1'^i) \rangle \mid i < n)$$

of generalized edges coming from $\tau_{r'} \cup \{e'\}$ such that

$$\langle (N_0'^0, \gamma_0'^0), (N_0'^1, \gamma_0'^1) \rangle = e'$$

and such that $\langle \vec{\mathcal{E}}', (\emptyset, \alpha) \rangle$ is a correct thread and such that, letting $\delta' = \min\{\delta_{N_0'^i} \mid i < n\}$ and $\bar{\alpha}' = \Psi_{\vec{\mathcal{E}}'}(\alpha)$, we have that $\rho < \delta'$ for every $(\rho, \nu) \in f_{q_0}(\alpha)$ with $\rho < \delta$, $\bar{\alpha}' < \alpha$, and that $r' \restriction \mu(\alpha)$ forces every two distinct nodes in $f_{r'}(\bar{\alpha}') \cap (\delta' \times \omega_1)$ to be incomparable in $\underset{\sim}{T}_{\mu(\alpha)}$. Moreover, $q_0 \in N_0'$, $q_0 \restriction N_1' = q_0 \restriction N_0' \cap N_1'$, and $f_{q_1}(\bar{\alpha}') \cap (\delta' \times \omega_1) = f_{q_1}(\alpha) \cap (\delta' \times \omega_1)$.

Proof. In order to find q_1, we first find e', $\vec{\mathcal{E}}'$ and r' in N_0 as in the statement. The existence of such objects is witnessed by $e \restriction \alpha$, $\vec{\mathcal{E}}$ and r, respectively, and can be expressed by a sentence over $(H(\kappa^+), \in, \Phi_{\alpha+1})$ with parameters in N_0.

We can now find a suitable condition q_1^- in $\mathbb{Q}_{\alpha+1} \cap N_0$ extending q_0 and such that $e' \in \tau_{q_1^-}$. Indeed, q_1^- is obtained by adding e' to τ_{q_0} and copying the relevant information coming from q_0 into N_1' via $\Psi_{N_0', N_1'}$ so as to make clauses (5) and (6) in the definition of condition hold for q_1^-; in other words, $q_1^- = q_0 \oplus (\emptyset, \{e'\})$.

The result of copying any piece of information carried by q_0 in N_0' into N_1' will not interfere with any piece of information previously carried by q_0 in N_1' as that information is also in N_0' and therefore fixed by $\Psi_{N_0', N_1'}$. Also, clause (7) in the definition of condition is ensured for e' at all ordinals $\xi + 1 \in N_0' \cap \alpha$ by the induction hypothesis applied to all $\xi \in (N_0' \cup N_1') \cap \alpha$. It then easily follows that q_1^- is a condition.

But now we may find q_1 as desired by simply copying the relevant information coming from q_0 at stage α via $\Psi_{N_0', N_1'}$, which again is possible since $q_0 \restriction N_1' = q_0 \restriction N_0' \cap N_1'$. □

Let us fix q_1, e', $\vec{\mathcal{E}}'$ and r' as given by the claim. By the choice of q_0 we may find a common extension $r_0^* \in \mathbb{Q}_\alpha$ of $q_1 \restriction \alpha$ and $r \restriction \alpha$, which we may assume is adequate by Lemma 3.19. By adequacy of r_0^* it follows that $f_{r_0^*}(\bar{\alpha}') \cap (\delta' \times \omega_1) = f_{r_0^*}(\bar{\alpha}) \cap (\delta' \times \omega_1)$. Let $q_2 \in N_0$ be the amalgamation, as given by Lemma 3.7, of $r_0^* \restriction N_0$ and q_1. In order to finish the proof it suffices to argue that (f, τ_{q_2}) is a condition in $\mathbb{Q}_{\alpha+1} \cap N_0$, where f is the function such that $f \restriction \alpha = f_{q_2} \restriction \alpha$ and $f(\alpha) = f_{q_0}(\alpha) \cup (f_{r_0^*}(\bar{\alpha}) \cap (\delta \times \omega_1))$, since then we can take r^* to be any condition in $\mathbb{Q}_{\alpha+1}^{\delta_{N_0}}$ extending both (f, τ_{q_2}) and r, which exists by the choice of q_0.

(f, τ_{q_2}) is of course in N_0. By our hypothesis, the only way (f, τ_{q_2}) could fail to be a condition in $\mathbb{Q}_{\alpha+1}$ is that there are distinct $x \in f_{q_0}(\alpha)$ and $y \in f_{r_0^*}(\bar{\alpha}) \cap (\delta' \times \omega_1)$ such that $q_2 \restriction \mu(\alpha)$ does not force x and y to be incomparable nodes in $\utilde{T}_{\mu(\alpha)}$. We can then extend $q_2 \restriction \mu(\alpha)$ to some $r' \in \mathbb{Q}_{\mu(\alpha)} \cap N_0$ forcing x and y to be $\utilde{T}_{\mu(\alpha)}$-comparable. By the κ-c.c. of $\mathbb{Q}_{\mu(\alpha)}$ we may of course assume that r' extends some $\bar{r} \in \mathbb{Q}_{\mu(\alpha)} \cap N_0'$ forcing x and y to be $\utilde{T}_{\mu(\alpha)}$-comparable. Once again by the choice of q_0, let $q \in \mathbb{Q}_\alpha$ be a common extension of r' and $r \restriction \mu(\alpha)$ which, thanks to Lemma 3.19, we may assume is adequate. But now $\Psi_{\vec{\mathcal{E}}'}(\bar{r})$ is a condition weaker than $q \restriction \mu(\bar{\alpha}')$ (by clauses (5) and (6) in the definition of condition applied to q) and forcing x and y to be $\utilde{T}_{\mu(\bar{\alpha}')}$-comparable, which of course is a contradiction since x, $y \in f_q(\bar{\alpha}')$. $\square$

4. The chain condition

This section is devoted to proving Lemma 4.1.

Lemma 4.1. *For each $\beta \leq \kappa^+$, $\mathbb{Q}_\beta$ has the κ-chain condition.*

As we will see, the weak compactness of κ is used crucially in order to prove Lemma 4.1. Let $\mathcal{F}$ be the weak compactness filter on κ, i.e., the filter on κ generated by the sets

$$\{\lambda < \kappa \mid (V_\lambda, \in, B \cap V_\lambda) \models \psi\},$$

where $B \subseteq V_\kappa$ and where ψ is a Π^1_1 sentence for the structure $(V_\kappa, \in, B)$ such that $(V_\kappa, \in, B) \models \psi$. $\mathcal{F}$ is a proper normal filter on κ. Let also $\mathcal{S}$ be the collection of $\mathcal{F}$-positive subsets of κ, i.e.,

$$\mathcal{S} = \{X \subseteq \kappa \mid X \cap C \neq \emptyset \text{ for all } C \in \mathcal{F}\}$$

We will call a model Q *suitable* if Q is an elementary submodel of cardinality κ of some high enough $H(\theta)$, closed under $<\kappa$-sequences, and such that $\langle \mathbb{Q}_\alpha \mid \alpha < \kappa^+ \rangle \in Q$. Given a suitable model Q, a bijection $\varphi : \kappa \to Q$, and an ordinal $\lambda < \kappa$, we will denote $\varphi``\lambda$ by M^φ_λ. It is easily seen that

$$\{\lambda < \kappa \mid M^\varphi_\lambda \prec Q,\ M^\varphi_\lambda \cap \kappa = \lambda \text{ and } {}^{<\lambda}M^\varphi_\lambda \subseteq M^\varphi_\lambda\} \in \mathcal{F}.$$

Definition 4.2 (strong chain condition). *Given $\beta \leq \kappa^+$, we will say that $\mathbb{Q}_\beta$* has the strong κ-chain condition *if for every $X \in \mathcal{S}$, every suitable model Q such that $\beta, X \in Q$, every bijection $\varphi : \kappa \to Q$, and every two sequences $(q^0_\lambda \mid \lambda \in X) \in Q$ and $(q^1_\lambda \mid \lambda \in X) \in Q$ of adequate $\mathbb{Q}_\beta$-conditions, if*

- *$M^\varphi_\lambda \cap \kappa = \lambda$ and*
- *$q^0_\lambda \restriction M^\varphi_\lambda = q^1_\lambda \restriction M^\varphi_\lambda$ for every $\lambda \in X$,*

then there is some $Y \in \mathcal{S}$, $Y \subseteq X$, together with sequences

$$(r^0_\lambda \mid \lambda \in Y)$$

and

$$(r^1_\lambda \mid \lambda \in Y)$$

of adequate $\mathbb{Q}_\beta$-conditions with the following properties.

(1) *$r^0_\lambda \leq_{\mathbb{Q}_\beta} q^0_\lambda$ and $r^1_\lambda \leq_{\mathbb{Q}_\beta} q^1_\lambda$ for every $\lambda \in Y$.*
(2) *For all $\lambda_0 < \lambda_1$ in Y, $r^0_{\lambda_0} \oplus r^1_{\lambda_1}$ is a common extension of $r^0_{\lambda_0}$ and $r^1_{\lambda_1}$.*

The following lemma is an immediate consequence of Lemma 3.19.

Lemma 4.3. *For every $\beta \leq \kappa^+$, if $\mathbb{Q}_\beta$ has the strong κ-chain condition, then $\mathbb{Q}_\beta$ has the κ-chain condition.*

Following [5], given $\beta \leq \kappa^+$, a suitable model Q such that $\beta \in Q$, a bijection $\varphi : \kappa \to Q$, $\lambda < \kappa$, and a $\mathbb{Q}_\beta$-condition $q \in Q$, let us say that *q is λ-compatible with respect to φ and β* if, letting $\mathbb{Q}^*_\beta = \mathbb{Q}_\beta \cap Q$, we have that

- $\mathbb{Q}^*_\beta \cap M^\varphi_\lambda \lessdot \mathbb{Q}^*_\beta$,
- $q \restriction M^\varphi_\lambda \in \mathbb{Q}^*_\beta$, and
- $q \restriction M^\varphi_\lambda$ forces in $\mathbb{Q}^*_\beta \cap M^\varphi_\lambda$ that q is in the quotient forcing $\mathbb{Q}^*_\beta / \dot{G}_{\mathbb{Q}^*_\beta \cap M^\varphi_\lambda}$; equivalently, for every $r \in \mathbb{Q}^*_\beta \cap M^\varphi_\lambda$, if $r \leq_{\mathbb{Q}^*_\beta \cap M^\varphi_\lambda} q \restriction M^\varphi_\lambda$, then r is compatible with q.[25]

Adopting the approach from [10], rather than proving Lemma 4.1 we will prove the following more informative lemma.

Lemma 4.4. *The following holds for every $\beta < \kappa^+$.*

$(1)_\beta$ *$\mathbb{Q}_\beta$ has the strong κ-chain condition.*

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[25] In [10], this situation is denoted by $*^\beta_\lambda(q_0, q_0 \restriction M^\varphi_\lambda)$.

$(2)_\beta$ *Suppose $D \in \mathcal{F}$, Q is a suitable model, β, $D \in Q$, $\varphi : \kappa \to Q$ is a bijection, and $(q^0_\lambda \mid \lambda \in D) \in Q$ and $(q^1_\lambda \mid \lambda \in D) \in Q$ are sequences of $\mathbb{Q}_\beta$-conditions. Then there is some $D' \in \mathcal{F}$, $D' \subseteq D$, such that for every $\lambda \in D'$ and for all $q^{0'}_\lambda \leq_{\mathbb{Q}_\beta} q^0_\lambda$ and $q^{1'}_\lambda \leq_{\mathbb{Q}_\beta} q^1_\lambda$, if $q^{0'}_\lambda \restriction M^\varphi_\lambda \in \mathbb{Q}_\beta$ and $q^{0'}_\lambda \restriction M^\varphi_\lambda = q^{1'}_\lambda \restriction M^\varphi_\lambda$, then there are conditions $r^0_\lambda \leq_{\mathbb{Q}_\beta} q^{0'}_\lambda$ and $r^1_\lambda \leq_{\mathbb{Q}_\beta} q^{1'}_\lambda$ such that*
(a) *$r^0_\lambda \restriction M^\varphi_\lambda = r^1_\lambda \restriction M^\varphi_\lambda$ and*
(b) *r^0_λ and r^1_λ are both λ-compatible with respect to φ and β.*

Corollary 4.5. *$\mathbb{Q}_{\kappa^+}$ has the κ-c.c.*

Proof. Suppose q_i, for $i < \kappa$, are conditions in $\mathbb{Q}_{\kappa^+}$. By Lemma 3.19, we may assume that each q_i, for $i < \kappa$, is adequate. We may then fix $\beta < \kappa^+$ such that $q_i \in \mathbb{Q}_\beta$ for all $i < \kappa$. But by Lemma 4.4 $(1)_\beta$ together with Lemma 4.3 there are $i \neq i'$ in κ such that q_i and $q_{i'}$ are compatible in $\mathbb{Q}_\beta$ and hence in $\mathbb{Q}_{\kappa^+}$. □

The rest of the section is devoted to proving the above lemma.

Proof. (of Lemma 4.4) The proof is by induction on β. Let $\beta < \kappa^+$ and suppose $(1)_\alpha$ and $(2)_\alpha$ hold for all $\alpha < \beta$. We will show that $(1)_\beta$ and $(2)_\beta$ hold as well.

There is nothing to prove for $\beta = 0$, and the case $\beta = 1$ is trivial, using the inaccessibility of κ and the fact that $\mathbb{Q}_1$ is essentially the Lévy collapse turning κ into $\aleph_2$.

Let us proceed to the case when $\beta > 1$. We start with the proof of $(1)_\beta$.

Let $X \in \mathcal{S}$ be given, together with a suitable model Q such that $\beta, X \in Q$, a bijection $\varphi : \kappa \to Q$, and sequences

$$\vec{\sigma}_0 = (q^0_\lambda \mid \lambda \in X) \in Q$$

and

$$\vec{\sigma}_1 = (q^1_\lambda \mid \lambda \in X) \in Q$$

of adequate $\mathbb{Q}_\beta$-conditions such that

$$M^\varphi_\lambda \cap \kappa = \lambda$$

and

$$q^0_\lambda \restriction M^\varphi_\lambda = q^1_\lambda \restriction M^\varphi_\lambda$$

for every $\lambda \in X$. We need to prove that there is some $Y \in \mathcal{S}$, $Y \subseteq X$, together with sequences

$$(r^0_\lambda \mid \lambda \in Y)$$

and

$$(r^1_\lambda \mid \lambda \in Y)$$

of $\mathbb{Q}_\beta$-conditions such that the following holds.

(1) $r^0_\lambda \leq_{\mathbb{Q}_\beta} q^0_\lambda$ and $r^1_\lambda \leq_{\mathbb{Q}_\beta} q^1_\lambda$ for every $\lambda \in Y$.
(2) For all $\lambda_0 < \lambda_1$ in Y, $r^0_{\lambda_0} \oplus r^1_{\lambda_1}$ is a common extension of $r^0_{\lambda_0}$ and $r^1_{\lambda_1}$.

We note that $\beta + 1 \subseteq Q$. In what follows, we will write M_λ instead of M^φ_λ.

Let $\mathbb{Q}^*_\alpha = \mathbb{Q}_\alpha \cap Q$ for every $\alpha \in \beta + 1$. By the induction hypothesis, $\mathbb{Q}_\alpha$ has the κ-c.c. for every $\alpha \in \beta$. Hence, since ${}^{<\kappa}Q \subseteq Q$, we have that $\mathbb{Q}^*_\alpha \lessdot \mathbb{Q}_\alpha$ for every such α; in particular, we have that for every $\alpha \in \mathcal{X} \cap \beta$, $\mathbb{Q}^*_\alpha$ forces over V that $\underset{\sim}{T}_\alpha$ does not have κ-branches.

Given

- conditions q^0, q^1 in $\mathbb{Q}_\beta$,
- nonzero stages $\alpha \in \mathrm{dom}(f_{q^0})$ and $\alpha' \in \mathrm{dom}(f_{q^1})$,[26]
- nodes $x = (\rho_0, \zeta_0)$ and $y = (\rho_1, \zeta_1)$ such that $x \in f_{q^0}(\alpha)$ and $y \in f_{q^1}(\alpha')$,[27] and
- $\lambda < \kappa$,

we will say that *x and y are separated below λ at stages $\mu(\alpha)$ and $\mu(\alpha')$ by q^0 and q^1 (via $\bar{x}$, $\bar{y}$)* if there are $\bar{\rho} < \lambda$ and $\zeta \neq \zeta'$ in ω_1 such that $\bar{x} = (\bar{\rho}, \zeta)$ and $\bar{y} = (\bar{\rho}, \zeta')$, and such that

(1) $q^0 \restriction \mu(\alpha)$ extends a condition in $A^{\mu(\alpha)}_{x,\bar{\rho}}$ forcing $\bar{x}$ to be below x in $\underset{\sim}{T}_{\mu(\alpha)}$ and
(2) $q^1 \restriction \mu(\alpha')$ extends a condition in $A^{\mu(\alpha)}_{y,\bar{\rho}}$ forcing $\bar{y}$ to be below y in $\underset{\sim}{T}_{\mu(\alpha')}$.

Definition 4.6. *Given $Y \in \mathcal{S}$ such that $Y \subseteq X$ and such that $M_\lambda \prec Q$, $M_\lambda \cap \kappa = \lambda$, and ${}^{<\lambda}M_\lambda \subseteq M_\lambda$ for all $\lambda \in Y$, and given two sequences $\vec{\sigma}^*_0 = (r^0_\lambda \mid \lambda \in Y)$, $\vec{\sigma}^*_1 = (r^1_\lambda \mid \lambda \in Y)$ of adequate $\mathbb{Q}^*_\beta$-conditions, we say that $\vec{\sigma}^*_0$, $\vec{\sigma}^*_1$* is a separating pair for $\vec{\sigma}_0$ and $\vec{\sigma}_1$ *if the following holds.*

(1) *For every $\lambda \in Y$, $r^0_\lambda \leq_{\mathbb{Q}_\beta} q^0_\lambda$, $r^1_\lambda \leq_{\mathbb{Q}_\beta} q^1_\lambda$, and $\mathrm{dom}(f_{r^0_\lambda}) = \mathrm{dom}(f_{r^1_\lambda})$.*
(2) *For every $\lambda \in Y$, every $\alpha \in \mathrm{dom}(f_{r^0_\lambda}) \cap M_\lambda$, every nonzero $\alpha' \in \mathrm{dom}(f_{r^1_\lambda})$ such that $\alpha' \leq \alpha$, and for all*

$$x \in f_{r^0_\lambda}(\alpha) \setminus (\lambda \times \omega_1)$$

and

$$y \in f_{r^1_\lambda}(\alpha') \setminus (\lambda \times \omega_1),$$

[26]Note that, by induction hypothesis, both $\mathbb{Q}_{\mu(\alpha)}$ and $\mathbb{Q}_{\mu(\alpha')}$ have the κ-c.c. and hence $\underset{\sim}{T}_{\mu(\alpha)}$ and $\underset{\sim}{T}_{\mu(\alpha')}$ are both defined.

[27]α and α' may or may not be equal and the same applies to x and y.

x and y are separated below λ at stages $\mu(\alpha)$ and $\mu(\alpha')$ by r^0_λ and r^1_λ via some pair $\chi_0(x,y,\alpha,\alpha',\lambda)$, $\chi_1(x,y,\alpha,\alpha',\lambda)$ of nodes.

(3) *The following holds for all $\lambda_0 < \lambda_1$ in Y.*
   (a) $r^0_{\lambda_0} \restriction M_{\lambda_0} = r^1_{\lambda_1} \restriction M_{\lambda_1}$.
   (b) $r^0_{\lambda_0} \in M_{\lambda_1}$.
   (c) *Let $\mathcal{N}_{\lambda_\epsilon}$ and Ξ_{λ_ϵ}, for $\epsilon \in \{0,1\}$, be defined as follows.*
      - *$\mathcal{N}_{\lambda_\epsilon}$ is the union of the sets of the form $N_0 \cup N_1$, where $\langle (N_0,\gamma_0),(N_1,\gamma_1)\rangle \in \tau_{r^0_{\lambda_\epsilon}} \cup \tau_{r^1_{\lambda_\epsilon}}$ and $\delta_{N_0} < \lambda_\epsilon$.*
      - *Ξ_{λ_ϵ} is the collection of ordinals of the form $\Psi_{\vec{\mathcal{E}}}(\bar\alpha)$, where $\langle \vec{\mathcal{E}}, (\rho,\bar\alpha)\rangle$ is a connected $\tau_{r^1_{\lambda_\epsilon}}$-thread such that $(\rho,\bar\alpha) \in N \cap (\kappa \times \kappa^+)$ for some model N of height less than λ_ϵ coming from some edge in $\tau_{r^1_{\lambda_\epsilon}}$.*

   Then
      (i) $\mathcal{N}_{\lambda_0} \cap M_{\lambda_0} = \mathcal{N}_{\lambda_1} \cap M_{\lambda_1}$ *and*
      (ii) $\Xi_{\lambda_0} \cap M_{\lambda_0} = \Xi_{\lambda_1} \cap M_{\lambda_1}$.

(4) *For all $\lambda_0 < \lambda_1$ in Y, all ordinals $\alpha \in \mathrm{dom}(f_{r^0_{\lambda_0}}) \cap M_{\lambda_0}$ and $\alpha' \in \mathrm{dom}(f_{r^1_{\lambda_1}})$ such that $0 < \alpha' \le \alpha$, and all nodes*

$$x \in f_{r^0_{\lambda_0}}(\alpha) \setminus (\lambda_0 \times \omega_1)$$

and

$$y' \in f_{r^1_{\lambda_1}}(\alpha') \setminus (\lambda_1 \times \omega_1)$$

there are
   - *a node $x' \in f_{r^0_{\lambda_1}}(\alpha) \setminus (\lambda_1 \times \omega_1)$,*
   - *a stage $\alpha^* \in \mathrm{dom}(f_{r^1_{\lambda_0}})$ such that $\alpha^* \le \alpha$, and*
   - *a node $y \in f_{r^1_{\lambda_0}}(\alpha^*) \setminus (\lambda_0 \times \omega_1)$*

such that

$$\chi_0(x,y,\alpha,\alpha^*,\lambda_0) = \chi_0(x',y',\alpha,\alpha',\lambda_1)$$

and

$$\chi_1(x,y,\alpha,\alpha^*,\lambda_0) = \chi_1(x',y',\alpha,\alpha',\lambda_1)$$

Let us now prove the following.

Claim 4.7. *Let $Y \in \mathcal{S}$ be such that $M_\lambda \cap \kappa = \lambda$ for all $\lambda \in Y$, and suppose $\vec\sigma^*_0 = (r^0_\lambda \mid \lambda \in Y)$, $\vec\sigma^*_1 = (r^1_\lambda \mid \lambda \in Y)$ is a separating pair for $\vec\sigma_0$ and $\vec\sigma_1$. Then for all $\lambda_0 < \lambda < \lambda_1$ in Y, $r^0_{\lambda_0} \oplus r^1_{\lambda_1}$ is a common extension of $r^0_{\lambda_0}$ and $r^1_{\lambda_1}$ in $\mathbb{Q}_\beta$.*

Proof. Suppose, towards a contradiction, that there are $\lambda_0 < \lambda < \lambda_1$ in Y such that $r^0_{\lambda_0} \oplus r^1_{\lambda_1}$ is not a common extension of $r^0_{\lambda_0}$ and $r^1_{\lambda_1}$. It then follows that $r^0_{\lambda_0} \oplus r^1_{\lambda_1}$ is not a condition. Hence, by Lemma 3.15, there is an ordinal $\alpha < \beta$ such that

$$q := (r^0_{\lambda_0} \restriction \alpha) \oplus (r^1_{\lambda_1} \restriction \alpha)$$

is a condition yet

$$q^+ := (r^0_{\lambda_0} \restriction \alpha + 1) \oplus (r^1_{\lambda_1} \restriction \alpha + 1)$$

is not. Assuming that we are in this situation, we will derive a contradiction by proving that q^+ is a condition after all.

To start with, note that $\alpha > 0$. We will need the following subclaim.

Subclaim 4.8. *Suppose*

1. $\beta_1 \leq \beta_0$ *are ordinals in* M_λ,
2. $\vec{\mathcal{E}} = (\langle (N^i_0, \gamma^i_0), (N^i_1, \gamma^i_1) \rangle \mid i \leq n)$ *is a sequence of generalized edges coming from* $\tau_{r^1_{\lambda_1}}$ *such that* $\langle \vec{\mathcal{E}}, (\emptyset, \beta_0) \rangle$ *is a* $\tau_{r^1_{\lambda_1}}$*-thread,*
3. $\beta_1 = \Psi_{\vec{\mathcal{E}}}(\beta_0)$, *and*
4. $\delta_{N^i_0} \geq \lambda_1$ *for all* $i \leq n$.

Then $\beta_1 = \beta_0$.

Proof. By correctness of $(N^0_0, \in, \Phi_0)$ within $(H(\kappa^+), \in, \Phi_0)$, we may pick some model $M \in N^0_0$ closed under $\vec{e}$ such that $\beta_0 \in M$, $\delta_M = \lambda$, and $|M| = \lambda$ (since M_λ is such a a model). Given that $\beta_1 \leq \beta_0$ are both in M_λ, $\delta_{M_\lambda} = \lambda$, and $\beta_0 \in M$, by the first part of Lemma 2.3 we then have that $\beta_1 \in M \subseteq N^0_0$. But that means that

$$(\Psi_{N^n_0, N^n_1} \circ \ldots \circ \Psi_{N^0_0, N^0_1})(\beta_0) = \beta_1$$

is in fact β_0 since $\beta_1 \in N^0_0 \cap N^n_1$ implies, by the second part of Lemma 2.3, that

$$\beta_0 = (\Psi_{N^n_1, N^n_0} \circ \ldots \circ \Psi_{N^0_1, N^0_0})(\beta_1) = \beta_1.$$

□

We will also be using the following subclaim.

Subclaim 4.9. *Suppose* $\alpha^* \in \mathrm{dom}(f_{r^0_{\lambda_0}}) \cup \mathrm{dom}(f_{r^1_{\lambda_1}})$, $x = (\rho, \zeta) \in f_{r^0_{\lambda_0}}(\alpha^*) \cup f_{r^1_{\lambda_1}}(\alpha^*)$, $\langle \vec{\mathcal{E}}_*, (\rho, \alpha^*) \rangle$ *is a connected* $\tau_{r^0_{\lambda_0}} \cup \tau_{r^1_{\lambda_1}}$*-thread, and all members* $\langle (N_0, \gamma_0), (N_1, \gamma_1) \rangle$ *of* $\vec{\mathcal{E}}_*$ *such that* $\delta_{N_0} \geq \lambda_1$ *are edges from* $\tau^1_{\lambda_1}$. *Then at least one of the following holds, where* $\bar{\alpha} = \Psi_{\vec{\mathcal{E}}_*}(\alpha^*)$.

1. $\bar{\alpha} \in \mathrm{dom}(f_{r^0_{\lambda_0}})$ *and* $x \in f_{r^0_{\lambda_0}}(\bar{\alpha})$.
2. $\bar{\alpha} \in \mathrm{dom}(f_{r^1_{\lambda_1}})$ *and* $x \in f_{r^1_{\lambda_1}}(\bar{\alpha})$.
3. *There is some* $\alpha^{**} \in \mathrm{dom}(f_{r^0_{\lambda_0}})$ *such that* $x \in f_{r^0_{\lambda_0}}(\alpha^{**})$ *and some connected* $\tau_{r^1_{\lambda_1}}$*-thread* $\langle \vec{\mathcal{E}}, (\rho, \alpha^{**}) \rangle$ *such that*

- *$\Psi_{\vec{\mathcal{E}}}(\alpha^{**}) = \bar{\alpha}$ and*
- *all members of $\vec{\mathcal{E}}$ are edges $\langle (N_0, \gamma_0), (N_1, \gamma_1)\rangle$ such that $\delta_{N_0} \geq \lambda_1$.*

Proof. We prove this by induction on $|\vec{\mathcal{E}}_*|$, which we may obviously assume is nonzero. Let

$$\vec{\mathcal{E}}_* = (\langle (N_0^i, \gamma_0^i), (N_1^i, \gamma_1^i) \mid i \leq m)$$

and

$$e = \langle (N_0^m, \gamma_0^m), (N_1^m, \gamma_1^m)\rangle.$$

By induction hypothesis, one of (1)–(3) holds for $\langle \vec{\mathcal{E}}_* \restriction m, (\rho, \alpha^*)\rangle$.[28] Let $\alpha^\dagger = \Psi_{\vec{\mathcal{E}}_* \restriction m}(\alpha^*)$.

Suppose (1) holds for $\langle \vec{\mathcal{E}}_* \restriction m, (\rho, \alpha^*)\rangle$. We have two cases. If $e \in \tau_{r^0_{\lambda_0}}$, then (1) holds trivially for $\langle \vec{\mathcal{E}}_*, (\rho, \alpha^*)\rangle$ by adequacy of $r^0_{\lambda_0}$. The other case is that $e \in \tau_{r^1_{\lambda_1}}$. If $\delta_{N_0^m} \geq \lambda_1$, then obviously (3) holds for $\langle \vec{\mathcal{E}}_*, (\rho, \alpha^*)\rangle$ as witnessed by $\alpha^\dagger$ and the thread $\langle (e), (\rho, \alpha^\dagger)\rangle$. We may thus assume that $\delta_{N_0^m} < \lambda_1$. We know that $\alpha^\dagger \in \mathcal{N}_{\lambda_1} \cap M_{\lambda_1}$ by clause (3)(b) in Definition 4.6 for the pair $r^0_{\lambda_1}$, $r^1_{\lambda_1}$. It follows that $\alpha^\dagger \in \mathcal{N}_{\lambda_0} \cap M_{\lambda_0}$ by (3)(c), and hence $\alpha^\dagger \in \mathrm{dom}(f_{r^1_{\lambda_1}})$ and $x \in f_{r^1_{\lambda_1}}(\alpha^\dagger)$ by (3)(a). But then $\bar{\alpha} \in \mathrm{dom}(f_{r^1_{\lambda_1}})$ and $x \in f_{r^1_{\lambda_1}}(\bar{\alpha})$ by adequacy of $r^1_{\lambda_1}$ and so (2) holds for $\langle \vec{\mathcal{E}}_*, (\rho, \alpha^*)\rangle$.

Next suppose (2) holds for $\langle \vec{\mathcal{E}}_* \restriction m, (\rho, \alpha^*)\rangle$. Suppose $e \in \tau_{r^0_{\lambda_0}}$. Since $e \in M_{\lambda_1}$ by clause (3)(b) in Definition 4.6, it follows from (3)(a) that $\alpha^\dagger \in \mathrm{dom}(f_{r^0_{\lambda_0}})$ and $x \in f_{r^0_{\lambda_0}}(\alpha^\dagger)$. But then, by adequacy of $r^0_{\lambda_0}$, $\bar{\alpha} \in \mathrm{dom}(r^0_{\lambda_0})$ and $x \in f_{r^0_{\lambda_0}}(\bar{\alpha})$. Hence (1) holds for $\langle \vec{\mathcal{E}}_*, (\rho, \alpha^*)\rangle$. If $e \in \tau_{r^1_{\lambda_1}}$, then (2) holds trivially for $\langle \vec{\mathcal{E}}_*, (\rho, \alpha^*)\rangle$, again by adequacy of $r^1_{\lambda_1}$.

Finally, suppose (3) holds for $\langle \vec{\mathcal{E}}_* \restriction m, (\rho, \alpha^*)\rangle$, as witnessed by $\alpha^{**} \in \mathrm{dom}(f_{r^0_{\lambda_0}})$ together with a connected $\tau_{r^1_{\lambda_1}}$-thread $\langle \vec{\mathcal{E}}, (\rho, \alpha^{**})\rangle$. Suppose $e \in \tau_{r^0_{\lambda_0}}$. Then, since all models occurring in the edges in $\vec{\mathcal{E}}$ are of height at least λ_1 and since, once again by clause (3)(b) in Definition 4.6, $r^0_{\lambda_0} \in M_\lambda$, we have by Subclaim 4.8 that $\alpha^\dagger = \alpha^{**}$. But then $\bar{\alpha} \in \mathrm{dom}(f_{r^0_{\lambda_0}})$ and $x \in f_{r^0_{\lambda_0}}(\bar{\alpha})$ by adequacy of $r^0_{\lambda_0}$, and so (1) holds for $\langle \vec{\mathcal{E}}_*, (\rho, \alpha^*)\rangle$. Finally, suppose $e \in r^1_{\lambda_1}$. If $\delta_{N_0^m} \geq \lambda_1$, then (3)

[28] Note that $\langle \vec{\mathcal{E}}_* \restriction m, (\rho, \alpha^*)\rangle$ is also a connected thread, so we may indeed apply the induction hypothesis to it. This is in contrast with the fact that it does not follow that $\langle \vec{\mathcal{E}}_* \restriction m, (\rho, \alpha^*)\rangle$ is a correct thread if we just assume that $\vec{\mathcal{E}}_*$ is correct.

holds trivially for $\langle \vec{\mathcal{E}}_*, (\rho, \alpha^*)\rangle$ as witnessed by $\langle \vec{\mathcal{E}}^\frown\langle e\rangle, (\rho, \alpha^{**})\rangle$. In the other case, by clauses (3)(c)(ii), (3)(b) and (3)(a) in Definition 4.6 for the pair $r^0_{\lambda_1}$, $r^1_{\lambda_1}$, we have that $\alpha^{**} \in \Xi_{\lambda_1} \cap M_{\lambda_1} = \Xi_{\lambda_0} \cap M_{\lambda_0}$, and therefore $\alpha^{**} \in \mathrm{dom}(f_{r^1_{\lambda_1}})$ and $x \in f_{r^1_{\lambda_1}}(\alpha^{**})$ again by (3)(a). But then $\bar{\alpha} \in \mathrm{dom}(f_{r^1_{\lambda_1}})$ and $x \in f_{r^1_{\lambda_1}}(\bar{\alpha})$ by adequacy of $r^1_{\lambda_1}$. Thus we have that (2) holds for $\langle \vec{\mathcal{E}}_*, (\rho, \alpha^*)\rangle$, which finishes the proof of the subclaim. □

Remember that

$$q^+ = (r^0_{\lambda_0} \restriction \alpha + 1) \oplus (r^1_{\lambda_1} \restriction \alpha + 1)$$

and that we are aiming to prove that $q^+ \in \mathbb{Q}_{\alpha+1}$. We also know that

$$q = (r^0_{\lambda_0} \restriction \alpha) \oplus (r^1_{\lambda_1} \restriction \alpha)$$

is a condition in $\mathbb{Q}_\alpha$. One way q^+ could fail to be a condition is that there are ϵ, $\epsilon' \in \{0, 1\}$, together with $\alpha_0 \in \mathrm{dom}(f_{r^\epsilon_{\lambda_\epsilon}})$, $x_0 = (\rho_0, \zeta_0) \in f_{r^\epsilon_{\lambda_\epsilon}}(\alpha_0)$, $\alpha_1 \in \mathrm{dom}(f_{r^{\epsilon'}_{\lambda_{\epsilon'}}})$, $x_1 = (\rho_1, \zeta_1) \in f_{r^{\epsilon'}_{\lambda_{\epsilon'}}}(\alpha_1)$, α_1, $\alpha_0 \le \alpha$, and a nonzero ordinal $\bar{\alpha}$ such that there are connected $\tau_{r^0_{\lambda_0} \oplus r^1_{\lambda_1}}$-threads $\langle \vec{\mathcal{E}}^*_0, (\rho_0, \alpha_0)\rangle$ and $\langle \vec{\mathcal{E}}^*_1, (\rho_1, \alpha_1)\rangle$, respectively, such that

- $\bar{\alpha} = \Psi_{\vec{\mathcal{E}}^*_0}(\alpha_0) = \Psi_{\vec{\mathcal{E}}^*_1}(\alpha_1)$,
- both $\vec{\mathcal{E}}^*_0$ and $\vec{\mathcal{E}}^*_1$ consist of edges in $\tau_{r^0_{\lambda_0} \oplus r^1_{\lambda_1}}$, and such that
- $q \restriction \mu(\bar{\alpha})$ does not force x_0 and x_1 to be incomparable in $\underset{\sim}{T}_{\mu(\bar{\alpha})}$.

By Lemma 3.14, we may replace $\langle \vec{\mathcal{E}}^*_0, (\rho_0, \alpha_0)\rangle$ and $\langle \vec{\mathcal{E}}^*_1, (\rho_1, \alpha_1)\rangle$ by connected $\tau_{r^0_{\lambda_0} \cup r^1_{\lambda_1}}$-threads $\langle \vec{\mathcal{E}}'_0, (\rho_0, \alpha_0)\rangle$ and $\langle \vec{\mathcal{E}}'_1, (\rho_1, \alpha_1)\rangle$ all of whose members involving models of height at least λ_1 are edges in $\tau_{r^1_{\lambda_1}}$.[29]

By Subclaim 4.9, applied to α_0, x_0 and the connected $\tau_{r^0_{\lambda_0} \cup r^1_{\lambda_1}}$-thread $\langle \vec{\mathcal{E}}'_0, (\rho_0, \alpha_0)\rangle$, at least one of the following holds.

(1)$_0$ $\bar{\alpha} \in \mathrm{dom}(f_{r^0_{\lambda_0}})$ and $x_0 \in f_{r^0_{\lambda_0}}(\bar{\alpha})$.

(2)$_0$ $\bar{\alpha} \in \mathrm{dom}(f_{r^1_{\lambda_1}})$ and $x_0 \in f_{r^1_{\lambda_1}}(\bar{\alpha})$.

(3)$_0$ There is some $\alpha^{**}_0 \in \mathrm{dom}(f_{r^0_{\lambda_0}})$ such that $x_0 \in f_{r^0_{\lambda_0}}(\alpha^{**}_0)$ and some connected $\tau_{r^1_{\lambda_1}}$-thread $\langle \vec{\mathcal{E}}''_0, (\rho_0, \alpha^{**}_0)\rangle$ such that
  - $\Psi_{\vec{\mathcal{E}}''_0}(\alpha^{**}_0) = \bar{\alpha}$ and
  - all members of $\vec{\mathcal{E}}''_0$ are edges $\langle (N_0, \gamma_0), (N_1, \gamma_1)\rangle$ such that $\delta_{N_0} \ge \lambda_1$.

Similarly, and by applying Subclaim 4.9 to α_1, x_1 and the connected $\tau_{r^0_{\lambda_0} \cup r^1_{\lambda_1}}$-thread $\langle \vec{\mathcal{E}}'_1, (\rho_1, \alpha_1)\rangle$, at least one of the following holds.

[29] $\vec{\mathcal{E}}'_0$ and $\vec{\mathcal{E}}'_1$ may of course involve anti-edges.

$(1)_1$ $\bar{\alpha} \in \mathrm{dom}(f_{r^0_{\lambda_0}})$ and $x_1 \in f_{r^0_{\lambda_0}}(\bar{\alpha})$.

$(2)_1$ $\bar{\alpha} \in \mathrm{dom}(f_{r^1_{\lambda_1}})$ and $x_1 \in f_{r^1_{\lambda_1}}(\bar{\alpha})$.

$(3)_1$ There is some $\alpha_1^{**} \in \mathrm{dom}(f_{r^0_{\lambda_0}})$ such that $x_1 \in f_{r^0_{\lambda_0}}(\alpha_1^{**})$ and some connected $\tau_{r^1_{\lambda_1}}$-thread $\langle \vec{\mathcal{E}}_1'', (\rho_1, \alpha_1^{**})\rangle$ such that
- $\Psi_{\vec{\mathcal{E}}_1''}(\alpha_1^{**}) = \bar{\alpha}$ and
- all members of $\vec{\mathcal{E}}_1''$ are edges $\langle (N_0, \gamma_0), (N_1, \gamma_1)\rangle$ such that $\delta_{N_0} \geq \lambda_1$.

After changing some of the above objects if necessary, we essentially reduce to one of the following situations.

(1) $\bar{\alpha} \in \mathrm{dom}(f_{r^0_{\lambda_0}})$ and $x_0, x_1 \in f_{r^0_{\lambda_0}}(\bar{\alpha})$.

(2) $\bar{\alpha} \in \mathrm{dom}(f_{r^1_{\lambda_1}})$ and $x_0, x_1 \in f_{r^1_{\lambda_1}}(\bar{\alpha})$.

(3) $\bar{\alpha} \in \mathrm{dom}(f_{r^0_{\lambda_0}}) \cap \mathrm{dom}(f_{r^1_{\lambda_1}})$, $x_0 \in f_{r^0_{\lambda_0}}(\bar{\alpha})$, and $x_1 \in f_{r^1_{\lambda_1}}(\bar{\alpha})$.

(4) $\alpha_0 \in \mathrm{dom}(f_{r^0_{\lambda_0}})$, $x_0 \in f_{r^0_{\lambda_0}}(\alpha_0)$, and there is a connected $\tau_{r^1_{\lambda_1}}$-thread $\langle \vec{\mathcal{E}}_0, (\rho_0, \alpha_0)\rangle$ such that
- $\Psi_{\vec{\mathcal{E}}_0}(\alpha_0) = \bar{\alpha}$ and
- all members of $\vec{\mathcal{E}}_0$ are edges involving models of height at least λ_1,

and such that one of the following holds.

(a) $\bar{\alpha} \in \mathrm{dom}(f_{r^0_{\lambda_0}})$ and $x_1 \in f_{r^0_{\lambda_0}}(\bar{\alpha})$.

(b) $\bar{\alpha} \in \mathrm{dom}(f_{r^1_{\lambda_1}})$ and $x_1 \in f_{r^1_{\lambda_1}}(\bar{\alpha})$.

(c) $\alpha_1 \in \mathrm{dom}(f_{r^0_{\lambda_0}})$, $x_1 \in f_{r^0_{\lambda_0}}(\alpha_1)$, and there is a connected $\tau_{r^1_{\lambda_1}}$-thread $\langle \vec{\mathcal{E}}_1, (\rho_1, \alpha_1)\rangle$ such that
- $\Psi_{\vec{\mathcal{E}}_1}(\alpha_1) = \bar{\alpha}$ and
- all members of $\vec{\mathcal{E}}_1$ are edges involving models of height at least λ_1.

We may clearly rule out (1) and (2) since $q \restriction \mu(\bar{\alpha})$ extends both of $r^0_{\lambda_0} \restriction \mu(\bar{\alpha})$ and $r^1_{\lambda_1} \restriction \mu(\bar{\alpha})$. Let us assume that (4) holds.[30] We will first consider the subcase when (a) holds. By Subclaim 4.8 applied to the fact that the height of all models occurring in $\vec{\mathcal{E}}_0$ is at least λ_1 and the fact that both $\bar{\alpha}$ and α_0 are in M_λ, we get that $\bar{\alpha} = \alpha_0$. But then we get a contradiction as in case (1).

Let us now consider the subcase when (b) holds. By adequacy of $r^1_{\lambda_1}$ it follows that $\alpha_0 \in \mathrm{dom}(f_{r^1_{\lambda_1}})$. If $\rho_1 < \lambda_1$, then by adequacy of $r^1_{\lambda_1}$ we get that $x_1 \in f^1_{\lambda_1}(\alpha_0)$ and then $x_1 \in f_{r^0_{\lambda_0}}(\alpha_0)$ by clauses (3)(a)-(b) in

[30]We are considering this case before the case that (3) holds since the proof in the latter case will be a simpler variant of an argument we are about to see.

Definition 4.6 for $r^0_{\lambda_0}$ and $r^1_{\lambda_1}$. But then $r^0_{\lambda_0} \restriction \mu(\alpha_0)$ extends conditions $r_0 \in A^{\mu(\alpha_0)}_{x_0,\bar\rho}$ and $r_1 \in A^{\mu(\alpha_0)}_{x_1,\bar\rho}$, for some $\bar\rho \leq \min\{\rho_0,\rho_1\}$, forcing x_0 and x_1 to be incomparable in $\underset{\sim}{T}_{\mu(\alpha_0)}$ and, by Lemma 3.12, $\Psi_{\vec{\mathcal{E}}_0}(r_0)$ and $\Psi_{\vec{\mathcal{E}}_0}(r_1)$ are conditions weaker than $q \restriction \mu(\bar\alpha)$ and forcing x_0 and x_1 to be incomparable in $\underset{\sim}{T}_{\mu(\bar\alpha)}$. We may thus assume that $\rho_1 \geq \lambda_1$. Suppose $\rho_0 < \lambda_0$. Then $x_0 \in f_{r^1_{\lambda_1}}(\alpha_0)$ by clause (3)(a) in Definition 4.6, and hence $x_0 \in f_{r^1_{\lambda_1}}(\bar\alpha)$ by adequacy of $r^1_{\lambda_1}$. We again reach a contradiction as in case (2). Hence we may assume $\lambda_0 \leq \rho_0$. The rest of the argument, in this case, is now essentially as in the corresponding proof in [10]. Since $\bar\alpha \leq \alpha_0$ due to the fact that all members of $\vec{\mathcal{E}}_0$ are edges, by an appropriate instance of clause (4) in Definition 4.6 we may pick

- a node $x'_0 = (\rho', \zeta'_0) \in f_{r^0_{\lambda_1}}(\alpha_0) \setminus (\lambda_1 \times \omega_1)$,
- a stage $\alpha^* \in \mathrm{dom}(f_{r^1_{\lambda_0}})$ such that $\alpha^* \leq \alpha_0$, and
- a node $x^*_1 = (\rho^*_1, \zeta^*_1) \in f_{r^1_{\lambda_0}}(\alpha^*) \setminus (\lambda_0 \times \omega_1)$

such that

$$\chi_0(x_0, x^*_1, \alpha_0, \alpha^*, \lambda_0) = \chi_0(x'_0, x_1, \alpha_0, \bar\alpha, \lambda_1)$$

and

$$\chi_1(x_0, x^*_1, \alpha_0, \alpha^*, \lambda_0) = \chi_1(x'_0, x_1, \alpha_0, \bar\alpha, \lambda_1)$$

(where χ_0 and χ_1 are the projections in Definition 4.6). Let $\bar\rho$ be such that

$$\chi_0(x_0, x^*_1, \alpha_0, \alpha^*, \lambda_0) = (\bar\rho, \bar\zeta_0)$$

and

$$\chi_1(x'_0, x_1, \alpha_0, \bar\alpha, \lambda_1) = (\bar\rho, \bar\zeta_1)$$

for some $\bar\zeta_0 \neq \bar\zeta_1$ in ω_1. We have that $q \restriction \mu(\alpha_0)$ extends a condition $r_0 \in A^{\mu(\alpha_0)}_{x_0,\bar\rho}$ forcing $\chi_0(x_0, x^*_1, \alpha_0, \alpha^*, \lambda_0)$ to be below x_0 in $\underset{\sim}{T}_{\mu(\alpha_0)}$ (because this is true about $r^0_{\lambda_0} \restriction \mu(\alpha_0)$). Also, $r^1_{\lambda_1} \restriction \mu(\bar\alpha)$ extends a condition $r_1 \in A^{\mu(\bar\alpha)}_{x_1,\bar\rho}$ forcing that $\chi_1(x'_0, x_1, \alpha_0, \bar\alpha, \lambda_1)$ is below x_1 in $\underset{\sim}{T}_{\mu(\bar\alpha)}$, and therefore so does $q \restriction \mu(\bar\alpha)$. We also have that $q \restriction \mu(\bar\alpha)$ extends $\Psi_{\vec{\mathcal{E}}_0}(r_0)$, and by Lemma 3.12 $\Psi_{\vec{\mathcal{E}}_0}(r_0)$ forces $\chi_0(x_0, x^*_1, \alpha_0, \alpha^*, \lambda_0)$ to be below x_0 in $\underset{\sim}{T}_{\mu(\bar\alpha)}$. But now we get a contradiction since $\bar\zeta_0 \neq \bar\zeta_1$ and hence $q \restriction \mu(\bar\alpha)$ forces x_0 and x_1 to be incomparable in $\underset{\sim}{T}_{\mu(\bar\alpha)}$.

It remains to consider the subcase that (c) holds. Since all models occurring in members of $\vec{\mathcal{E}}_0$ or of $\vec{\mathcal{E}}_1$ are of height at least λ_1 and both α_0 and α_1 are in M_λ, by Subclaim 4.8 we get that $\alpha_0 = \alpha_1$. But now we get a contradiction by the same argument we have already encountered using Lemma 3.12.

We finally handle the case when (3) holds. In this case we may assume that both $\rho_0 \geq \lambda_0$ and $\rho_1 \geq \lambda_1$ hold, as otherwise we get, by an application of clause (3)(a) in Definition 4.6, that at least one of x_0, x_1 is in $f_{r^0_{\lambda_0}}(\bar{\alpha}) \cap f_{r^1_{\lambda_1}}(\bar{\alpha})$, which immediately yields a contradiction. But now, since $\rho_0 \geq \lambda_0$ and $\rho_1 \geq \lambda_1$, we obtain a contradiction by a separation argument using clause (4) in Definition 4.6—with both α and α', in that definition, being $\bar{\alpha}$—like the one we have already seen.

We will now prove that clause (7) in the definition of $\mathbb{Q}_{\alpha+1}$-condition holds for q^+. This will conclude the proof that q^+ is a condition (the verification of all remaining clauses in the definition of $\mathbb{Q}_{\alpha+1}$-condition is immediate), and will therefore complete the proof of the claim.

Suppose $\bar{\alpha} < \alpha + 1$ and $e = \langle (N_0, \bar{\alpha} + 1), (N_1, \gamma_1) \rangle$ is a generalized edge coming from $\tau_{q^+} \restriction \bar{\alpha} + 1$. We must show that the following holds.

Let $r \in \mathbb{Q}^{\delta_{N_0}}_{\bar{\alpha}+1}$ be such that $e \restriction \bar{\alpha}$ comes from τ_r, and suppose

$$\vec{\mathcal{E}} = (\langle (N^i_0, \gamma^i_0), (N^i_1, \gamma^i_1) \rangle \mid i < n)$$

is a sequence of generalized edges coming from $\tau_r \cup \{e\}$ such that $\langle (N^0_0, \gamma^0_0), (N^0_1, \gamma^0_1) \rangle = e$ and $\langle \vec{\mathcal{E}}, (\emptyset, \bar{\alpha}) \rangle$ is a correct thread. Let $\delta = \min\{\delta_{N^i_0} \mid i < n\}$ and $\alpha' = \Psi_{\vec{\mathcal{E}}}(\bar{\alpha})$. Suppose $r \restriction \mu(\bar{\alpha})$ forces every two distinct nodes in $f_r(\alpha') \cap (\delta \times \omega_1)$ to be incomparable in $\underset{\sim}{T}_{\mu(\bar{\alpha})}$. Then there is an extension $r^* \in \mathbb{Q}^{\delta_{N_0}}_{\bar{\alpha}+1}$ of r such that

(1) $f_{r^*}(\bar{\alpha}) \cap (\delta \times \omega_1) \subseteq f_{r^*}(\alpha') \cap (\delta \times \omega_1)$, and
(2) $r^* \restriction \mu(\bar{\alpha})$ forces every two distinct nodes in

$$(f_r(\alpha') \cap (\delta \times \omega_1)) \cup f_r(\bar{\alpha})$$

to be incomparable in $\underset{\sim}{T}_{\mu(\bar{\alpha})}$.

We may assume that e does not come from either $\tau_{r^0_{\lambda_0}}$ or $\tau_{r^1_{\lambda_1}}$, as otherwise we would be done since both $r^0_{\lambda_0}$ and $r^1_{\lambda_1}$ are conditions. The crucial point is now that, thanks to Lemma 3.4, the above is a fact about e that can be expressed over $(H(\kappa^+), \in, \Phi_{\bar{\alpha}+1})$ with e as parameter. Letting now $\alpha^\dagger = \max\{\bar{\alpha}+1, \gamma_1\}$, $e = \Psi_{\vec{\mathcal{E}}}(e^*)$ for some generalized edge e^* coming from $(\tau_{r^0_{\lambda_0}} \restriction \alpha^* + 1) \cup (\tau_{r^1_{\lambda_1}} \restriction \alpha^* + 1)$, for some α^*, and some appropriate connected $\tau_{r^0_{\lambda_0}} \cup \tau_{r^1_{\lambda_1}}$-thread $\langle \vec{\mathcal{E}}, (e^*, \alpha^*) \rangle$ such that $\Psi_{\vec{\mathcal{E}}}(\alpha^*) = \alpha^\dagger$. The corresponding fact holds in $(H(\kappa^+), \in, \Phi_{\alpha^*})$ about e^* since both $r^0_{\lambda_0} \restriction \alpha^* + 1$ and $r^1_{\lambda_1} \restriction \alpha^* + 1$ are $\mathbb{Q}_{\alpha^*+1}$-conditions. But then the desired fact holds about e in $(H(\kappa^+), \in, \Phi_{\bar{\alpha}+1})$ by correctness of $\vec{\mathcal{E}}$, using the fact that $\Psi_{\vec{\mathcal{E}}^{-1}}(\bar{\alpha} + 1) \leq \alpha^*$. This concludes the proof of Claim 4.7. □

The following technical fact appears essentially in [10].

Claim 4.10. *Suppose* $Z \in \mathcal{S}$, $(p^0_\lambda \mid \lambda \in Z) \in Q$ *and* $(p^1_\lambda \mid \lambda \in Z) \in Q$ *are sequences of conditions in* $\mathbb{Q}^*_\beta$, *and suppose that for every* $\lambda \in Z$,

- $p^0_\lambda \restriction M_\lambda$ *and* $p^1_\lambda \restriction M_\lambda$ *are compatible conditions in* $\mathbb{Q}^*_\beta \cap M_\lambda$,
- p^0_λ *and* p^1_λ *are* λ*-compatible with respect to* φ *and* α *for all* $\alpha < \beta$,
- $\alpha_\lambda \in \mathrm{dom}(f_{p^0_\lambda}) \cap M_\lambda$,
- $\alpha'_\lambda \in \mathrm{dom}(f_{p^1_\lambda})$ *is a nonzero ordinal such that* $\alpha'_\lambda \leq \alpha_\lambda$, *and*
- $x_\lambda = (\rho^0_\lambda, \zeta^0_\lambda)$ *and* $y_\lambda = (\rho^1_\lambda, \zeta^1_\lambda)$ *are nodes of level at least* λ *such that* $x_\lambda \in f_{p^0_\lambda}(\alpha_\lambda)$ *and* $y_\lambda \in f_{p^1_\lambda}(\alpha'_\lambda)$.

Then there is $D \in \mathcal{F}$, *together with two sequences* $(p^2_\lambda \mid \lambda \in Z \cap D)$, $(p^3_\lambda \mid \lambda \in Z \cap D)$ *of conditions in* $\mathbb{Q}^*_\beta$ *such that*

(1) *for each* $\lambda \in Z \cap D$, $p^2_\lambda \leq p^0_\lambda$ *and* $p^3_\lambda \leq p^1_\lambda$,
(2) *for each* $\lambda \in Z \cap D$, $p^2_\lambda \restriction M_\lambda$ *and* $p^3_\lambda \restriction M_\lambda$ *are compatible in* $\mathbb{Q}^*_\beta \cap M_\lambda$, *and*
(3) *for each* $\lambda \in Z \cap D$, x_λ *and* y_λ *are separated below* λ *at stages* $\mu(\alpha_\lambda)$ *and* $\mu(\alpha'_\lambda)$ *by* $p^2_\lambda \restriction \mu(\alpha_\lambda)$ *and* $p^3_\lambda \restriction \mu(\alpha'_\lambda)$.

Proof. Let $B \subseteq V_\kappa$ code φ, $(\mathbb{Q}^*_\alpha)_{\alpha \in (\beta+1) \cap Q}$, the collection of maximal antichains of $\mathbb{Q}^*_\alpha$, for $\alpha \in \beta \cap Q$, and $(\underset{\sim}{T}_\alpha)_{\alpha \in \mathcal{X} \cap \beta \cap Q}$. By a reflection argument with an appropriate Π^1_1 sentence over the structure $(V_\kappa, \in, B)$, together with the fact that $\mathbb{Q}^*_\alpha$ has the κ-c.c. for every $\alpha \in \beta \cap Q$, there is a set $D \in \mathcal{F}$ consisting of inaccessible cardinals $\lambda < \kappa$ for which M_λ is a model such that $M_\lambda \cap \kappa = \lambda$, M_λ is closed under $<\lambda$-sequences, and such that for every $\alpha \in M_\lambda \cap \beta$,

(1) $\mathbb{Q}^*_{\mu(\alpha)} \cap M_\lambda$ forces, over V, that $\underset{\sim}{T}_{\mu(\alpha)} \cap M_\lambda$ has no λ-branches,
(2) $\mathbb{Q}^*_\alpha \cap M_\lambda$ has the λ-c.c., and
(3) $\mathbb{Q}^*_\alpha \cap M_\lambda \lessdot \mathbb{Q}^*_\alpha$

Fix $\lambda \in Z \cap D$. Thanks to Lemma 3.7, it suffices to show that there are extensions p^2_λ and p^3_λ of $p^0_\lambda \restriction \alpha_\lambda$ and $p^1_\lambda \restriction \alpha'_\lambda$, respectively, such that $p^2_\lambda \restriction M_\lambda$ and $p^3_\lambda \restriction M_\lambda$ are compatible in $\mathbb{Q}^*_{\alpha_\lambda} \cap M_\lambda$, and such that x_λ and y_λ are separated below λ at stages $\mu(\alpha_\lambda)$ and $\mu(\alpha'_\lambda)$ by $p^2_\lambda \restriction \mu(\alpha_\lambda)$ and $p^3_\lambda \restriction \mu(\alpha'_\lambda)$. By (3) we may view $\mathbb{Q}^*_{\alpha_\lambda}$ as a two-step forcing iteration $(\mathbb{Q}^*_{\alpha_\lambda} \cap M_\lambda) * \underset{\sim}{\mathbb{S}}$. By λ-compatibility we may then identify $p^0_\lambda \restriction \alpha_\lambda$ and $p^1_\lambda \restriction \alpha_\lambda$ with, respectively, $\langle r^0, \underset{\sim}{s}^0 \rangle$ and $\langle r^1, \underset{\sim}{s}^1 \rangle$, both in $(\mathbb{Q}^*_{\alpha_\lambda} \cap M_\lambda) * \underset{\sim}{\mathbb{S}}$.

Note that r^0 and r^1 are compatible in $\mathbb{Q}^*_{\alpha_\lambda} \cap M_\lambda$. Working in an $(\mathbb{Q}^*_{\alpha_\lambda} \cap M_\lambda)$-generic extension $V[G]$ of V containing r^0 and r^1, we note that there have to be

- extensions $\langle r^{00}, \underset{\sim}{s}^{00} \rangle$ and $\langle r^{01}, \underset{\sim}{s}^{01} \rangle$ of $\langle r^0, \underset{\sim}{s}^0 \rangle$ and
- an extension $\langle r^3, \underset{\sim}{s}^3 \rangle$ of $\langle r^1, \underset{\sim}{s}^1 \rangle$

such that r^{00}, r^{01} and r^3 are all in G, together with some $\bar{\rho} < \lambda$ for which there is a pair $\zeta^{00} \neq \zeta^{01}$ of ordinals in ω_1 and there is $\zeta^3 \in \omega_1$

such that, identifying $\underset{\sim}{T}_{\mu(\alpha_\lambda)}$ and $\underset{\sim}{T}_{\mu(\alpha'_\lambda)}$ with $(\mathbb{Q}^*_{\alpha_\lambda} \cap M_\lambda) * \underset{\sim}{\mathbb{S}}$-names,[31] we have the following.

- $\langle r^{00}, \underset{\sim}{s}^{00}\rangle$ extends a condition in $A^{\mu(\alpha_\lambda)}_{x_\lambda,\bar\rho}$ forcing that $(\bar\rho, \zeta^{00})$ is below x_λ in $\underset{\sim}{T}_{\mu(\alpha_\lambda)}$.
- $\langle r^{01}, \underset{\sim}{s}^{01}\rangle$ extends a condition in $A^{\mu(\alpha_\lambda)}_{x_\lambda,\bar\rho}$ forcing that $(\bar\rho, \zeta^{01})$ is below x_λ in $\underset{\sim}{T}_{\mu(\alpha_\lambda)}$.
- $\langle r^3, \underset{\sim}{s}^3\rangle$ extends a condition in $A^{\mu(\alpha'_\lambda)}_{y_\lambda,\bar\rho}$ forcing that $(\bar\rho, \zeta^3)$ is below y_λ in $\underset{\sim}{T}_{\mu(\alpha'_\lambda)}$.

Indeed, any condition $\langle r, \underset{\sim}{s}\rangle$ in $(\mathbb{Q}^*_{\alpha_\lambda} \cap M_\lambda) * \underset{\sim}{\mathbb{S}}$ such that $r \in G$ can be extended, for any $\bar\rho < \lambda$, to a condition $\langle r^+, \underset{\sim}{s}^+\rangle$ such that

- $\langle r^+, \underset{\sim}{s}^+\rangle$ is stronger than some condition in $A^{\mu(\alpha_\lambda)}_{x_\lambda,\bar\rho}$ deciding some node $(\bar\rho, \zeta)$ to be below x_λ in $\underset{\sim}{T}_{\mu(\alpha_\lambda)}$, and
- $r^+ \in G$,

and similarly with y_λ and $\underset{\sim}{T}_{\mu(\alpha'_\lambda)}$ in place of x_λ and $\underset{\sim}{T}_{\mu(\alpha_\lambda)}$. Hence, if the above were to fail, then the following would hold.

- For every $\bar\rho < \lambda$ there is exactly one $\zeta < \omega_1$ for which there is some condition $\langle r, \underset{\sim}{s}\rangle$ in $(\mathbb{Q}^*_{\alpha_\lambda} \cap M_\lambda) * \underset{\sim}{\mathbb{S}}$ stronger than $\langle r^0, \underset{\sim}{s}^0\rangle$ with $r \in G$, and such that $\langle r, \underset{\sim}{s}\rangle$ extends a condition in $A^{\mu(\alpha_\lambda)}_{x_\lambda,\bar\rho}$ forcing that $(\bar\rho, \zeta)$ is below x_λ in $\underset{\sim}{T}_{\mu(\alpha_\lambda)}$.

It would then follow that $\underset{\sim}{T}_{\mu(\alpha_\lambda)}$ has a λ-branch in $V[G]$, which contradicts (1).

Let $\zeta^3 < \omega_1$ be such that some condition $\langle r^3, \underset{\sim}{s}^3\rangle$ extending $\langle r^1, \underset{\sim}{s}^1\rangle$ is such that

- $\langle r^3, \underset{\sim}{s}^3\rangle \restriction \mu(\alpha'_\lambda)$ extends a condition in $A^{\mu(\alpha'_\lambda)}_{y_\lambda,\bar\rho}$ forcing $(\bar\rho, \zeta^3)$ to be below y_λ in $\underset{\sim}{T}_{\mu(\alpha'_\lambda)}$, and
- $r^3 \in G$

But now, given conditions $\langle r^{0i}, \underset{\sim}{s}^{0i}\rangle$ as above (for $i \in \{0, 1\}$) there must be $i \in \{0, 1\}$ such that $\zeta^{0i} \neq \zeta^3$. We may then set $p^2_\lambda = \langle r^{0i}, \underset{\sim}{s}^{0i}\rangle$ and $p^3_\lambda = \langle r^3, \underset{\sim}{s}^3\rangle$. □

By Claim 4.7, in order to conclude the proof of the current instance of $(1)_\beta$, it suffices to prove the following.

Claim 4.11. *There is a separating pair for $\vec\sigma_0$ and $\vec\sigma_1$.*

Proof. This follows from first applying Claim 4.10 and $(2)_\alpha$, for $\alpha < \beta$, countably many times, using the normality of $\mathcal{F}$, and then running a pressing-down argument again using the normality of $\mathcal{F}$.

[31] $\underset{\sim}{T}_{\mu(\alpha'_\lambda)}$ is of course a $\mathbb{Q}^*_{\alpha_\lambda}$-name since $\mathbb{Q}^*_{\alpha'_\lambda} \subseteq \mathbb{Q}^*_{\alpha_\lambda}$, so this identification makes sense.

To be more specific, we start by building sequences

$$\vec{\sigma}_{0,n} = (q^0_{\lambda,n} \mid \lambda \in X \cap D_n)$$

and

$$\vec{\sigma}_{1,n} = (q^1_{\lambda,n} \mid \lambda \in X \cap D_n),$$

for a $\subseteq$-decreasing sequence $(D_n)_{n<\omega}$ of sets in $\mathcal{F}$, such that $\vec{\sigma}_{0,0} = \vec{\sigma}_0$ and $\vec{\sigma}_{1,0} = \vec{\sigma}_1$, and such that for every $n < \omega$, $\vec{\sigma}_{0,n+1}$ and $\vec{\sigma}_{1,n+1}$ are obtained from $\vec{\sigma}_{0,n}$ and $\vec{\sigma}_{1,n}$ in the following way.

We first let

$$\vec{\sigma}_{0,n,+} = (q^0_{\lambda,n,+} \mid \lambda \in X \cap D_n)$$

and

$$\vec{\sigma}_{1,n,+} = (q^1_{\lambda,n,+} \mid \lambda \in X \cap D_n)$$

be sequences of $\mathbb{Q}^*_\beta$-conditions such that for every $\lambda \in X \cap D_n$,

- $q^0_{\lambda,n,+} \leq_{\mathbb{Q}_\beta} q^0_{\lambda,n}$ and $q^0_{\lambda,n,+} \leq_{\mathbb{Q}_\beta} q^1_{\lambda,n}$,
- $q^0_{n,+} \restriction M_\lambda$ and $q^1_{n,+} \restriction M_\lambda$ are compatible conditions in $\mathbb{Q}^*_\beta \cap M_\lambda$, and
- $q^0_{\lambda,n,+}$ and $q^1_{\lambda,n,+}$ are both λ-compatible with respect to φ and α for every $\alpha < \beta$.

Recall that $e_\beta : \kappa \longrightarrow \beta$ is a surjection. Let also $D_{-1} = \kappa$. We may take D_n to be the diagonal intersection $\Delta_{\xi<\kappa} D^n_\xi$, where for each $\alpha < \kappa$, D^n_ξ witnesses $(2)_{e_\beta(\xi)}$ for $\vec{\sigma}_{0,n}$, $\vec{\sigma}_{1,n}$, φ, and D_{n-1}, i.e., $D^n_\xi \in \mathcal{F}$ is such that $D^n_\xi \subseteq D_{n-1}$ and such that for every $\lambda \in D^n_\xi$ and for all $q^{0'}_\lambda \leq_{\mathbb{Q}_{e_\beta(\xi)}} q^0_{\lambda,n}$ and $q^{1'}_\lambda \leq_{\mathbb{Q}_{e_\beta(\xi)}} q^1_{\lambda,n}$, if $q^{0'}_\lambda \restriction M_\lambda \in \mathbb{Q}_{e_\beta(\xi)}$ and $q^{0'}_\lambda \restriction M^\varphi_\lambda = q^{1'}_\lambda \restriction M_\lambda$, then there are conditions $r^0_\lambda \leq_{\mathbb{Q}_{e_\beta(\xi)}} q^{0'}_\lambda$ and $r^1_\lambda \leq_{\mathbb{Q}_{e_\beta(\xi)}} q^{1'}_\lambda$ such that

(1) $r^0_\lambda \restriction M_\lambda = r^1_\lambda \restriction M_\lambda$ and
(2) r^0_λ and r^1_λ are both λ-compatible with respect to φ and $e_\beta(\xi)$.

Given $\lambda \in X \cap D_n$, we need to construct $q^0_{\lambda,n,+}$ and $q^1_{\lambda,n,+}$. For this, let $W^\epsilon_{\lambda,n}$ be, for each $\epsilon \in \{0,1\}$, the set of ordinals $\alpha \in M_\lambda \cap \beta$ such that

- $\alpha \in \mathrm{dom}(f_{q^\epsilon_{\lambda,n}})$ or
- there is a connected $\tau_{q^\epsilon_{\lambda,n}}$-thread $\langle \vec{\mathcal{E}}, \alpha \rangle$ with $\Psi_{\vec{\mathcal{E}}}(\alpha) \in \mathrm{dom}(f_{q^\epsilon_{\lambda,n}})$ and such that $\vec{\mathcal{E}}$ consists of edges.

We of course have that $|W^\epsilon_{\lambda,n}| \leq \aleph_0$. We may assume that both $W^0_{\lambda,n}$ and $W^0_{\lambda,n}$ are nonempty (the proof in the case when at least one of $W^0_{\lambda,n}$ and $W^0_{\lambda,n}$ is empty is an easier variation of the proof in the other case). Let $\epsilon \in \{0,1\}$ be such that $\sup(W^\epsilon_{\lambda,n}) = \max\{\sup(W^0_{\lambda,n}), \sup(W^1_{\lambda,n})\}$. We will assume that $\sup(W^\epsilon_{\lambda,n})$ has countable cofinality (the proof when $W^\epsilon_{\lambda,n}$ is empty or has a maximum is an easier variant of the proof in the case that $\mathrm{cf}(\sup(W^{\epsilon}_{\lambda,n})) = \omega$). Let $(\beta_m)_{m<\omega}$ be a strictly increasing sequence of ordinals in M_λ converging to $\sup(W^{\epsilon}_{\lambda,n})$. We construct $q^0_{\lambda,n,+}$ and $q^0_{\lambda,n,+}$ as the greatest lower bound of $(q^0_{\lambda,n,m})_{m<\omega}$ and $(q^1_{\lambda,n,m})_{m<\omega}$, respectively, where $q^0_{\lambda,n,0} = q^0_{\lambda,n}$ and $q^1_{\lambda,n,0} = q^1_{\lambda,n}$ and where, for each $m < \omega$, $q^0_{\lambda,n,m+1} = r^0_{\lambda,n,m} \oplus q^0_{\lambda,n,m}$ and $q^1_{\lambda,n,m+1} = r^1_{\lambda,n,m} \oplus q^1_{\lambda,n,m}$, where $r^0_{\lambda,n,m}$, $r^1_{\lambda,n,m} \in \mathbb{Q}_{\beta_m}$ are conditions extending $q^0_{\lambda,n,m}$ and $q^1_{\lambda,n,m}$, respectively, and such that

(1) $r^0_{\lambda,n,m} \restriction M_\lambda = r^1_{\lambda,n,m} \restriction M_\lambda$ and
(2) $r^0_{\lambda,n,m}$ and $r^1_{\lambda,n,m}$ are both λ-compatible with respect to φ and β_m.

We may assume each $r^0_{\lambda,n,m}$ and $r^1_{\lambda,n,m}$ to be in Q, so that $q^0_{\lambda,n,+}$ and $q^1_{\lambda,n,+}$ are both in $\mathbb{Q}^*_\beta$.

Now we find D_{n+1} and $\vec{\sigma}_{0,n+1}$, $\vec{\sigma}_{1,n+1}$ by an application of Claim 4.10 to $\vec{\sigma}_{0,n,+}$ and $\vec{\sigma}_{1,n,+}$ with an appropriate sequence $\alpha_{\lambda,n}$, $\alpha'_{\lambda,n}$, $x_{\lambda,n}$, $y_{\lambda,n}$ (for $\lambda \in X \cap D_n$). By extending $q^0_{\lambda,n+1}$ and $q^1_{\lambda,n+1}$ if necessary for $\lambda \in X \cap D_{n+1}$ we may assume that for every such λ, $q^0_{\lambda,n+1}$ and $q^1_{\lambda,n+1}$ are both adequate conditions, and $\mathrm{dom}(f_{q^0_{\lambda,n+1}}) = \mathrm{dom}(f_{q^1_{\lambda,n+1}})$.

Let r^0_λ and r^1_λ be the greatest lower bound of, respectively, $(q^0_{\lambda,n})_{n<\omega}$ and $(q^1_{\lambda,n})_{n<\omega}$, for $\lambda \in X \cap \bigcap_n D_n$.

By construction we have that for all $\lambda \in X \cap \bigcap_n D_n$, r^0_λ and r^1_λ are both adequate conditions, and $\mathrm{dom}(f_{r^0_\lambda}) = \mathrm{dom}(f_{r^1_\lambda})$. Also, by a standard book-keeping argument we can ensure that all relevant objects $\alpha_{\lambda,n}$, $\alpha'_{\lambda,n}$, $x_{\lambda,n}$, $y_{\lambda,n}$ (for $n < \omega$ and $\lambda \in X \cap D_n$) have been chosen in such a way that in the end

$$(r^0_\lambda \mid \lambda \in X \cap \bigcap_n D_n)$$

and

$$(r^1_\lambda \mid \lambda \in X \cap \bigcap_n D_n)$$

satisfy clause (2) in Definition 4.6 as well. Finally, by a standard pressing-down argument using the normality of $\mathcal{F}$, we may find $Y \in \mathcal{S}$, $Y \subseteq X \cap \bigcap_n D_n$, such that $\vec{\sigma}^*_0 = (r^0_\lambda \mid \lambda \in Y)$ and $\vec{\sigma}^*_1 = (r^1_\lambda \mid \lambda \in Y)$ satisfy clauses (3) and (4) in Definition 4.6. □

We are left with proving $(2)_\beta$. This is established with an argument similar to the one in the corresponding proof from [10]. Suppose $D \in \mathcal{F}$, Q is a suitable model such that $\beta, D \in Q$, $\varphi : \kappa \to Q$ is a bijection, and $(q^0_\lambda \mid \lambda \in D) \in Q$ and $(q^1_\lambda \mid \lambda \in D) \in Q$ are sequences of $\mathbb{Q}_\beta$-conditions. By shrinking D if necessary we may assume that $M_\lambda \cap \kappa = \lambda$ for each $\lambda \in D$. It suffices to show that there is some $D' \in \mathcal{F}$, $D' \subseteq D$, with the property that for every $\lambda \in D'$, if $q^{0'}_\lambda \leq_\beta q^0_\lambda$ and $q^{1'}_\lambda \leq_\beta q^1_\lambda$ are such that $q^{0'}_\lambda \restriction M_\lambda \in \mathbb{Q}_\beta$ and $q^{0'}_\lambda \restriction M_\lambda = q^{1'}_\lambda \restriction M_\lambda$, then there is a condition $r_\lambda \leq_{\mathbb{Q}_\beta} q^{0'}_\lambda \restriction M_\lambda$ such that every condition in $\mathbb{Q}_\beta \cap M_\lambda$ extending r_λ is compatible with both $q^{0'}_\lambda$ and $q^{1'}_\lambda$.

The case when β is a limit ordinal follows from the induction hypothesis, using the normality of $\mathcal{F}$ (cf. the proof in [10]). Specifically, we fix an increasing sequence $(\beta_i)_{i<\mathrm{cf}(\beta)}$ of ordinals in Q converging to β. If $\mathrm{cf}(\beta) = \kappa$, we take each β_i to be $\sup(M_\lambda \cap \beta)$ for some $\lambda \in D$. For each $i < \mathrm{cf}(\beta)$ we fix some $D_i \in \mathcal{F}$, $D_i \subseteq D$, witnessing $(2)_{\beta_i}$ for $(q^0_\lambda \restriction \beta_i \mid \lambda \in D)$ and $(q^1_\lambda \restriction \beta_i \mid \lambda \in D)$. We make sure that $(D_i)_{i<\mathrm{cf}(\beta)}$ is $\subseteq$-decreasing. If $\mathrm{cf}(\beta) < \kappa$, then $D' = \bigcap_{i<\mathrm{cf}(\beta)} D_i$ will witness $(2)_\beta$ for $(q^0_\lambda \mid \lambda \in D)$ and $(q^1_\lambda \mid \lambda \in D)$, and if $\mathrm{cf}(\beta) = \kappa$, $D' = \Delta_{i<\kappa} D_i$ will witness $(2)_\beta$ for these objects. This can be easily shown, using the fact that each M_λ is closed under ω-sequences in the case when $\mathrm{cf}(\beta) = \omega$.

To see this, suppose $\lambda \in D'$, $q^{0'}_\lambda \leq_{\mathbb{Q}_\beta} q^0_\lambda$, $q^{1'}_\lambda \leq_{\mathbb{Q}_\beta} q^1_\lambda$, $q^{0'}_\lambda \restriction M_\lambda \in \mathbb{Q}_\beta$, and $q^{0'}_\lambda \restriction M_\lambda = q^{1'}_\lambda \restriction M_\lambda$. Suppose first that $\mathrm{cf}(\beta) > \omega$. In this case, we pick any $i \in \mathrm{cf}(\beta) \cap M_\lambda$ such that β_i is

- above $(\mathrm{dom}(f_{q^{0'}_\lambda}) \cup \mathrm{dom}(f_{q^{1'}_\lambda})) \cap \sup(M_\lambda \cap \beta)$ and
- above every ordinal $\alpha \in M_\lambda \cap \beta$ such that $\Psi_{\vec{\mathcal{E}}}(\alpha) \in \mathrm{dom}(f_{q^{0'}_\lambda}) \cup \mathrm{dom}(f_{q^{1'}_\lambda})$, for some connected $\tau_{q^{0'}_\lambda} \cup \tau_{q^{1'}_\lambda}$-thread $\langle \vec{\mathcal{E}}, \alpha \rangle$ such that $\vec{\mathcal{E}}$ consists of edges,[32]

and find a condition $r \in M_\lambda \cap \mathbb{Q}_{\beta_i}$ with the property that every condition in $\mathbb{Q}_{\beta_i} \cap M_\lambda$ is compatible with both $q^{0'}_\lambda \restriction \beta_i$ and $q^{1'}_\lambda \restriction \beta_i$. Let r_λ be any condition in $\mathbb{Q}_\beta \cap M_\lambda$ extending r and $q^{0'}_\lambda \restriction M_\lambda$. It then follows that every condition in $\mathbb{Q}_\beta \cap M_\lambda$ extending r_λ is compatible with both $q^{0'}_\lambda$ and $q^{1'}_\lambda$.

Now suppose β has countable cofinality. Since $\beta \in M_\lambda$, we may build sequences $(q^{0,i}_\lambda \mid i < \omega)$, $(q^{1,i}_\lambda \mid i < \omega)$ and $(r^i_\lambda \mid i < \omega)$ such that for each i,

(1) $q^{0,i}_\lambda$ and $q^{1,i}_\lambda$ are conditions in $\mathbb{Q}_{\beta_i}$ extending $q^{0'}_\lambda \restriction \beta_i$ and $q^{1'}_\lambda \restriction \beta_i$, respectively,
(2) $r^i_\lambda \in \mathbb{Q}_{\beta_i} \cap M_\lambda$,
(3) every condition in $\mathbb{Q}_{\beta_i} \cap M_\lambda$ extending r^i_λ is compatible with both $q^{0,i}_\lambda$ and $q^{1,i}_\lambda$,
(4) $q^{0,i+1}_\lambda \restriction \beta_i$ extends $q^{0,i}_\lambda$ and r^i_λ,
(5) $q^{1,i+1}_\lambda \restriction \beta_i$ extends $q^{1,i}_\lambda$ and r^i_λ, and
(6) $r^{i+1}_\lambda \restriction \beta_i$ extends r^i_λ.

[32]Cf. the proof of Claim 4.11.

Let $r_\lambda \in \mathbb{Q}_\beta$ be the greatest lower bound of $\{r^i_\lambda \mid i < \omega\}$, and note that $r_\lambda \in M_\lambda$ since M_λ is closed under sequences of length ω. But now it is straightforward to verify that every condition in $\mathbb{Q}_\beta \cap M_\lambda$ extending r_λ is compatible with $q^{0'}_\lambda$ and $q^{1'}_\lambda$.

It remains to consider the case that β is a successor ordinal, $\beta = \beta_0 + 1$. Assuming the desired conclusion fails, there is some $X \in \mathcal{S}$, $X \subseteq D$, together with sequences $(q^{0'}_\lambda \mid \lambda \in X)$ and $(q^{0'}_\lambda \mid \lambda \in X)$ of conditions in $\mathbb{Q}_\beta$ such that for every $\lambda \in X$,

- $q^{0'}_\lambda$ extends q^0_λ and $q^{1'}_\lambda$ extends q^1_λ,
- $q^{0'}_\lambda \restriction M_\lambda \in \mathbb{Q}_\beta$,
- $q^{0'}_\lambda \restriction M_\lambda = q^{1'}_\lambda \restriction M_\lambda$, and
- for every condition r in $\mathbb{Q}_\beta \cap M_\lambda$ extending $q^{0'}_\lambda \restriction M_\lambda$ there is a condition in $\mathbb{Q}_\beta \cap M_\lambda$ extending r and incompatible with at least one of $q^{0'}_\lambda$, $q^{1'}_\lambda$.

Thanks to the induction hypothesis applied to β_0 and to the fact that $(1)_\beta$ holds we may assume, after shrinking X to some $Y \in \mathcal{S}$ and extending the corresponding conditions if necessary, that for each $\lambda \in Y$,

- $q^{0'}_\lambda \restriction \beta_0$ and $q^{1'}_\lambda \restriction \beta_0$ are both λ-compatible with respect to φ and β_0, and
- $q^{0'}_\lambda \oplus q^{1'}_{\lambda^*}$ is a condition for each $\lambda^* \in Y$, $\lambda^* > \lambda$.

By our assumption above we may then assume, after shrinking Y if necessary, that for each $\lambda \in Y$ there is a maximal antichain A_λ of $\mathbb{Q}_\beta \cap M_\lambda$ below $q^0_\lambda \restriction M_\lambda$ consisting of conditions r such that at least one of the following statements holds.

$\theta_{r,0,\lambda}$: r is incompatible with $q^{0'}_\lambda$.
$\theta_{r,1,\lambda}$: r is incompatible with $q^{1'}_\lambda$.

By the definition of $\mathcal{F}$ coupled with an appropriate Π^1_1-reflection argument, we may further assume that each A_λ is in fact a maximal antichain of $\mathbb{Q}_\beta$ below $q^{0'}_\lambda \restriction M_\lambda$ and that it has cardinality less than λ (cf. the proof of Claim 4.10). Hence, after shrinking Y one more time using the normality of $\mathcal{F}$, we may assume, for all $\lambda < \lambda^*$ in Y, that

- $A_\lambda = A_{\lambda^*}$ and that
- for every $r \in A_\lambda$, $\theta_{r,0,\lambda}$ holds if and only if $\theta_{r,0,\lambda^*}$ does, and $\theta_{r,1,\lambda}$ holds if and only if $\theta_{r,1,\lambda^*}$ does.

Let us now fix any $\lambda < \lambda^*$ in Y. Since A_λ is a maximal antichain of $\mathbb{Q}_\beta$ below $q^{0'}_\lambda \restriction M_\lambda$, we may find some $r \in A_\lambda$ compatible with $q^{0'}_\lambda \oplus q^{1'}_{\lambda^*}$. We have that $\theta_{r,0,\lambda}$ cannot hold since $q^{0'}_\lambda \oplus q^{1'}_{\lambda^*}$ extends $q^{0'}_\lambda$. Therefore $\theta_{r,1,\lambda}$ holds, and hence also $\theta_{r,1,\lambda^*}$ does. But that is also a contradiction since $q^{0'}_\lambda \oplus q^{1'}_{\lambda^*}$ extends $q^{1'}_{\lambda^*}$.

This contradiction concludes the proof of $(2)_\beta$, and hence the proof of the lemma. $\square$

It may be worthwhile observing that, as opposed to what is usually the case in forcing constructions incorporating models as side conditions, our use of side conditions does not interfere with the κ-chain condition. The underlying reason is of course the fact that our pure side condition forcing is trivial (Lemma 2.9).

5. Completing the proof of Theorem 1.2

In this final section we conclude the proof of Theorem 1.2. By Lemma 3.10, $\mathbb{Q}_{\kappa^+}$ does not add new ω-sequences of ordinals and hence it preserves CH. We will start this section by proving that $\mathbb{Q}_{\kappa^+}$ also preserves $2^{\aleph_1} = \aleph_2$. Of course, the reason we have incorporated edges in our construction is precisely to make this proof work.

Lemma 5.1. $\Vdash_{\mathbb{Q}_{\kappa^+}} 2^{\aleph_1} = \kappa$

Proof. Suppose, towards a contradiction, that there is a condition $q \in \mathbb{Q}_{\kappa^+}$ and a sequence $(\underset{\sim}{r}_i)_{i<\kappa^+}$ of $\mathbb{Q}_{\kappa^+}$-names for subsets of ω_1 such that

$$q \Vdash_{\mathbb{Q}_{\kappa^+}} \underset{\sim}{r}_i \neq \underset{\sim}{r}_{i'} \text{ for all } i < i' < \kappa^+$$

By Lemma 4.1 we may assume, for each i, that $\underset{\sim}{r}_i \in H(\kappa^+)$ and $\underset{\sim}{r}_i$ is a $\mathbb{Q}_{\beta_i}$-name for some $\beta_i < \kappa^+$.

Let θ be a large enough regular cardinal. For each $i < \kappa^+$ let $N_i^* \preceq H(\theta)$ be such that

1. (1) $|N_i^*| = |N_i^* \cap \kappa|$,
2. (2) N_i^* is closed under sequences of length less than $|N_i^*|$,
3. (3) q, $\underset{\sim}{r}_i$, β_i, $(\Phi_\alpha)_{\alpha<\kappa^+}$, $(\mathbb{Q}_\alpha)_{\alpha<\kappa^+} \in N_i^*$, and
4. (4) $\mathbb{Q}_\alpha \cap N_i^* \lessdot \mathbb{Q}_\alpha$ for every $\alpha \in \kappa^+ \cap N_i^*$.

N_i^* can be found by a Π^1_1-reflection argument, using the weak compactness of κ and the κ-chain condition of each $\mathbb{Q}_\alpha$, as in the proof of Claim 4.10. Let $N_i = N_i^* \cap H(\kappa^+)$ for each i.

Let now P be the satisfaction predicate for the structure

$$\langle H(\kappa^+), \in, \vec{\Phi}\rangle,$$

where $\vec{\Phi} \subseteq H(\kappa^+)$ codes $(\Phi_\alpha)_{\alpha<\kappa^+}$ in some canonical way, and let M be an elementary submodel of $H(\theta)$ containing q, $\underset{\sim}{r}_i$, $(\beta_i)_{i<\kappa^+}$, $(\mathbb{Q}_\alpha)_{\alpha\leq\kappa^+}$, $(N_i^*)_{i<\kappa^+}$ and P, and such that $|M| = \kappa$ and ${}^{<\kappa}M \subseteq M$.

Let $i_0 \in \kappa^+ \setminus M$. By a standard reflection argument we may find $i_1 \in \kappa^+ \cap M$ for which there exists an isomorphism

$$\Psi : (N_{i_0}, \in, P, \underset{\sim}{r}_{i_0}, \beta_{i_0}, q) \cong (N_{i_1}, \in, P, \underset{\sim}{r}_{i_1}, \beta_{i_1}, q),$$

such that $\Psi(\xi) \leq \xi$ for every ordinal $\xi \in N_{i_0}$. Indeed, the existence of such an i_1 follows from the correctness of M in $H(\theta)$ about an appropriate statement with parameters $(N_i)_{i<\kappa^+}$, q, P, $(\beta_i)_{i<\kappa^+}$, $(\underset{\sim}{r}_i)_{i<\kappa^+}$, $N_{i_0} \cap M$, and the isomorphism type of the structure

$$(N_{i_0}, \in, P, \underset{\sim}{r}_{i_0}, \beta_{i_0}, q),$$

all of which are in M.

Let $\bar{q} = (f_q, \tau_{\bar{q}})$, where

$$\tau_{\bar{q}} = \tau_q \cup \{\langle (N_{i_0}, \beta_{i_0} + 1), (N_{i_1}, \beta_{i_1} + 1)\rangle\}.$$

It follows, using Lemma 3.20, that $\bar{q} \in \mathbb{Q}_{\kappa^+}$. We now show that $\bar{q} \Vdash_{\mathbb{Q}_{\kappa^+}} \underset{\sim}{r}_{i_0} = \underset{\sim}{r}_{i_1}$.

Suppose not, and we will derive a contradiction. Thus we can find $\nu < \omega_1$ and $q' \leq_{\kappa^+} \bar{q}$ such that

$$q' \Vdash_{\mathbb{Q}_{\kappa^+}} \text{“}\nu \in \underset{\sim}{r}_{i_0} \iff \nu \notin \underset{\sim}{r}_{i_1}\text{”}.$$

Let us assume, for concreteness, that $q' \Vdash_{\mathbb{Q}_{\kappa^+}}$ “$\nu \in \underset{\sim}{r}_{i_0}$ and $\nu \notin \underset{\sim}{r}_{i_1}$” (the proof in the case that $q' \Vdash_{\mathbb{Q}_{\kappa^+}}$ “$\nu \in \underset{\sim}{r}_{i_1}$ and $\nu \notin \underset{\sim}{r}_{i_0}$” is exactly the same). By correctness of $N^*_{i_0}$ we have that this model contains a maximal antichain A of conditions in $\mathbb{Q}_{\beta_{i_0}}$ deciding the statement “$\nu \in \underset{\sim}{r}_{i_0}$”. By Lemma 4.1 we know that $|A| < \kappa$ and hence, since $N^*_{i_0} \cap \kappa \in \kappa$, $A \subseteq N^*_{i_0} \cap H(\kappa^+) = N_{i_0}$ (cf. the proof of Lemma 3.12). Hence, we may find a common extension q'' of q' and some $r \in N_{i_0} \cap A$ such that $r \Vdash_{\mathbb{Q}_{\kappa^+}}$ “$\nu \in \underset{\sim}{r}_{i_0}$”.

Also, note that, since Ψ is an isomorphism between the structures $(N_{i_0}, \in, P, \underset{\sim}{r}_{i_0}, \beta_{i_0}, q)$ and $(N_{i_1}, \in, P, \underset{\sim}{r}_{i_1}, \beta_{i_1}, q)$, and by the choice of P, we have that

$$\Psi(r) \Vdash_{\mathbb{Q}_{\beta_{i_1}}} \text{“}\nu \in \Psi(\underset{\sim}{r}_{i_0}) = \underset{\sim}{r}_{i_1}\text{”}$$

But then, by clauses (5) and (6) in the definition of condition, we have that $q'' \leq \Psi(r)$. We thus obtain that $q'' \Vdash_{\mathbb{Q}_{\kappa^+}}$ “$\nu \in \underset{\sim}{r}_{i_1}$”, which is impossible as $q' \Vdash_{\mathbb{Q}_{\kappa^+}}$ “$\nu \notin \underset{\sim}{r}_{i_1}$” and $q'' \leq q'$.

We get a contradiction and the lemma follows.[33] □

Corollary 5.2. $\mathbb{Q}_{\kappa^+}$ *forces GCH.*

Lemma 5.3, which completes the proof of Theorem 1.2, follows immediately from earlier lemmas, together with a standard density argument.

Lemma 5.3. $\mathbb{Q}_{\kappa^+}$ *forces* $\mathrm{SATP}_{\aleph_2}$.

[33]Note the resemblance of this proof with the proof of Lemma 3.12.

Proof. Let G be $\mathbb{Q}_{\kappa^+}$-generic over V. Since CH holds in $V[G]$, there are $\aleph_2$-Aronszajn trees there. Hence, it suffices to prove that, in $V[G]$, every $\aleph_2$-Aronszajn tree is special.

Let $T \in V[G]$ be an $\aleph_2$-Aronszajn tree. Note that $\aleph_2 = \kappa$ in $V[G]$ by Lemmas 3.5 and 4.1. We need to prove that T is special in $V[G]$. Let us go down to V and let us note there that, by the κ-chain condition of $\mathbb{Q}_{\kappa^+}$ together with the choice of Φ, we may find some nonzero $\alpha \in \mathcal{X}$ such that $\Phi(\alpha)$ is a $\mathbb{Q}_\alpha$-name for an $\aleph_2$-Aronszajn tree such that $\Phi(\alpha)_G = T$. We then have that $\underset{\sim}{T}_\alpha = \Phi(\alpha)$.

For every $\nu < \omega_1$, let $A_\nu = \bigcup\{f_q(\alpha+\nu) \mid q \in G\}$. By the definition of the forcing, we have that A_ν is an antichain of T. Also, given any condition $q \in \mathbb{Q}_{\kappa^+}$ and any node $x \in \kappa \times \omega_1$ such that $x \notin f_q(\alpha+\nu)$ for any $\nu < \omega_1$, it is easy to see that we may extend q to a condition q^* such that $x \in f_{q^*}(\alpha+\nu)$ for some $\nu < \omega_1$; indeed, it suffices for this to pick any $\nu < \omega_1$ such that $\alpha' + \nu \notin \mathrm{dom}(f_q)$ for any $\alpha' \in \mathcal{X}$, which of course is possible since $\mathrm{dom}(f_q)$ is countable, extend f_q to a function f such that $\alpha+\nu \in \mathrm{dom}(f)$ and $f(\alpha+\nu) = \{x\}$, and close under the relevant (restrictions of) functions Ψ_{N_0,N_1} for edges $\langle (N_0,\gamma_0),(N_1,\gamma_1)\rangle \in \tau_q$.[34] The above density argument shows that every node in T is in some A_ν. It follows that T is special in $V[G]$, which concludes the proof. □

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References

[1] Asperó, David; Mota, Miguel Angel, *Forcing consequences of PFA together with the continuum large*, Trans. Amer. Math. Soc., vol. 367 (2015), no. 9, 6103–6129.

[34] In other words, we may take $q^* = q \oplus q'$, for $q' = (f, \emptyset)$, where $\mathrm{dom}(f) = \{\alpha+\nu\}$ and $f_q(\alpha+\nu) = \{x\}$.

[2] Asperó, David; Mota, Miguel Angel, *Few new reals*, to appear in Journal of Math. Logic (2024).
[3] Baumgartner, J.; Malitz, J.; Reinhardt, W., *Embedding trees in the rationals*, Proc. Nat. Acad. Sci. U.S.A., vol. 67 (1970), 1748–1753.
[4] Devlin, Keith J.; Johnsbraten, Havard, *The Souslin problem*, Lecture Notes in Mathematics, vol. 405, Springer–Verlag, Berlin, New York, 1974. viii+132 pp.
[5] Golshani, Mohammad; Hayut, Yair, *The Special Aronszajn tree property*, J. Math. Logic., vol. 20 (2020), no. 1, 2050003.
[6] Gregory, John, *Higher Souslin trees and the generalized continuum hypothesis*, J. Symbolic Logic, vol. 41 (1976), no. 3, 663–671.
[7] Jensen, Ronald, *The fine structure of the constructible hierarchy. With a section by Jack Silver*, Ann. Math. Logic, vol. 4 (1972), 229–308; erratum, ibid. 4 (1972), 443.
[8] A. Kanamori; M. Magidor, *The evolution of large cardinal axioms in set theory*, in Higher set theory (Proc. Conf. Math. Forschungsinst., Oberwolfach, 1977), Lecture Notes in Mathematics, 669, Springer, 9–275.
[9] Kurepa, Djuro, *Ensembles ordonnés et ramifiés. Thèse, Paris.* Published as: Publications mathematiques de l'Université de Belgrade, vol. 4 (1935). 1–138.
[10] Laver, Richard; Shelah, Saharon, *The $\aleph_2$-Souslin hypothesis*, Trans. Amer. Math. Soc., vol. 264 (1981), no. 2, 411–417.
[11] Rinot, Assaf, *Jensen's diamond principle and its relatives*, in Set theory and its applications, 125–156, Contemp. Math., 533, Amer. Math. Soc., Providence, RI, 2011.
[12] Rinot, Assaf, *Higher Souslin trees and the GCH, revisited*, Adv. Math., vol. 311 (2017), 510–531.
[13] Shelah, Saharon, *Proper and improper forcing. Second edition*, Perspectives in Mathematical Logic, Springer-Verlag, Berlin, 1998. xlviii+1020 pp.
[14] Shelah, Saharon, *On what I do not understand (and have something to say). I*, in Saharon Shelah's anniversary issue, Fund. Math. 166 (2000), no. 1–2, 1–82.
[15] Shelah, Saharon; Stanley, Lee, *Weakly compact cardinals and nonspecial Aronszajn trees*, Proc. of the Amer. Math. Soc., vol. 104 (1988), no. 3, 887–897.
[16] R. Solovay; S. Tennenbaum, *Iterated Cohen extensions and Souslins problem*, Ann. of Math. (2) 94 (1971), 201-245.
[17] E. Specker, *Sur un problème de Sikorski*, Colloquium Math. 2 (1949), 9–12.
[18] S. Todorčević, *Trees and linearly ordered sets*, in Handbook of set-theoretic topology, pp. 235–293, North-Holland, Amsterdam, 1984.
[19] S. Todorčević, *Partitioning pairs of countable ordinals*, Acta Math., vol. 159(3-4):261–294, 1987.

David Asperó, School of Mathematics, University of East Anglia, Norwich NR4 7TJ, UK
Email address: `d.aspero@uea.ac.uk`

Mohammad Golshani, School of Mathematics, Institute for Research in Fundamental Sciences (IPM), P.O. Box: 19395–5746, Tehran, Iran.
Email address: `golshani.m@gmail.com`