THE PROPER FORCING AXIOM FOR $\aleph_1$-SIZED POSETS, PŘÍKRÝ-TYPE PROPER FORCING, AND THE SIZE OF THE CONTINUUM

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Abstract. We show that the Proper Forcing Axiom for forcing notions of size $\aleph_1$ is consistent with the continuum being arbitrarily large. In fact, assuming GCH holds and $\kappa \geq \omega_2$ is a regular cardinal, we prove that there is a proper and $\aleph_2$-c.c. forcing $\mathbb{P}$ giving rise to a model of this forcing axiom together with $2^{\aleph_0} = \kappa$ and which, in addition, satisfies all statements of the form $\mathscr{H}(\aleph_2) \models \forall x \exists y \varphi(x, y)$, where $\varphi(x, y)$ is a restricted formula with the property that for every $a \in \mathscr{H}(\aleph_2)$ and every inner model M of CH with $a \in M$ there is, in M, a suitably nice proper poset adding some b such that $\varphi(a, b)$. In particular, $\mathbb{P}$ forces Moore's Measuring principle, Baumgartner's Axiom for $\aleph_1$-dense sets of reals, Todorčević's Open Colouring Axiom for sets of size $\aleph_1$, and Todorěvić's P-ideal Dichotomy for $\aleph_1$-generated ideals on ω_1, among other statements. In particular, all these statements are simultaneously compatible with a large continuum.

§ 0. Introduction

Forcing axioms can be considered as generalizations of the Baire category theorem and spell out one version of the idea that the universe of sets should be rich. The first example of a forcing axiom, introduced by Martin (see [25]), is known as Martin's Axiom. In this paper we concentrate on a generalization of Martin's Axiom known as the Proper Forcing Axiom (PFA). PFA was introduced by Baumgartner [8] (and Shelah [21]) and states that given a proper forcing notion $\mathbb{P}$ and a collection $\mathcal{D}$ of $\aleph_1$-many dense subsets

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of $\mathbb{P}$ there exists a filter $G \subseteq \mathbb{P}$ meeting all the members of $\mathcal{D}$. PFA is consistent modulo the existence of a supercompact cardinal and has many consequences for the structure of the universe; in particular, by the work of Todorčević and Veličković (see for example [9] and [29]), it implies that the size of the continuum is $\aleph_2$.

Throughout this paper let us denote by $\mathsf{PFA}(\aleph_1)$ the restriction of PFA to posets of cardinality at most $\aleph_1$; i.e., $\mathsf{PFA}(\aleph_1)$ is the statement that if $\mathbb{P}$ is a proper forcing notion such that $|\mathbb{P}| \leq \aleph_1$ and $\mathcal{D}$ is a collection of $\aleph_1$-many dense subsets of $\mathbb{P}$, then there is a filter $G \subseteq \mathbb{P}$ meeting all members of $\mathcal{D}$.

$\mathsf{PFA}(\aleph_1)$ clearly implies Martin's Axiom for collection of $\aleph_1$-many dense sets (this is usually denoted by $\mathsf{MA}_{\aleph_1}$). It properly extends $\mathsf{MA}_{\aleph_1}$ as various non-c.c.c. proper forcings of size $\aleph_1$ fall in its range—for example Baumgartner's forcing for adding a club of ω_1 with finite conditions, or natural forcings for adding, by finite conditions, various instances of the negation of Club Guessing at ω_1. Thus, it is easy to see, for example, that $\mathsf{PFA}(\aleph_1)$ implies $\neg\,\mathsf{WCG}$, where WCG denotes weak Club Guessing.[1] It is also worth mentioning that $\mathsf{PFA}(\aleph_1)$ implies that every two normal Aronszajn trees T and U are club-isomorphic, i.e., there is a club $C \subseteq \omega_1$ such that the subtrees $T \restriction C = \bigcup_{\alpha\in C}\{t \in T \,:\, \mathrm{ht}_T(t) = \alpha\}$ and $U \restriction C = \bigcup_{\alpha\in C}\{u \in U \,:\, \mathrm{ht}_U(u) = \alpha\}$ are isomorphic (s. [26], Theorem 5.10).

Shelah [21] showed that the consistency of $\mathsf{PFA}(\aleph_1)$ does not need any large cardinal hypotheses. In fact, starting with a model of GCH, one can easily force $\mathsf{PFA}(\aleph_1)$ by means of a suitable countable support iteration of proper forcings of size $\aleph_1$. In this model $2^{\aleph_0} = \aleph_2$ holds. The question whether this forcing axiom decides the value of the continuum remained an open problem.

[1] Weak Club Guessing is the statement that there is a ladder system $\langle C_\delta \,:\, \delta \in \mathrm{Lim}(\omega_1)\rangle$ (i.e., each C_δ is a cofinal subset of δ of order type ω) such that every club of ω_1 has infinite intersection with some C_δ. It is easy to see that $\mathsf{MA}_{\aleph_1}$ is compatible with WCG since WCG is preserved by c.c.c. forcing.

There is a wide range of works showing the consistency of some forcing axioms or their consequences with the continuum being larger than $\aleph_2$; see for example [3], [4], [10], [12], [13], [14], [18], [19] and [20]. For instance, and most to the point for us in the present paper, it is shown in [3] using forcing with side conditions that PFA restricted to certain classes of posets with the $\aleph_2$-chain condition is consistent with $2^{\aleph_0} > \aleph_2$.[2]

In this paper we prove that $\mathsf{PFA}(\aleph_1)$ is consistent with the continuum being arbitrary large, thereby answering the above question. Given a class Γ of forcing notions and a cardinal λ, the forcing axiom $\mathsf{FA}(\Gamma)_\lambda$ is the statement that for every $\mathbb{B} \in \Gamma$ and every collection $\{D_i : i < \lambda\}$ of dense subsets of $\mathbb{P}$ there is a filter $G \subseteq \mathbb{P}$ such that $G \cap D_i \neq \emptyset$ for all $i < \lambda$. In fact, we will prove the consistency of $2^{\aleph_0}$ being an arbitrarily fixed regular cardinal κ together with $\mathsf{PFA}(\aleph_1)_{<\kappa}$, where $\mathsf{PFA}(\aleph_1)_{<\kappa}$ is $\mathsf{FA}(\{\mathbb{P} : \mathbb{P} \text{ proper}, |\mathbb{P}| = \aleph_1\})_\lambda$ for all $\lambda < \kappa$.

Our first main theorem is the following.

Theorem 0.1. *Assume GCH. Let $\kappa \geq \aleph_2$ be a regular cardinal. Then there is a proper partial order $\mathbb{P}$ with the $\aleph_2$-chain condition and forcing the following statements.*

(1) $2^{\aleph_0} = \kappa$
(2) $\mathit{PFA}(\aleph_1)_{<\kappa}$

The forcing witnessing Theorem 0.1 is a finite-support iteration with systems of models with markers as side conditions, in the style of the constructions in [3], [4] or [5]. However, we were also inspired by Shelah's memory iteration technique (s. for example [23], [24], [17], [16] and [13]).

[2]Some form of side condition is usually necessary in the constructions we are referring to. At any rate, and as is well-known, any countable support iteration $\langle \mathbb{P}_\alpha : \alpha \leq \lambda \rangle$ of nontrivial forcing notions will collapse $(2^{\aleph_0})^{\mathbf{V}^{\mathbb{P}_\alpha}}$ to $\aleph_1$ for every stage α with $\alpha + \omega_1 \leq \lambda$, which renders this method useless in the construction of models of forcing axioms with large continuum.

Familiarity with proper forcing should be enough to follow the paper. Some familiarity with the method of forcing with symmetric systems, as presented for example in [3], and with some of the arguments from [5], might also be useful. Our notation is standard (see for example [15]); in particular, given a forcing notion $\mathbb{P}$ and two forcing condition p, $q \in \mathbb{P}$, we use $q \leq_{\mathbb{P}} p$ to mean that q is stronger than p. Also, given forcing notions $\mathbb{P}$ and $\mathbb{Q}$, we write $\mathbb{P} \lessdot \mathbb{Q}$ to denote that $\mathbb{P}$ is a complete suborder of $\mathbb{Q}$ (i.e., $\mathbb{P}$ is a suborder of $\mathbb{Q}$, any two incompatible conditions in $\mathbb{P}$ are incompatible in $\mathbb{Q}$, and any maximal antichain in $\mathbb{P}$ is in fact also a maximal antichain in $\mathbb{Q}$).

As it turns out, a small variant of our construction gives rise to a model satisfying a useful generic absoluteness statement, in addition to the statements in the conclusion in Theorem 0.1 (this is pursued in Section 2). This form of generic absoluteness implies, among other principles, Todorčević's Open Colouring Axiom for sets of size $\aleph_1$, Todorěvić's P-ideal Dichotomy for $\aleph_1$-generated ideals on ω_1, Moore's Measuring principle, and Baumgartner's Thinning-out Principle. Hence, all these principles are simultaneously compatible with $2^{\aleph_0} > \aleph_2$.

Theorem 0.1—as well as the results from [3], [4], etc., and all known derivations of $2^{\aleph_0} = \aleph_2$ from forcing axioms—strongly suggest that the restriction of PFA to a class $\mathcal{K}$ of proper posets should decide the size of the continuum to be $\aleph_2$ only if $\mathcal{K}$ contains enough forcing notions collapsing cardinals to $\aleph_1$. This motivates the following general question.

Question 0.2. Is the Proper Forcing Axiom restricted to the class of cardinal-preserving posets compatible with $2^{\aleph_0} > \aleph_2$? Is even the Proper Forcing Axiom restricted to the class of posets with the $\aleph_2$-chain condition compatible with $2^{\aleph_0} > \aleph_2$?

Remark 0.3. In [6] it is proved that the forcing axiom

$$\mathsf{FA}(\{\{\mathbb{P} : \mathbb{P} \text{ preserves stationary subsets of } \omega_1 \text{ and has the } \aleph_2\text{-c.c.}\})_{\aleph_2}$$

is inconsistent.

The rest of the paper is structured as follows. In Section 1 we prove Theorem 0.1. In Section 2 we consider a slight variation of the construction from Theorem 0.1 and show that it satisfies a useful form of generic absoluteness.

§ 1. A model of $\mathrm{PFA}(\aleph_1)_{<2^{\aleph_0}}$ and $2^{\aleph_0}$ arbitrarily large

In this section we prove Theorem 0.1.

Assume GCH holds and let $\kappa \geq \aleph_2$ be a regular cardinal. Let $\mathscr{H}(\kappa)$ denote the set of all sets that are hereditarily of cardinality less than κ and let $\phi : \kappa \to \mathscr{H}(\kappa)$ be a function such that the set $\phi^{-1}(x) \subseteq \kappa$ is unbounded for every $x \in \mathscr{H}(\kappa)$.[3] The function ϕ will be our book-keeping function.

Given a set N, δ_N is defined as $N \cap \omega_1$. δ_N is sometimes called *the height of* N. If N_0 and N_1 are $\in$-isomorphic models of the Axiom of Extensionality, we refer to the unique isomorphism $\Psi : (N_0; \in) \to (N_1; \in)$ by Ψ_{N_0,N_1}.

The following notion is defined in [4].

Definition 1.1. Given a predicate $\Phi \subseteq \mathscr{H}(\kappa)$, a finite set $\mathcal{N} \subseteq [\mathscr{H}(\kappa)]^{\aleph_0}$ is a Φ-*symmetric system* if the following holds.

(1) For every $N \in \mathcal{N}$, $(N; \in, \Phi \cap N) \prec (\mathscr{H}(\kappa); \in, \Phi)$.

(2) For all N_0, $N_1 \in \mathcal{N}$, if $\delta_{N_0} = \delta_{N_1}$, then $(N_0; \in, \Phi \cap N_0) \cong (N_1; \in, \Phi \cap N_1)$. Moreover, Ψ_{N_0,N_1} is the identity on $N_0 \cap N_1$.

(3) For all N_0, $N_1 \in \mathcal{N}$ and all $M \in \mathcal{N} \cap N_0$, if $\delta_{N_0} = \delta_{N_1}$, then $\Psi_{N_0,N_1}(M) \in \mathcal{N}$.

(4) For all N_0, $N_1 \in \mathcal{N}$, if $\delta_{N_0} < \delta_{N_1}$, then there is some $N_1' \in \mathcal{N}$ such that $\delta_{N_1'} = \delta_{N_1}$ and $N_0 \in N_1'$.

The following amalgamation lemma is proved in [4].

[3] ϕ exists since $|\mathscr{H}(\kappa)| = \kappa$ by GCH.

Lemma 1.2. *Let $\Phi \subseteq \mathscr{H}(\kappa)$, let $\mathcal{N}$ be a Φ-symmetric system, $N \in \mathcal{N}$, and let $\mathcal{M} \in N$ be a Φ-symmetric system such that $\mathcal{N} \cap N \subseteq \mathcal{M}$. Let*

$$\mathcal{W} = \mathcal{N} \cup \mathcal{M} \cup \{\Psi_{N,N'}(M) \,:\, N' \in \mathcal{N},\, \delta_{N'} = \delta_N,\, M \in \mathcal{M}\}$$

Then $\mathcal{W}$ is a Φ-symmetric system.

Our forcing witnessing Theorem 0.1 will be $\mathbb{P}_\kappa$ for a certain sequence $\langle \mathbb{P}_\alpha \,:\, \alpha \leq \kappa\rangle$ of forcing notions we will soon define by recursion on α. Together with $\langle \mathbb{P}_\alpha \,:\, \alpha \leq \kappa\rangle$, we will define a sequence $\langle \Phi_\alpha \,:\, \alpha < \kappa\rangle$ of predicates of $\mathscr{H}(\kappa)$. To start with, Φ_0 is the satisfaction predicate for $(\mathscr{H}(\kappa); \in, \phi)$. Given $\alpha < \kappa$, and assuming $\mathbb{P}_\alpha$ has been defined, we let $\Phi_{\alpha+1}$ be a predicate of $\mathscr{H}(\kappa)$ encoding, in some fixed canonical way, the satisfaction predicate of

$$(\mathscr{H}(\kappa); \in, \{(\beta, x) \,:\, \beta < \alpha, x \in \Phi_\beta\}, \mathbb{P}_\alpha, \Vdash^*_\alpha).$$

Here, $\Vdash^*_\alpha$ is the forcing relation for $\mathbb{P}_\alpha$ for formulas involving names in $\mathscr{H}(\kappa)$. Also, if $\alpha < \kappa$ is a limit ordinal and Φ_β has been defined for all $\beta < \alpha$,

$$\Phi_\alpha = \{(\beta, x) \,:\, \beta < \alpha, x \in \Phi_\beta\}.$$

A *model with marker* is a pair (N, ρ), where

- $N \in [\mathscr{H}(\kappa)]^{\aleph_0}$,
- $\rho \in N \cap \kappa$, and
- $N = N^* \cap \mathscr{H}(\kappa)$ for some countable $N^* \prec (\mathscr{H}(\kappa^+); \in)$ such that $\phi \in N^*$ and $\mathbb{P}_\beta \in N^*$ for each $\beta \in N \cap \rho$.[4]

We define a *symmetric system of models with makers* to be a collection Δ of models with markers such that:

[4]The (recursive) definition of $\mathbb{P}_\alpha$ will involve models with markers (N, ρ) with $\rho \leq \alpha$. $\mathbb{P}_\beta$ will already be defined for $\beta < \alpha$, so the definition will make sense.

- $\mathrm{dom}(\Delta) = \{N : (N, \rho) \in \Delta \text{ for some } \rho\}$ is a Φ_0-symmetric system;
- for every $\rho < \kappa$ and all (N_0, ρ), $(N_1, \rho) \in \Delta$, if $\delta_{N_0} = \delta_{N_1}$, then Ψ_{N_0,N_1} is in fact an isomorphism between the structures $(N_0; \in, \Phi_\rho \cap N_0)$ and $(N_1; \in, \Phi_\rho \cap N_1)$.

Given $\alpha < \kappa$, $\mathbb{P}_\alpha$ will consist of pairs $p = (F_p, \Delta_p)$, where:

(1) F_p is a finite function with $\mathrm{dom}(F_p) \subseteq \alpha$ and such that for each $\beta \in \mathrm{dom}(F_p)$, $F_p(\beta) \in \omega_1$.

(2) Δ_p is a symmetric system of models with markers (N, ρ) such that $\rho \leq \alpha$.

Remark 1.3. Modulo some notational changes, our construction will in fact be a small variation of (a simple version of) the main forcing construction from [3]. One important change with respect to that construction is that now we drop the requirement that if $(N, \rho) \in \Delta_p$ and $\bar{\rho} \in N \cap \rho$, then also $(N, \bar{\rho}) \in \Delta_p$. Another essential change is that, given a model with marker $(N, \beta + 1) \in \Delta_p$, we require that the working condition $F_p(\beta)$ be forced to be, not $(N[\underset{\sim}{G}_{\mathbb{P}_\beta}], \underset{\sim}{\mathbb{Q}}_\beta)$-generic, but $(N[\underset{\sim}{G}_{\mathbb{P}_\beta \restriction \mathcal{U}^\beta}], \underset{\sim}{\mathbb{Q}}_\beta)$-generic for a certain appropriate complete suborder $\mathbb{P}_\beta \restriction \mathcal{U}^\beta$ of $\mathbb{P}_\beta$. We will then of course have that $\underset{\sim}{\mathbb{Q}}_\beta$ is in fact a $\mathbb{P}_\beta \restriction \mathcal{U}^\beta$-name.

The above specification defines the universe of $\mathbb{P}_0$.

For all $\alpha < \kappa$, $p \in \mathbb{P}_\alpha$, and $\beta < \alpha$, $p \restriction \beta = (F_p \restriction \beta, \Delta_p \restriction \beta)$, where

$$\Delta_p \restriction \beta = \{(N, \rho) \in \Delta_p : \rho \leq \beta\}$$

Given $\alpha < \kappa$, the extension relation on $\mathbb{P}_\alpha$ will be denoted by $\leq_\alpha$.

Given p_0, $p_1 \in \mathbb{P}_0$, $p_1 \leq_0 p_0$ iff $\Delta_{p_0} \subseteq \Delta_{p_1}$.

Given any $\alpha < \kappa$, we associate to α sets $\overline{\mathcal{U}}^\alpha, \mathcal{U}^\alpha \in [\alpha]^{\leq \aleph_1}$ defined by letting $\overline{\mathcal{U}}^\alpha = \mathcal{U}^\alpha = \emptyset$ if $\phi(\alpha)$ is not a sequence $\langle A_i^\alpha : i < \omega_1 \rangle$ of antichains of $\mathbb{P}_\alpha$, each of size $\leq \aleph_1$, and, in the other case, letting $\overline{\mathcal{U}}^\alpha$ and $\mathcal{U}^\alpha$ be the following sets, where $\phi(\alpha) = \langle A_i^\alpha : i < \omega_1 \rangle$:

- $\overline{\mathcal{U}}^\alpha = \bigcup\{\mathrm{dom}(F_p) : p \in A_i^\alpha,\, i < \omega_1\} \cup \bigcup\{N \cap \rho : (N, \rho) \in \Delta_p,\, p \in A_i^\alpha,\, i < \omega_1\}$.

- $\mathcal{U}^\alpha = \overline{\mathcal{U}}^\alpha \cup \bigcup\{\mathcal{U}^\beta : \beta \in \overline{\mathcal{U}}^\alpha\}$.

Given any $\alpha < \kappa$, and assuming $\mathbb{P}_\alpha$ has been defined, we define $\mathbb{P}_\alpha \restriction \mathcal{U}^\alpha$ to be the suborder of $\mathbb{P}_\alpha$ consisting of those $p \in \mathbb{P}_\alpha$ such that

- $\mathrm{dom}(F_p) \subseteq \mathcal{U}^\alpha$ and
- $\rho \in \mathcal{U}^\alpha$ for all $(N, \rho) \in \Delta_p$.

Given a $\mathbb{P}_\alpha$-condition p, $p \restriction \mathcal{U}^\alpha$ is defined as

$$(F_p \restriction \mathcal{U}^\alpha, \{(N, \rho) : (N, \rho) \in \Delta_p,\, \rho \in \mathcal{U}^\alpha\}).$$

We note that $p \restriction \mathcal{U}^\alpha$ is a condition in $\mathbb{P}_\alpha \restriction \mathcal{U}^\alpha$.

Suppose $\underset{\sim}{\mathbb{Q}}$ is a $\mathbb{P}_\alpha \restriction \mathcal{U}^\alpha$-name for a partial order on $\omega_1^{\mathbf{V}}$. We say that $\underset{\sim}{\mathbb{Q}}$ *is forced to be* $(\dot{G}_{\mathbb{P}_\alpha \restriction \mathcal{U}^\alpha}, \mathbb{P}_\alpha)$*-proper* in case $\mathbb{P}_\alpha \restriction \mathcal{U}^\alpha$ forces that there is a club E of $[\mathscr{H}(\kappa)]^{\aleph_0}$ in $\mathbf{V}$ with the property that for all $N \in E$, if there is some $q \in \mathbb{P}_\alpha$ such that $q \restriction \mathcal{U}^\alpha \in \dot{G}_{\mathbb{P}_\alpha \restriction \mathcal{U}^\alpha}$ and $(N, \rho) \in \Delta_q$ for all $\rho \in N \cap (\alpha + 1)$, then for every $\nu \in \omega_1^{\mathbf{V}} \cap N[\dot{G}_{\mathbb{P}_\alpha \restriction \mathcal{U}^\alpha}]$ there is some $(N[\dot{G}_{\mathbb{P}_\alpha \restriction \mathcal{U}^\alpha}], \underset{\sim}{\mathbb{Q}})$-generic condition $\nu^* \in \omega_1^{\mathbf{V}}$ such that $\nu^* \leq_{\underset{\sim}{\mathbb{Q}}} \nu$.

We also define $\underset{\sim}{\mathbb{Q}}_\alpha$ to be the following $\mathbb{P}_\alpha \restriction \mathcal{U}^\alpha$-name:

- if $\phi(\alpha)$ is a sequence $\langle A_i^\alpha : i < \omega_1\rangle$ of antichains of $\mathbb{P}_\alpha$, each of size $\leq \aleph_1$, such that $\bigcup_{i<\omega_1} \{i\} \times A_i^\alpha$, viewed as a nice $\mathbb{P}_\alpha \restriction \mathcal{U}^\alpha$-name for a subset of ω_1, canonically encodes a forcing notion on $\omega_1^{\mathbf{V}}$ which $\mathbb{P}_\alpha \restriction \mathcal{U}^\alpha$ forces to be $(\dot{G}_{\mathbb{P}_\alpha \restriction \mathcal{U}^\alpha}, \mathbb{P}_\alpha)$-proper, then $\underset{\sim}{\mathbb{Q}}_\alpha$ is a $\mathbb{P}_\alpha \restriction \mathcal{U}^\alpha$-name for this forcing notion;
- in the other case, $\underset{\sim}{\mathbb{Q}}_\alpha$ is a $\mathbb{P}_\alpha \restriction \mathcal{U}^\alpha$-name for trivial forcing $\{\emptyset\}$.

Now we are in a position to define $\mathbb{P}_\alpha$ for $\alpha > 0$. A condition in $\mathbb{P}_\alpha$ is a pair $p = (F_p, \Delta_p)$ satisfying (1) and (2) above, together with the following.

(3) For each $\beta < \alpha$ and $(N, \beta + 1) \in \Delta_p$, $\mathcal{U}_\beta \cap N \subseteq \{\rho : (N, \rho) \in \Delta_p\}$.

(4) For each $\beta \in \mathrm{dom}(F_p)$ and $(N, \beta + 1) \in \Delta_p$, $p \restriction \mathcal{U}^\beta$ forces in $\mathbb{P}_\beta \restriction \mathcal{U}^\beta$ that $F_p(\beta)$ is $(N[\underset{\sim}{G}_{\mathbb{P}_\beta \restriction \mathcal{U}^\beta}], \underset{\sim}{\mathbb{Q}}_\beta)$-generic.

Given p_0 and p_1, $\mathbb{P}_\alpha$-conditions, $p_1 \leq_\alpha p_0$ iff

- for each $\beta < \alpha$, $p_1 \restriction \beta \leq_\beta p_0 \restriction \beta$;
- $\{N : (N, \alpha) \in \Delta_{p_0}\} \subseteq \{N : (N, \alpha) \in \Delta_{p_1}\}$;
- if $\alpha = \alpha_0 + 1$ and $\alpha_0 \in \mathrm{dom}(F_{p_0})$, then
  - (i) $\alpha_0 \in \mathrm{dom}(F_{p_1})$ and
  - (ii) $p_1 \restriction \alpha_0 \Vdash_{\mathbb{P}_{\alpha_0}} F_{p_1}(\alpha_0) \leq_{\underset{\sim}{\mathbb{Q}}_{\alpha_0}} F_{p_0}(\alpha_0)$.

Finally, we define $\mathbb{P}_\kappa = \bigcup_{\alpha<\kappa} \mathbb{P}_\alpha$.

It is immediate to see that $\langle \mathbb{P}_\alpha : \alpha \leq \kappa \rangle$ is a forcing iteration, in the sense that $\mathbb{P}_\alpha \lessdot \mathbb{P}_\beta$ holds for all $\alpha < \beta$. In fact, we have the following.

Lemma 1.4. *For all $\beta < \alpha < \kappa$, $p \in \mathbb{P}_\alpha$, and $q \in \mathbb{P}_\beta$, if $q \leq_\beta p \restriction \beta$, then*

$$(F_q \cup (F_p \restriction [\beta, \alpha)), \Delta_p \cup \Delta_q)$$

is a condition in $\mathbb{P}_\alpha$ extending both p and q.

We also have, for any given $\alpha < \kappa$, that $\mathbb{P}_\alpha \restriction \mathcal{U}^\alpha$ is a complete suborder of $\mathbb{P}_\alpha$. In fact, as the following easy lemma shows, the function sending $p \in \mathbb{P}_\alpha$ to $p \restriction \mathcal{U}^\alpha$ is a projection from $\mathbb{P}_\alpha$ onto $\mathbb{P}_\alpha \restriction \mathcal{U}^\alpha$.

Lemma 1.5. *Let $\alpha < \kappa$, $p \in \mathcal{P}_\alpha$, and let $q \in \mathbb{P}_\alpha \restriction \mathcal{U}^\alpha$ be a condition extending $p \restriction \mathcal{U}^\alpha$. Let*

$$r = ((F_p \restriction \alpha \setminus \mathcal{U}^\alpha) \cup F_q, \Delta_p \cup \Delta_q).$$

Then r is a condition in $\mathbb{P}_\alpha$ extending both p and q.

In what follows we will use Lemmas 1.4 and 1.5 repeatedly, often without mention.

We will next show, by a standard Δ-system argument using CH, that each $\mathbb{P}_\alpha$ has the $\aleph_2$-chain condition. In fact we will show that $\mathbb{P}_\alpha$ is $\aleph_2$-Knaster; in other words, for each sequence $(p_i : i < \omega_2)$ of conditions in $\mathbb{P}_\alpha$ there is $I \subseteq \omega_2$ of size $\aleph_2$ such that p_{i_0} and p_{i_1} are compatible in $\mathbb{P}_\alpha$ for all i_0, $i_1 \in I$.

Lemma 1.6. *For each $\alpha \leq \kappa$, $\mathbb{P}_\alpha$ is $\aleph_2$-Knaster.*

Proof. Let $p_i \in \mathbb{P}_\alpha$ for each $i < \omega_2$. For each i let M_i be a countable elementary submodel of $\mathscr{H}(\kappa^+)$ containing ϕ and p_i, and let $\mathcal{M}_i$ be a structure with universe M_i coding p_i and $\{(\beta, x) : \beta \in M_i \cap (\alpha+1), x \in \Phi_\beta \cap M_i\}$ in some canonical way. By CH, we may find $I \in [\omega_2]^{\aleph_2}$ and $R \in [\mathscr{H}(\kappa^+)]^{\aleph_0}$ such that $\{M_i : i \in I\}$ forms a Δ-system with root R. By further shrinking I if necessary, we may assume that for all i_0, $i_1 \in I$, $\mathcal{M}_{i_0} \cong \mathcal{M}_{i_1}$ and $\Psi_{M_{i_0}, M_{i_1}}$ is the identity on $M_{i_0} \cap M_{i_1}$.

It is then straightforward to see that if i_0, $i_1 \in I$, then $(F_{p_{i_0}} \cup F_{p_{i_1}}, \Delta_{p_{i_0}} \cup \Delta_{p_{i_1}})$, is a condition in $\mathbb{P}_\alpha$ extending both p_{i_0} and p_{i_1}. $\square$

The following is our properness lemma.

Lemma 1.7. *Let $\alpha < \kappa$. Then the following holds.*

- $(1)_\alpha$ *Let $p \in \mathbb{P}_\alpha$, $\beta \leq \alpha$, $n < \omega$, and let (N_i, β), for $i \leq n$, be models with markers. Suppose for all $i_0 \neq i_1$, $(N_{i_0}; \in, \Phi_\beta \cap N_{i_0}) \cong (N_{i_1}; \in, \Phi_\beta \cap N_{i_1})$ and $\Psi_{N_{i_0}, N_{i_1}}$ is the identity on $N_{i_0} \cap N_{i_1}$. Suppose $p \in N_0$. Then there is an extension $p^* \in \mathbb{P}_\alpha$ of p such that $(N_i, \rho) \in \Delta_{p^*}$ for all $i \leq n$ and all $\rho \in N_i \cap (\beta+1)$.*
- $(2)_\alpha$ *Let $p \in \mathbb{P}_\alpha$, let N^* be a countable elementary submodel of $\mathscr{H}(\kappa^+)$ such that ϕ, $\mathbb{P}_\alpha$, $p \in N^*$, let $N = N^* \cap \mathscr{H}(\kappa)$, and suppose $(N, \rho) \in \Delta_p$ for all $\rho \in N \cap (\alpha+1)$. Then p is $(N^*, \mathbb{P}_\alpha)$-generic.*
- $(3)_\alpha$ *Let $p \in \mathbb{P}_\alpha \restriction \mathcal{U}^\alpha$, let N^* be a countable elementary submodel of $\mathscr{H}(\kappa^+)$ such that ϕ, $\mathbb{P}_\alpha \restriction \mathcal{U}^\alpha$, $p \in N^*$, let $N = N^* \cap \mathscr{H}(\kappa)$, and suppose $(N, \rho) \in \Delta_p$ for all $\rho \in N \cap \mathcal{U}^\alpha$. Then p is $(N^*, \mathbb{P}_\alpha \restriction \mathcal{U}^\alpha)$-generic.*

Proof. The proof will be by induction on α. Let us start with the proof of $(1)_\alpha$. We may assume that $\mathrm{dom}(F_p) \cap \beta \neq \emptyset$ as otherwise we may simply let $p^* = (F_p, \Delta_p \cup \Delta_*)$, where

$$\Delta_* = \{(N_i, \gamma) : i \leq n, \gamma \in N_i \cap (\beta+1)\} \cup \{(\Psi_{N_0, N_i}(M), 0) : M \in \mathrm{dom}(\Delta_p),\ 0 < i \leq n\}.$$

Let $\bar{\alpha} = \max(\mathrm{dom}(F_p) \cap \beta)$. By induction hypothesis $(1)_{\bar{\alpha}}$ we know that there is an extension $\bar{p}$ of $p \restriction \bar{\alpha}$ such that $(N_i, \gamma) \in \Delta_{\bar{p}}$ for each i and $\gamma \in N_i \cap \bar{\alpha}$. (This is trivial if $\mathrm{dom}(F_p) \cap \bar{\alpha} = \emptyset$; if $\bar{\alpha}_0 = \max(\mathrm{dom}(F_p) \cap \bar{\alpha})$, then we get $\bar{p}$ by applying $(1)_{\bar{\alpha}_0+1}$ and thus obtaining some $\bar{p}_0 \in \mathbb{P}_{\bar{\alpha}_0+1}$, and then adding (N_i, γ) to $\Delta_{\bar{p}_0}$ for all i and $\gamma \in N_i \cap (\bar{\alpha}_0, \bar{\alpha})$.) Let $I = \{i \leq n : \bar{\alpha} \in N_i\}$. For each $i \in I$, let $N_i^* \prec \mathscr{H}(\kappa^+)$ be a countable model such that ϕ, $\mathbb{P}_{\bar{\alpha}+1} \in N_i^*$, and $N_i = N_i^* \cap \mathscr{H}(\kappa)$. We note that N_0^* contains a $\mathbb{P}_{\bar{\alpha}} \restriction \mathcal{U}^{\bar{\alpha}}$-name $\underset{\sim}{E}$ for a club of $[\mathscr{H}(\kappa)]^{\aleph_0}$ in $\mathbf{V}$ witnessing that $\underset{\sim}{\mathbb{Q}}_{\bar{\alpha}}$ is a $(\underset{\sim}{G}_{\mathbb{P}_{\bar{\alpha}} \restriction \mathcal{U}^{\bar{\alpha}}}, \mathbb{P}_{\bar{\alpha}})$-proper poset. By induction hypothesis $(3)_{\bar{\alpha}}$, $\bar{p}$ is $(N_i^*, \mathbb{P}_{\bar{\alpha}} \restriction \mathcal{U}^{\bar{\alpha}})$-generic for each $i \in I$. In particular, $\bar{p}$ forces that $N_0 = N_0^* \cap \mathscr{H}(\kappa) \in \underset{\sim}{E}$. Since $(N_0, \rho) \in \Delta_{\bar{p}}$ for all $\rho \in N_0 \cap (\bar{\alpha}+1)$, we may then extend $\bar{p} \restriction \mathcal{U}^{\bar{\alpha}}$ to a condition $p' \in \mathbb{P}_{\bar{\alpha}} \restriction \mathcal{U}^{\bar{\alpha}}$ for which there is some ν such that p' forces ν to be an $(N_0[\underset{\sim}{G}_{\mathbb{P}_{\bar{\alpha}} \restriction \mathcal{U}^{\bar{\alpha}}}], \underset{\sim}{\mathbb{Q}}_{\bar{\alpha}})$-generic condition in $\underset{\sim}{\mathbb{Q}}_{\bar{\alpha}}$ stronger than $F_p(\bar{\alpha})$. In order to finish the proof of $(1)_\alpha$ it will be enough to prove that ν is in fact forced by p' to be $(N_i[\underset{\sim}{G}_{\mathbb{P}_{\bar{\alpha}} \restriction \mathcal{U}^{\bar{\alpha}}}], \underset{\sim}{\mathbb{Q}}_{\bar{\alpha}})$-generic for all $i \in I$. This follows immediately from Claim 1.8.

Claim 1.8. *For every $t \in \mathbb{P}_{\bar{\alpha}+1}$, $(t \restriction \bar{\alpha}) \restriction \mathcal{U}^{\bar{\alpha}}$ forces in $\mathbb{P}_{\bar{\alpha}} \restriction \mathcal{U}^{\bar{\alpha}}$ that*

$$\{X \cap \delta_{M^0} : X \in \mathcal{P}(\omega_1)^{M^0[\dot{G}_{\mathbb{P}_{\bar{\alpha}} \restriction \mathcal{U}^{\bar{\alpha}}}]}\} = \{X \cap \delta_{M^1} : X \in \mathcal{P}(\omega_1)^{M^1[\dot{G}_{\mathbb{P}_{\bar{\alpha}} \restriction \mathcal{U}^{\bar{\alpha}}}]}\}$$

holds for all models with marker $(M^0, \bar{\alpha}+1)$, $(M^1, \bar{\alpha}+1)$ such that $\delta_{M^0} = \delta_{M^1}$ and such that $(M^0, \rho_0) \in \Delta_t$ and $(M^1, \rho_1) \in \Delta_t$ for all $\rho_0 \in M^0 \cap (\bar{\alpha}+1)$ and $\rho_1 \in M^1 \cap (\bar{\alpha}+1)$.

Proof. Let us first notice that for every $s \in (\mathbb{P}_{\bar{\alpha}} \restriction \mathcal{U}^{\bar{\alpha}}) \cap M^0$ such that $(t \restriction \bar{\alpha}) \restriction \mathcal{U}^{\bar{\alpha}} \leq_{\bar{\alpha}} s$, we also have that $(t \restriction \bar{\alpha}) \restriction \mathcal{U}^{\bar{\alpha}} \Vdash_{\mathbb{P}_{\bar{\alpha}} \restriction \mathcal{U}^{\bar{\alpha}}} \Psi_{M^0, M^1}(s) \in \dot{G}_{\mathbb{P}_{\bar{\alpha}} \restriction \mathcal{U}^{\bar{\alpha}}}$. For this, suppose towards a contradiction that this is not true. After extending t if necessary we may assume that $(t \restriction \bar{\alpha}) \restriction \mathcal{U}^{\bar{\alpha}}$ is incompatible with $\Psi_{M^0, M^1}(s)$. But using $(1)_\beta$ for suitable $\beta < \bar{\alpha}$ we can extend $(t \restriction \bar{\alpha}) \restriction \mathcal{U}^{\bar{\alpha}}$ to a condition extending $\Psi_{M_0, M_1}(s)$. For this note that $\Psi_{M_0, M_1}(F_s) = F_s$ since $\mathrm{dom}(F_s) \subseteq \mathcal{U}^{\bar{\alpha}} \in M^0 \cap M^1$ and $|\mathcal{U}^{\bar{\alpha}}| \leq \aleph_1$ and that for every $(M, \rho) \in \Delta_s$, $\Psi_{M^0, M^1}(\rho) = \rho$ again using the fact that $\rho \in \mathcal{U}^{\bar{\alpha}} \in M^0 \cap M^1$ and $|\mathcal{U}^{\bar{\alpha}}| \leq \aleph_1$.

Using the above facts, together with the induction hypothesis $(3)_{\bar{\alpha}}$, we get the conclusion of the claim. Specifically, suppose $\underaccent{\sim}{X} \in M^0$ is a $\mathbb{P}_{\bar{\alpha}} \restriction \mathcal{U}^{\bar{\alpha}}$-name for a subset of ω_1 and $\xi \in \delta_{M^0}$ is such that some condition $t' \in \mathbb{P}_{\bar{\alpha}}$ such that $t' \leq_{\mathbb{P}_{\bar{\alpha}}} t \restriction \bar{\alpha}$ forces $\xi \in \underaccent{\sim}{X}$. By $(3)_{\bar{\alpha}}$ we may assume, after extending t' if necessary, that t' extends some $s \in \mathbb{P}_{\bar{\alpha}} \restriction \mathcal{U}^{\bar{\alpha}} \cap M^0$ such that $s \Vdash_{\mathbb{P}_{\bar{\alpha}} \restriction \mathcal{U}^{\bar{\alpha}}} \xi \in \underaccent{\sim}{X}$.

Let $\Vdash^*$ denote the forcing relation for $\mathbb{P}_{\bar{\alpha}} \restriction \mathcal{U}^{\bar{\alpha}}$ restricted to formulas with names in $\mathscr{H}(\kappa)$. Since $\Vdash^*$ is definable from $\Vdash^*_{\bar{\alpha}}$ and $\mathbb{P}_{\bar{\alpha}}$ and Ψ_{M^0,M^1} is an isomorphism between $(M^0; \in, \Phi_{\bar{\alpha}+1} \cap M^0)$ and $(M^1; \in, \Phi_{\bar{\alpha}+1} \cap M^1)$, it follows that

$$\Psi_{M^0,M^1}(s) \Vdash_{\mathbb{P}_{\bar{\alpha}} \restriction \mathcal{U}^{\bar{\alpha}}} \Psi_{M^0,M^1}(\xi) = \xi \in \Psi_{M^0,M^1}(\underaccent{\sim}{X})$$

Since $t' \Vdash_{\mathbb{P}_{\bar{\alpha}} \restriction \mathcal{U}^{\bar{\alpha}}} \Psi_{M^0,M^1}(s) \in \dot{G}_{\mathbb{P}_{\bar{\alpha}} \restriction \mathcal{U}^{\bar{\alpha}}}$, it follows that $t' \Vdash_{\mathbb{P}_{\bar{\alpha}} \restriction \mathcal{U}^{\bar{\alpha}}} \xi \in \psi_{M^0,M^1}(\underaccent{\sim}{X})$. And similarly, if $t' \Vdash_{\mathbb{P}_{\bar{\alpha}} \restriction \mathcal{U}^{\bar{\alpha}}} \xi \notin \underaccent{\sim}{X}$, then also $t' \Vdash_{\mathbb{P}_{\bar{\alpha}} \restriction \mathcal{U}^{\bar{\alpha}}} \xi \notin \Psi_{M^0,M^1}(\underaccent{\sim}{X})$.

The above argument shows that $t \Vdash_{\mathbb{P}_{\bar{\alpha}} \restriction \mathcal{U}^{\bar{\alpha}}} \underaccent{\sim}{X} \cap \delta_{M^0} = \Psi_{M^0,M^1}(\underaccent{\sim}{X}) \cap \delta_{M^1}$, which establishes the claim since every $\mathbb{P}_{\bar{\alpha}} \restriction \mathcal{U}^{\bar{\alpha}}$-name in M^1 for a subset of ω_1 is of the form $\Psi_{M^0,M^1}(\underaccent{\sim}{X})$ for some $\underaccent{\sim}{X} \in M^0$ as above. □

Finally, let $p^* = (F_*, \Delta_*)$, where

- $F_* = F_{p'} \cup (F_{\bar{p}} \restriction \bar{\alpha} \setminus \mathcal{U}^{\bar{\alpha}}) \cup \{(\bar{\alpha}, \nu)\} \cup (F_p \restriction \alpha \setminus \beta)$ and
- $\Delta_* = \Delta_{p'} \cup \Delta_{\bar{p}} \cup \Delta_p \cup \{(N_i, \gamma) : i \leq n, \gamma \in N_i \cap (\beta + 1)\}$.

Then p^* is an extension of p as desired.

We now proceed to the proof of $(2)_\alpha$. For $\alpha = 0$, the conclusion follows trivially from Lemma 1.2. We may thus assume in what follows that $\alpha > 0$.

Suppose first that α is a successor ordinal, $\alpha = \alpha_0 + 1$. Let $D \in N^*$ be an open dense subset of $\mathbb{P}_\alpha$ and let $p' \in \mathbb{P}_\alpha$ be a condition extending p. By further extending p' if necessary, we may assume that $p' \in D$. We may assume that $\alpha_0 \in \mathrm{dom}(F_{p'})$ as otherwise the proof is a simpler version of the proof in this case.

Let G be any $\mathbb{P}_{\alpha_0}$-generic filter such that $p' \restriction \alpha_0 \in G$, let $G_0 = G \cap \mathbb{P}_{\alpha_0} \restriction \mathcal{U}^{\alpha_0}$, and let us note that G_0 is $\mathbb{P}_{\alpha_0} \restriction \mathcal{U}^{\alpha_0}$-generic over $\mathbf{V}$ by Lemma 1.5 (or by $(3)_{\alpha_0}$). Working in $N^*[G]$, let E_0 be the set of $\nu < \omega_1$ for which there is some $r \in D$ such that:

(1) $r \restriction \mathcal{U}^{\alpha_0} \in G_0$;

(2) $\alpha_0 \in \mathrm{dom}(F_r)$ and $F_r(\alpha_0) = \nu$.

Let $\mathbb{Q} = (\underset{\sim}{\mathbb{Q}}_{\alpha_0})_{G_0}$ and let E be the set of $\nu \in \omega_1$ such that

- $\nu \in E_0$, or else
- ν is incompatible in $\mathbb{Q}$ with all conditions in E_0.

E is of course in $N^*[G_0]$, and it is trivially a predense subset of $\mathbb{Q}$. Also, given any $\nu_0 \in E_0$, every $\nu < \omega_1$ such that $\nu \leq_{\mathbb{Q}} \nu_0$ is also in E_0.

Since $F_{p'}(\alpha_0)$ is $(N[G_0], \mathbb{Q})$-generic, by Lemma 1.6 it is also $(N^*[G_0], \mathbb{Q})$-generic. Hence, there is some $\nu^* \in \delta_N \cap E$ which is $\mathbb{Q}$-compatible with $F_{p'}(\alpha_0)$. Since $p' \in D$, it follows that in fact $\nu^* \in E_0$. Let us fix $r^* \in G_0$ for which there is some $r \in D$ such that $r^* = r \restriction \mathcal{U}^{\alpha_0}$, $\alpha_0 \in \mathrm{dom}(F_r)$, and $F_r(\alpha_0) = \nu^*$. Let $\mathbb{P}_{\alpha_0}/G_0 = \{t \in \mathbb{P}_{\alpha_0} : t \restriction \mathcal{U}^{\alpha_0} \in G_0\}$. By induction hypothesis $(2)_{\alpha_0}$, $p' \restriction \alpha_0$ is $(N^*, \mathbb{P}_{\alpha_0})$-generic. Since $\mathbb{P}_{\alpha_0} \restriction \mathcal{U}^{\alpha_0}$ is a complete suborder of $\mathbb{P}_{\alpha_0}$, it follows that

(1) $p' \restriction \mathcal{U}^{\alpha_0}$ is $(N^*, \mathbb{P}_{\alpha_0} \restriction \mathcal{U}^{\alpha_0})$-generic[5] and

(2) $p' \restriction \alpha_0$ is $(N^*[G_0], \mathbb{P}_{\alpha_0}/G_0)$-generic.

By (1), we may assume that $r^* \in N$. But then, by (2) and since $p' \restriction \alpha_0 \in G$ and $r^* \in N$, we may find $r_0 \in N \cap D$ such that:

- $r_0 \restriction \alpha_0 \in G$;
- $r_0 \restriction \mathcal{U}^{\alpha_0} = r^*$;
- $\alpha_0 \in \mathrm{dom}(F_{r_0})$ and $F_{r_0}(\alpha_0) = r^*$.

[5]This of course follows also from $(3)_{\alpha_0}$.

Let p'' be a condition in $\mathbb{P}_{\alpha_0}$ forcing the above for r_0 and forcing some $\nu^{**} < \omega_1$ to be a common extension of ν^* and $F_{p'}(\alpha_0)$ in $\undertilde{\mathbb{Q}}$. Then

$$(F_{p''} \cup \{(\alpha_0, \nu^{**})\}, \Delta_{p''} \cup \Delta_{p'} \cup \Delta_{r_0})$$

is a condition in $\mathbb{P}_\alpha$ and it extends both p' and r_0. This finishes the proof in this case.

Let us now consider the case that α is a limit ordinal. Again, let $D \in N^*$ be an open dense subset of $\mathbb{P}_\alpha$ and let $p' \in \mathbb{P}_\alpha$ be a condition extending p, which we may assume is in D. Let $\alpha_0 \in N \cap \alpha$ be high enough so that $\mathrm{dom}(F_{p'}) \cap [\alpha_0, \alpha) \cap N = \emptyset$, and let $G \in \mathbb{P}_{\alpha_0}$ be generic over $\mathbf{V}$ and such that $p' \restriction \alpha_0 \in G$. If $\mathrm{cf}(\alpha_0) = \omega$ we may assume that in fact $\mathrm{dom}(F_{p'}) \setminus \alpha_0 = \emptyset$, and if $\mathrm{cf}(\alpha) \geq \omega_1$ we may assume that $\sup(M \cap \alpha) < \alpha_0$ for every $M \in \mathrm{dom}(\Delta_{p'}) \cap N$.

Suppose first that $\mathrm{cf}(\alpha) = \omega$. In this case, working in $N^*[G]$ we may find a condition $r \in D$ such that

(1) $r \restriction \alpha_0 \in G$ and
(2) $\mathrm{dom}(F_r) \subseteq \alpha_0$.

By induction hypothesis $(2)_{\alpha_0}$ we have that $p' \restriction \alpha_0$ is $(N^*, \mathbb{P}_{\alpha_0})$-generic. Hence we may assume that $r \in N$. Let $p'' \in G$ be a condition extending $p' \restriction \alpha_0$ and $r \restriction \alpha_0$ and deciding r. Then $(F_{p''}, \Delta_{p''} \cup \Delta_{p'} \cup \Delta_r)$ is a condition in $\mathbb{P}_\alpha$ and it extends both p' and r, which finishes the proof in this subcase.

Finally, suppose $\mathrm{cf}(\alpha) \geq \omega_1$. This time, working in $N^*[G]$ we may find a condition $r \in D$ such that $r \restriction \alpha_0 \in G$. As in the previous subcase, we may assume that $r \in N$. Let $p'' \in G$ be a condition extending $p' \restriction \alpha_0$ and $r \restriction \alpha_0$ and deciding r. Let $(\beta_i)_{i<n}$ be the strictly increasing enumeration of $\mathrm{dom}(F_r) \setminus \alpha_0$. We may assume that $n > 0$ since otherwise we can finish as in the previous subcase.

Let $q = (F_{p''}, \Delta_{p''} \cup \Delta_{p'} \restriction \beta_0 \cup \Delta_r \restriction \beta_0)$. We now build a certain decreasing sequence $(q_i^+)_{i<n}$ of conditions extending q. For this, at step i of the construction we first fix a condition q_i in $\mathbb{P}_{\beta_i}$ such that

(1) q_i extends q,

(2) $(\Delta_{p'} \cup \Delta_r) \restriction \beta_i \subseteq \Delta_{q_i}$,

(3) q_i extends q_{i-1}^+ if $i > 0$, and such that

(4) for some $\nu_i < \omega_1$, q_i forces ν_i to be $(M[\dot{G}_{\mathbb{P}_{\beta_i} \restriction \mathcal{U}^{\beta_i}}], M)$-generic for every M such that $(M, \beta_i + 1) \in \Delta_{p'}$.

Then we let $q_i^+ = (F_{q_i} \cup \{(\beta_i, \nu_i)\}, \Delta_{q_i} \cup (\Delta_{p'} \cup \Delta_r) \restriction \beta_{i+1})$, where $\beta_n = \alpha$.

It is trivial to find q_i satisfying (1)–(3). The fact that q_i can be found so that it satisfies (4) as well follows from iteratedly applying the argument from the proof of $(1)_{\beta_i}$ m times, where m is the number of heights of models M such that $(M, \beta_i + 1) \in \Delta_{p'}$. We then have that q_{n-1}^+ is a condition in $\mathbb{P}_\alpha$ extending both p' and r, which concludes the proof of $(2)_\alpha$ in this subcase.

Finally, the proof of $(3)_\alpha$ is essentially the same as the proof of $(2)_\alpha$. $\square$

Lemma 1.7, together with Lemma 1.6 (for the case $\alpha = \kappa$), yields the following corollary.

Corollary 1.9. *For each $\alpha \leq \kappa$, $\mathbb{P}_\alpha$ is proper.*

The following lemma shows that $\mathbb{P}_\kappa$ produces a model of the relevant forcing axiom.

Lemma 1.10. *$\mathbb{P}_\kappa$ forces PFA$_{<\kappa}(\aleph_1)$.*

Proof. Let $\underset{\sim}{\mathbb{Q}}$ be a $\mathbb{P}_\kappa$-name for a proper forcing on $\omega_1^{\mathbf{V}[\dot{G}_{\mathbb{P}_\kappa}]}$ $(= \omega_1^{\mathbf{V}})$ and let $\{\underset{\sim}{D}_i \,:\, i < \lambda\}$ be, for some $\lambda < \kappa$, $\mathbb{P}_\kappa$-names for dense subsets of $\underset{\sim}{\mathbb{Q}}$. By Lemma 1.6 and the fact that $\kappa \geq \omega_2$ is regular, there is some $\alpha_0 < \kappa$ such that $\underset{\sim}{\mathbb{Q}}$ and $\underset{\sim}{D}_i$, for all $i < \lambda$, are $\mathbb{P}_\alpha$-names for all α, $\alpha_0 \leq \alpha < \kappa$. Letting $\langle A_i \,:\, i < \omega_1 \rangle$ be an ω_1-sequence of antichains of $\mathbb{P}_{\alpha_0}$ such that $\bigcup_{i<\omega_1}\{i\}\times A_i$ is a nice $\mathbb{P}_{\alpha_0}$-name for a subset of ω_1 canonically encoding $\underset{\sim}{\mathbb{Q}}$, we may find $\alpha\geq\alpha_0$ such that $\phi(\alpha)=\langle A_i\,:\,i<\omega_1\rangle$.

Claim 1.11. $\mathbb{P}_\alpha\restriction\mathcal{U}^\alpha$ *forces that* $\underset{\sim}{\mathbb{Q}}$ *is* $(\dot{G}_{\mathbb{P}_\alpha\restriction\mathcal{U}^\alpha},\mathbb{P}_\alpha)$*-proper.*

Proof. Let $f:[\mathscr{H}(\kappa)]^{<\omega}\to\mathscr{H}(\kappa)$ be a function with the property that for every countable $N\subseteq\mathscr{H}(\kappa)$, if $f``[N]^{<\omega}\subseteq N$, then $N=N^*\cap\mathscr{H}(\kappa)$ for some countable $N^*\prec\mathscr{H}(\kappa^+)$ such that ϕ, $\langle\mathbb{P}_\gamma\,:\,\gamma<\kappa\rangle\in N^*$. Let G be $\mathbb{P}_\alpha\restriction\mathcal{U}^\alpha$-generic, let $\mathbb{Q}=\underset{\sim}{\mathbb{Q}}_G$, let N be such that $f``[N]^{<\omega}\subseteq N$ and such that, for some $p\in\mathbb{P}_\alpha$ with $p\restriction\mathcal{U}^\alpha\in G$, $(N,\rho)\in\Delta_p$ for all $\rho\in N\cap(\alpha+1)$, and let $\nu\in\delta_N$. It will suffice to show that there is some $\nu^*\leq_{\mathbb{Q}}\nu$ which is $(N[G],\mathbb{Q})$-generic.

Suppose, for a contradiction, that this is not the case. By extending p if necessary, we may then assume that $p\restriction\mathcal{U}^\alpha$ forces that there is no $\nu^*\leq_{\underset{\sim}{\mathbb{Q}}}\nu$ which is $(N[\dot{G}_{\mathbb{P}_\alpha\restriction\mathcal{U}^\alpha}],\underset{\sim}{\mathbb{Q}})$-generic. Let now $\nu^*\in\omega_1$ and $q\leq_\alpha p$ be such that $(N,\beta)\in\Delta_q$ for each $\beta\in N\cap\kappa$ and such that q forces that $\nu^*\leq_{\underset{\sim}{\mathbb{Q}}}\nu$ and that ν^* is $(N[\dot{G}_{\mathbb{P}_\kappa}],\underset{\sim}{\mathbb{Q}})$-generic. Such a q exists by Lemma 1.7 $(1)_\beta$, for relevant $\beta\in N\cap\kappa$, together with the choice of $\underset{\sim}{\mathbb{Q}}$ as a name for a proper forcing. We have that q forces $\delta_{N[\dot{G}_{\mathbb{P}_\kappa}]}=\delta_{N[\dot{G}_{\mathbb{P}_\alpha\restriction\mathcal{U}^\alpha}]}=\delta_N$ by Lemma 1.7 $(2)_\beta$, for $\beta\in N\cap\kappa$, together with Lemma 1.6. Since $\mathbb{P}_\alpha\restriction\mathcal{U}^\alpha\lessdot\mathbb{P}_\kappa$, it then follows that $q\restriction\mathcal{U}^\alpha$ forces ν^* to be $(N[\dot{G}_{\mathbb{P}_\alpha\restriction\mathcal{U}^\alpha}],\underset{\sim}{\mathbb{Q}}_\alpha)$-generic. But that is a contradiction since $q\restriction\mathcal{U}^\alpha$ extends $p\restriction\mathcal{U}^\alpha$ in $\mathbb{P}_\alpha\restriction\mathcal{U}^\alpha$. $\square$

By the claim, together with the choice of $\langle A_i\,:\,i<\omega_1\rangle$, we have that $\underset{\sim}{\mathbb{Q}}_\alpha=\underset{\sim}{\mathbb{Q}}$. This finishes the proof of the lemma since $\mathbb{P}_{\alpha+1}$ forces

$$H=\{\nu<\omega_1\,:\,\nu\in F_p(\alpha)\text{ for some }p\in\dot{G}_{\mathbb{P}_{\alpha+1}}\text{ with }\alpha\in\mathrm{dom}(F_p)\}$$

to be a generic filter for $\underset{\sim}{\mathbb{Q}}$ over $\mathbf{V}[\dot{G}_{\mathbb{P}_\alpha}]$ and hence such that $H\cap\underset{\sim}{D}_i\neq\emptyset$ for each $i<\lambda$. $\square$

Finally, we prove that $\mathbb{P}_\kappa$ forces the right cardinal arithmetic.

Lemma 1.12. $\mathbb{P}_\kappa$ *forces* $2^{\aleph_0}=\kappa$.

Proof. $\Vdash_{\mathbb{P}_\kappa} 2^{\aleph_0} \geq \kappa$ follows, for example, from $\Vdash_{\mathbb{P}_\kappa} \mathsf{FA}(\{\text{Cohen}\})_{<\kappa}$. $\Vdash_{\mathbb{P}_\kappa} 2^{\aleph_0} \leq \kappa$ follows from the fact that there are $(\kappa^{\aleph_1})^{\aleph_0} = \kappa$ nice $\mathbb{P}_\kappa$-names for subsets of ω. $\square$

Lemma 1.12 concludes the proof of the theorem.

§ 2. A Π_2-rich model of $\mathrm{PFA}(\aleph_1)_\kappa$ for large κ

Let us say that $\mathbb{Q}$ is a *Příkrý-type* partial order in case there is a set $\mathrm{Res}(\mathbb{Q})$ such that:

(1) $\mathbb{Q}$ is a partial order with conditions being ordered pairs (s, A) with $A \in \mathrm{Res}(\mathbb{Q})$;
(2) for all A_0, $A_1 \in \mathrm{Res}(\mathbb{Q})$, $A_0 \cap A_1 \in \mathrm{Res}(\mathbb{Q})$;
(3) for every $(s, A_0) \in \mathbb{Q}$ and every $A_1 \in \mathrm{Res}(\mathbb{Q})$, if $A_1 \subseteq A_0$, then (s, A_1) is a condition in $\mathbb{Q}$ extending (s, A_0).

In the above situation we will sometimes refer to s as the *stem* of (s, A) and to A as its *reservoir*. Given a set X, we will say that a Příkrý-type partial order $\mathbb{Q}$ has *stems in* X if for all $(s, A) \in \mathbb{Q}$, $s \in X$.

Clearly, if $\mathbb{Q}$ is a Příkrý-type partial order $\mathbb{Q}$ with stems in X, then $\mathbb{Q}$ has the $|X|^+$-chain condition and in fact is $|X|$-centred, in the sense that there is a decomposition $\mathbb{Q} = \bigcup_{i<|X|} \mathbb{Q}_i$ such that each $\mathbb{Q}_i$ is a centred suborder of $\mathbb{Q}$ (i.e., every finite subset of $\mathbb{Q}_i$ has a lower bound in $\mathbb{Q}_i$).

Let *Local* CH be the statement that every set in $\mathscr{H}(\aleph_2)$ belongs to some ground model satisfying CH.

In this section we will be mostly concerned with proper Příkrý-type forcing notions with stems in ω_1.[6] In particular, we will be interested in the following forcing axiom-like principle.

Definition 2.1. *Příkrý-type* BPFA *from ground models of* CH, CH-Pr_{ω_1}-BPFA, is the conjunction of the following two statements.

[6]This property of forcing notions is of a very similar flavour to having the $\aleph_2$-p.i.c. At the moment it is not clear to us exactly how related are these two properties.

(1) Local CH

(2) Suppose $\varphi(x,y)$ is a restricted formula in the language of set theory. Suppose for every $a \in \mathscr{H}(\aleph_2)$ and every ground model M, if $a \in M$ and $M \models \mathsf{CH}$, then it holds in M that there is a proper Příkrý-type forcing notion $\mathbb{Q}$ with stems in ω_1 such that $\mathbb{Q}$ forces $\mathscr{H}(\aleph_2) \models \exists y \varphi(a,y)$. Then $\mathscr{H}(\aleph_2) \models \forall x \exists y \varphi(x,y)$.

Note that, by the first order definability of the generic multiverse, CH-Pr_{ω_1}-BPFA, and hence also Local CH, are first order statements in the language of set theory.

The following is the main theorem in this section.

Theorem 2.2. *Assume GCH. Let $\kappa \geq \aleph_2$ be a regular cardinal. Then there is a proper partial order $\mathbb{P}$ with the $\aleph_2$-chain condition and forcing the following statements.*

(1) $2^{\aleph_0} = \kappa$

(2) *PFA*$(\aleph_1)_{<\kappa}$

(3) *CH*-Pr_{ω_1}-*BPFA*

In the next subsection we will prove this theorem and in Subsection 2.2 we will give some applications of CH-Pr_{ω_1}-BPFA.

§ 2.1. **Proving Theorem 2.2.** Let us assume GCH and let $\kappa \geq \aleph_2$ be a regular cardinal. The forcing $\mathbb{P}$ witnessing Theorem 2.2 will be obtained by a modification of the construction from the proof of Theorem 0.1, which we will also refer to as $\langle \mathbb{P}_\alpha : \alpha \leq \kappa \rangle$. In fact, we will recycle much of the notation from that proof. Also, most of the verifications of the relevant points will be the same as in the proof of Theorem 0.1 so we will only give details of those arguments which are new.

Let $\mathrm{Fml}_{\Sigma_0}(x,y)$ be the set of restricted formulas in the language of set theory with free variables among x, y. There are four differences in the present construction with respect to the one from the proof of Theorem 0.1:

Given $\alpha < \kappa$, and assuming $\mathbb{P}_\alpha$ has been defined, we define $\mathcal{U}^\alpha$ by letting $\mathcal{U}^\alpha = \emptyset$ if $\phi(\alpha)$ is not a pair of the form $(p_\alpha, \langle A^\alpha_i : i < \omega_1\rangle)$, where $p_\alpha \in \{0\} \cup \mathrm{Fml}_{\Sigma_0}(x, y)$ and where $\langle A^\alpha_i : i < \omega_1\rangle$ is a sequence of antichains of $\mathbb{P}_\alpha$, all of them of size $\leq \aleph_1$, and, in the other case, letting $\mathcal{U}^\alpha$ be defined in exactly the same way as in the proof of Theorem 0.1.

The second difference is in the definition of $\underset{\sim}{\mathbb{Q}}_\alpha$. This is now a $\mathbb{P}_\alpha \restriction \mathcal{U}^\alpha$-name which is defined as follows:

- If $\phi(\alpha)$ is of the form $(0, \langle A^\alpha_i : i < \omega_1\rangle)$, where $\langle A^\alpha_i : i < \omega_1\rangle$ is a sequence of antichains of $\mathbb{P}_\alpha$, all of them of size $\leq \aleph_1$, such that $\bigcup_{i<\omega_1}\{i\} \times A^\alpha_i$, viewed as a nice $\mathbb{P}_\alpha \restriction \mathcal{U}^\alpha$-name for a subset of ω_1, canonically encodes a forcing notion on $\omega_1^{\mathbf{V}}$ which $\mathbb{P}_\alpha \restriction \mathcal{U}^\alpha$ forces to be $(\dot{G}_{\mathbb{P}_\alpha \restriction \mathcal{U}^\alpha}, \mathbb{P}_\alpha)$-proper, then $\underset{\sim}{\mathbb{Q}}_\alpha$ is a $\mathbb{P}_\alpha \restriction \mathcal{U}^\alpha$-name for this forcing notion.
- If $\phi(\alpha)$ is of the form $(\varphi(x, y), \langle A^\alpha_i : i < \omega_1\rangle)$, with $\varphi(x, y) \in \mathrm{Fml}_{\Sigma_0}(x, y)$ and $\langle A^\alpha_i : i < \omega_1\rangle$ a sequence of antichains of $\mathbb{P}_\alpha$, all of them of size $\leq \aleph_1$, then $\underset{\sim}{\mathbb{Q}}_\alpha$ is a $\mathbb{P}_\alpha \restriction \mathcal{U}^\alpha$-name for a partial order forced to be as follows:
  - Suppose $\underset{\sim}{a}_\alpha = \bigcup_{i<\omega_1}\{i\} \times A^\alpha_i$, viewed as a nice $\mathbb{P}_\alpha \restriction \mathcal{U}^\alpha$-name for a subset of ω_1, is such that there is a Příkrý-type proper partial order $\mathbb{Q} \subseteq \mathscr{H}(\kappa)^{\mathbf{V}[\dot{G}_{\mathbb{P}_\alpha \restriction \mathcal{U}^\alpha}]}$ with stems in ω_1 and forcing $\exists y \varphi(y, \underset{\sim}{a}_\alpha)$. Then $\underset{\sim}{\mathbb{Q}}_\alpha$ is such a partial order.
  - In the other case, $\underset{\sim}{\mathbb{Q}}_\alpha$ is trivial forcing $\{\emptyset\}$.
- In the remaining case, $\underset{\sim}{\mathbb{Q}}_\alpha$ is a $\mathbb{P}_\alpha \restriction \mathcal{U}^\alpha$-name for trivial forcing $\{\emptyset\}$.

The third difference is that now clause (1) in the definition of $\mathbb{P}_\alpha$ gets replaced with the following.

(1)* F_p is a finite function with $\mathrm{dom}(F_p) \subseteq \alpha$ and such that for each $\beta \in \mathrm{dom}(F_p)$,

  (a) $F_p(\beta)$ is of the form $(\nu, \underset{\sim}{A})$, with $\nu \in \omega_1$ and $\underset{\sim}{A} \in \mathscr{H}(\kappa)$ a $\mathbb{P}_\beta \restriction \mathcal{U}^\beta$-name for a subset of ω_1, and

(b) $p \restriction \mathcal{U}^\beta \Vdash_{\mathbb{P}_\beta \restriction \mathcal{U}^\beta} (\nu, \underset{\sim}{A}) \in \underset{\sim}{\mathbb{Q}}_\beta$.

Remark 2.3. We can obviously view a poset on ω_1 as a Příkrý-type poset with stems in ω_1 by identifying a condition $\nu \in \omega_1$ with the pair $(\nu, \emptyset)$.

The last difference is that we have the following additional symmetry clause.

(5) For every $p \in \mathbb{P}_\alpha$ and every $\rho < \alpha$, $\{N : (N, \rho+1) \in \Delta_p\}$ forms a symmetric system.

This completes the specification of the present forcing construction.

The proofs of the corresponding versions of Lemmas 1.4 and 1.5 are exactly the same as the proofs of the original lemmas, and the corresponding version of Lemma 1.6 is also almost the same. The only difference is that, in the situation of that proof, letting $\alpha_0 < \ldots < \alpha_{n-1}$ be the ordinals in $\mathrm{dom}(F_{p_{i_0}}) \cap \mathrm{dom}(F_{p_{i_1}})$, which we may assume is a nonempty sequence, we recursively build a decreasing sequence $(q_i)_{i<n}$ of conditions extending $((F_{p_{i_0}} \cup F_{p_{i_1}}) \restriction \alpha_0, (\Delta_{p_{i_0}} \cup \Delta_{p_{i_1}}) \restriction \alpha_0)$. At each step i, q_i is a condition in $\mathbb{P}_{\alpha_i}$ extending q_{i-1} (if $i > 0$) and deciding some $\underset{\sim}{A}$ such that $(\nu, \underset{\sim}{A}) \leq_{\underset{\sim}{\mathbb{Q}}_{\alpha_i}} (\nu, \underset{\sim}{A}_{i_0}), (\nu, \underset{\sim}{A}_{i_1})$ (where $F_{p_{i_0}}(\alpha_i) = (\nu, \underset{\sim}{A}_{i_0})$ and $F_{p_{i_1}}(\alpha_i) = (\nu, \underset{\sim}{A}_{i_1})$). We then amalgamate q_{n-1}, p_{i_0} and p_{i_1} in the obvious way.

The current version of Lemma 1.7 is also proved by induction on $\alpha < \kappa$. The proof of $(1)_\alpha$ is now very similar to the proof in the original lemma. In the situation of the proof of Lemma 1.7 $(1)_\alpha$, when $\bar{\alpha}$ is such that $\phi(\bar{\alpha})$ is of the form $(0, \langle A_i^{\bar{\alpha}} : i < \omega_1 \rangle)$, then we continue exactly as in that proof. There is only a new point in the proof in the case when $\phi(\bar{\alpha})$ is of the form $(\varphi(x, y), \langle A_i^{\bar{\alpha}} : i < \omega_1 \rangle)$ with $\varphi(x, y)$ a restricted formula and $\underset{\sim}{\mathbb{Q}}_{\bar{\alpha}}$ given as in the second bullet point in the definition of this name. In this case, given $(\nu_0, \underset{\sim}{A}_0)$ which p' forces to be an $(N_0^*[\dot{G}_{\mathbb{P}_{\bar{\alpha}} \restriction \mathcal{U}^{\bar{\alpha}}}], \underset{\sim}{\mathbb{Q}}_{\bar{\alpha}})$-generic condition, we need to prove that p' forces $(\nu_0, \underset{\sim}{A}_0)$ to be $(N_i^*[\underset{\sim}{G}_{\mathbb{P}_{\bar{\alpha}} \restriction \mathcal{U}^{\bar{\alpha}}}], \underset{\sim}{\mathbb{Q}}_{\bar{\alpha}})$-generic for all $i \in I$. For this, let $i \in I$, $\underset{\sim}{D} \in N_i^*$ be a $\mathbb{P}_{\bar{\alpha}} \restriction \mathcal{U}^{\bar{\alpha}}$-name for a dense subset of $\underset{\sim}{\mathbb{Q}}_{\bar{\alpha}}$, $p'' \in \mathbb{P}_{\bar{\alpha}} \restriction \mathcal{U}^{\bar{\alpha}}$ extend p', $\xi \in \delta_{N_i^*}$, let $\underset{\sim}{A} \in N_0^*$ be a $\mathbb{P}_{\bar{\alpha}} \restriction \mathcal{U}^{\bar{\alpha}}$-name for a subset of ω_1, and suppose p'' is such that for some $(\nu^\dagger, \underset{\sim}{A}^\dagger)$,

(1) $p'' \Vdash_{\mathbb{P}_{\bar{\alpha}} \restriction \mathcal{U}^{\bar{\alpha}}} (\nu^\dagger, \underset{\sim}{A}^\dagger) \leq_{\underset{\sim}{\mathbb{Q}}_{\bar{\alpha}}} (\nu_0, \underset{\sim}{A}_0)$, $(\xi, \underset{\sim}{A})$, and

(2) $p'' \Vdash_{\mathbb{P}_{\bar{\alpha}} \restriction \mathcal{U}^{\bar{\alpha}}} (\xi, \underset{\sim}{A}) \in \Psi_{N_i^*, N_0^*}(\underset{\sim}{D})$.

Let $\underset{\sim}{A}_0^\dagger$ be a $\mathbb{P}_{\bar{\alpha}} \restriction \mathcal{U}^{\bar{\alpha}}$-name for $\underset{\sim}{A}^\dagger \cap \Psi_{N_0^*, N_1^*}(\underset{\sim}{A})$. Then $p'' \Vdash_{\mathbb{P}_{\bar{\alpha}} \restriction \mathcal{U}^{\bar{\alpha}}} (\nu^\dagger, \underset{\sim}{A}_0^\dagger) \in \underset{\sim}{\mathbb{Q}}_{\bar{\alpha}}$ since, by a standard properness argument using $(3)_{\bar{\alpha}}$ together with the argument in the proof of Claim 1.8, we have that $p'' \Vdash_{\mathbb{P}_{\bar{\alpha}} \restriction \mathcal{U}^{\bar{\alpha}}} \Psi_{N_0^*, N_1^*}(\underset{\sim}{A}) \in \mathrm{Res}(\underset{\sim}{\mathbb{Q}}_{\bar{\alpha}})$. It suffices to show that p'' forces $(\nu^\dagger, \underset{\sim}{A}_0^\dagger)$ to be compatible with a condition in $\underset{\sim}{D} \cap N_i^*$. But $p'' \Vdash_{\mathbb{P}_{\bar{\alpha}} \restriction \mathcal{U}^{\bar{\alpha}}} (\xi, \Psi_{N_0^*, N_i^*}(\underset{\sim}{A})) \in \underset{\sim}{D} \cap N_i^*$, again by the usual properness argument using $(3)_{\bar{\alpha}}$ (together with the argument in the proof of Claim 1.8), and $p'' \Vdash_{\mathbb{P}_{\bar{\alpha}} \restriction \mathcal{U}^{\bar{\alpha}}} (\nu^\dagger, \underset{\sim}{A}_0^\dagger) \leq_{\underset{\sim}{\mathbb{Q}}_{\bar{\alpha}}} (\xi, \Psi_{N_0^*, N_i^*}(\underset{\sim}{A}))$.

The proof of $(2)_\alpha$ (and of $(3)_\alpha$) is the same as in the original lemma when α is 0, a successor ordinal, or an ordinal of countable cofinality. When $\mathrm{cf}(\alpha) \geq \omega_1$, we fix α_0, G, r and q as in the old proof, and let us fix $(\beta_i)_{i<n}$ as we did there. We then build a decreasing sequence $(q_i^+)_{i<n}$ as in that proof. In order to argue that q_i^+ can be taken to satisfy (4) we again apply $(1)_{\beta_i}$ successively m times, where m is the number of heights δ_k of models M such that $(M, \beta_i + 1) \in \Delta_{p'}$, using the properties (2) and (3) from the definition of Příkrý-type forcing as in the proof of $(1)_\alpha$, together with the symmetry of $\{M : (M, \beta_i + 1) \in \Delta_{p'}\}$ — so that at each step $k < m$ of the construction such that $k + 1 < m$, a $\mathbb{P}_{\beta_i} \restriction \mathcal{U}^{\beta_i}$-name for the current condition belongs to some relevant model of the next height δ_{k+1} to be considered (namely some M with $\delta_M = \delta_{k+1}$, $(M, \beta_i + 1) \in \Delta_{p'}$, and such that $\overline{M} \in M$ for some M such that $\delta_{\overline{M}} = \delta_k$, $(M, \beta_i + 1) \in \Delta_{p'}$, and such that at step k we fixed a name for a $(\overline{M}[\dot{G}_{\mathbb{P}_{\beta_i} \restriction \mathcal{U}^{\beta_i}}], \underset{\sim}{\mathbb{Q}}_{\beta_i})$-generic condition). In the end we of course take a name for the intersection of all the (finitely many) relevant reservoirs, as in the proof of $(1)_\alpha$.

The proof of the corresponding versions of Corollary 1.9 and Lemmas 1.10 and 1.12 is the same, using what have already established, as for the original results.

Given any $\alpha_0 < \kappa$ and any sequence $\vec{A} = \langle A_i : i < \omega_1 \rangle$ of antichains of $\mathbb{P}_{\alpha_0}$, we associate to $\vec{A}$ sets $\overline{\mathcal{U}}_{\vec{A}}$ and $\mathcal{U}_{\vec{A}} \in [\alpha_0]^{\leq \aleph_1}$ very much as in the construction of $\overline{\mathcal{U}}^\alpha$ and $\mathcal{U}^\alpha$ for $\alpha < \kappa$. The definition is the following.

- $\overline{\mathcal{U}}_{\vec{A}} = \bigcup \{\mathrm{dom}(F_p) : p \in A_i, i < \omega_1\} \cup \bigcup \{N \cap \rho : (N, \rho) \in \Delta_p, p \in A_i, i < \omega_1\}$.
- $\mathcal{U}^\alpha = \overline{\mathcal{U}}_{\vec{A}} \cup \bigcup \{\mathcal{U}^\beta : \beta \in \overline{\mathcal{U}}_{\vec{A}}\}$.

Given any $\beta < \kappa$ such that $\beta \leq \alpha_0$, we define $\mathbb{P}_\beta \restriction \mathcal{U}_{\vec{A}}$ in exactly the same way as how we defined $\mathbb{P}_\alpha \restriction \mathcal{U}^\alpha$ from $\mathcal{U}^\alpha$. And given any $p \in \mathbb{P}_\beta$, we also define the restriction $p \restriction \mathcal{U}_{\vec{A}}$ in the same way as how we defined $p \restriction \mathcal{U}^\alpha$.

We have the following version of Lemma 1.5.

Lemma 2.4. *For every $\alpha_0 < \kappa$, every sequence $\vec{A} = \langle A_i : i < \omega_1 \rangle$ of antichains of $\mathbb{P}_{\alpha_0}$, and every $\beta < \kappa$ such that $\alpha_0 \leq \beta$, the function sending $p \in \mathbb{P}_\beta$ to $p \restriction \mathcal{U}_{\vec{A}}$ is a projection of $\mathbb{P}_\beta$ onto $\mathbb{P}_{\alpha_0} \restriction \mathcal{U}_{\vec{A}}$.*

The following CH-preservation lemma is proved by a symmetry argument very much as in the proof of Claim 1.8.[7]

Lemma 2.5. *For every $\alpha < \kappa$ and every sequence $\vec{A} = \langle A_i : i < \omega_1 \rangle$ of antichains of $\mathbb{P}_\alpha$, $\mathbb{P}_\alpha \restriction \mathcal{U}_{\vec{A}}$ forces CH.*

Proof. Let $\underset{\sim}{r}_i$, for $i < \omega_2$, be $\mathbb{P}_\alpha \restriction \mathcal{U}_{\vec{A}}$-names for subsets of ω and suppose, towards a contradiction, that $p \in \mathbb{P}_\alpha \restriction \mathcal{U}_{\vec{A}}$ forces $\underset{\sim}{r}_i \neq \underset{\sim}{r}_{i'}$ for all $i \neq i'$. By the $\aleph_2$-c.c. of $\mathbb{P}_\alpha \restriction \mathcal{U}_{\vec{A}}$ we may of course assume that these names are all in $\mathscr{H}(\kappa)$. For each i let M_i^* be a countable elementary submodel of $\mathscr{H}(\kappa^+)$ containing everything relevant, which includes $\vec{A}$, p, and $\underset{\sim}{r}_i$. Let $\mathcal{M}_i$ be a structure with universe $M_i := M_i^* \cap \mathscr{H}(\kappa)$ coding $\vec{A}$, p, $\underset{\sim}{r}_i$ and $\langle (i, x) : \beta \in M_i \cap \kappa, x \in \Phi_\beta \cap M_i \rangle$ in some canonical way. By CH we may find $i_0 \neq i_1$ such that $\mathcal{M}_{i_0} \cong \mathcal{M}_{i_1}$ and $\Psi_{M_{i_0}, M_{i_1}}$ is the identity on $M_{i_0} \cap M_{i_1}$.

[7] See for example [5] for a similar argument.

By the current version of Lemma 1.7 $(1)_\alpha$ we may extend $p \in \mathbb{P}_\alpha$ to a condition $p^* \in \mathbb{P}_\alpha$ such that (M_{i_0}, ρ_0), $(M_{i_1}, \rho_1) \in \Delta_{p^*}$ for al $\rho_0 \in M_{i_0} \cap (\alpha+1)$ and $\rho_1 \in M_{i_1} \cap (\alpha+1)$. By an argument as in the proof of Claim 1.8 we have that for every $s \in (\mathbb{P}_\alpha \restriction \mathcal{U}_{\vec{A}}) \cap M_{i_0}$ and every $p' \in \mathbb{P}_\alpha$ extending p^*, $p' \leq_{\mathbb{P}_\alpha} s$ if and only if $p' \leq_{\mathbb{P}_\alpha \restriction \mathcal{U}^\alpha} \Psi_{M_{i_0}, M_{i_1}}(s) \in \dot{G}_{\mathbb{P}_\alpha \restriction \mathcal{U}^\alpha}$.

Finally, by the current version of Lemma 1.7 $(2)_\alpha$, for every condition $p' \in \mathbb{P}_\alpha$ extending p^* and every $n < \omega$, if p' extends a condition deciding the truth value of the statement $n \in \underset{\sim}{r}_{i_0}$, then p' can be extended to a condition $s \in (\mathbb{P}_\alpha \restriction \mathcal{U}_{\vec{A}}) \cap M_{i_0}$ deciding this statement (of course in the same way). Hence, by the previous paragraph and the fact that

$$\Psi_{M_{i_0}, M_{i_1}} : (M_{i_0}; \in, \Phi_{\alpha+1}, \vec{A}) \to (M_{i_1}; \in, \Phi_{\alpha+1}, \vec{A})$$

is an isomorphism sending $\underset{\sim}{r}_{i_0}$ to $\underset{\sim}{r}_{i_1}$, we get that $p^* \Vdash_{\mathbb{P}_\alpha} \underset{\sim}{r}_{i_0} = \Psi_{M_{i_0}, M_{i_1}}(\underset{\sim}{r}_{i_0}) = \underset{\sim}{r}_{i_1}$. This of course is a contradiction since $p^* \leq_\alpha p$. $\square$

The following is an immediate consequence of Lemmas 2.4, 2.5, together with the $\aleph_2$-c.c. of $\mathbb{P}_\kappa$.

Corollary 2.6. *$\mathbb{P}_\kappa$ forces Local CH.*

The following easy reflection lemma will be used in the proof of Lemma 2.8. The second part of the lemma follows from the characterization of properness as preservation of stationary subsets of $[X]^{\aleph_0}$, for all sets X.

Lemma 2.7. *Let λ be an infinite cardinal and suppose $\mathbb{Q}$ is a forcing notion with the λ^+-chain condition. Let θ be a cardinal such that $\mathbb{Q} \in \mathscr{H}(\theta)$ and let $N \prec \mathscr{H}(\theta)$ be such that $\mathbb{Q} \in N$ and ${}^\lambda N \subseteq N$. Then $\mathbb{Q} \cap N$ is a complete suborder of $\mathbb{Q}$. Hence, if $\mathbb{Q}$ is proper, then so is $\mathbb{Q} \cap N$.*

Lemma 2.8. *$\mathbb{P}_\kappa$ forces CH-Pr_{ω_1}-BPFA.*

Proof. Let G be $\mathbb{P}_\kappa$-generic over $\mathbf{V}$, let $a \in \mathscr{H}(\aleph_2)^{\mathbf{V}[G]}$, let $\varphi(x, y)$ be a restricted formula, and suppose for every ground model M of $\mathbf{V}[G]$ such that $a \in M$ and $M \models \mathsf{CH}$ it holds in M that there is a proper Příkrý-tpye forcing $\mathbb{Q}$ with stems in ω_1 such that $\mathbb{Q}$ forces $\mathscr{H}(\aleph_2) \models \exists y \varphi(a, y)$. By Corollary 2.6, it will be enough to show that there is some $b \in \mathbf{V}[G]$ such that $\mathbf{V}[G] \models \varphi(a, b)$.

Let $\underset{\sim}{a}$ be a nice $\mathbb{P}_\kappa$-name for a subset of ω_1 coding a and let $\vec{A} = \langle A_i \,:\, i < \omega \rangle$ be a sequence of antichains of $\mathbb{P}_\kappa$ such that $\underset{\sim}{a} = \bigcup_{i<\omega_1} \{i\} \times A_i$. Using the $\aleph_2$-c.c. of $\mathbb{P}_\kappa$ and the choice of ϕ, we may find α such that $\phi(\alpha) = (\varphi(x, y), \langle A_i \,:\, i < \omega_1 \rangle)$. Let $G_0 = G \cap \mathbb{P}_\alpha \restriction \mathcal{U}^\alpha$ and let $M = \mathbf{V}[G_0]$. Then $a \in M$ and, by the proof of Lemma 2.6, $M \models \mathsf{CH}$. It thus suffices to prove that in M there is a proper Příkrý-type forcing $\mathbb{Q} \subseteq \mathscr{H}(\kappa)^M$ with stems in ω_1 adding a witness to $\exists y \varphi(a, y)$ since then some condition in G_0 will force $\underset{\sim}{\mathbb{Q}}_\alpha$ to be such such a forcing, from which it will follow that $\mathbf{V}[G] \models \exists y \varphi(y, a)$ as desired.

The existence in M of a $\mathbb{Q}$ as above follows from the fact that in M there is some proper Příkrý-type forcing $\mathbb{Q}^*$ with stems in ω_1 such that $\mathbb{Q}^*$ forces $\mathscr{H}(\aleph_2) \models \exists y \varphi(a, y)$ together with Lemma 2.7. Indeed, working in M, let N be an elementary submodel of some large enough $\mathscr{H}(\theta)$ containing $\mathbb{Q}^*$ such that ${}^{\omega_1}N \subseteq N$ and $|\{x \in N \,:\, x \in y\}| < \kappa$ for each $y \in N$ (such an N exists by the $\aleph_2$-c.c. of $\mathbb{P}_\alpha \restriction \mathcal{U}^\alpha \subseteq \mathscr{H}(\kappa)$ in $\mathbf{V}$). By Lemma 2.7 and the $\aleph_2$-c.c. of $\mathbb{Q}^*$ we then have that $\mathbb{Q}_0 = \mathbb{Q}^* \cap N$ is a proper suborder of $\mathbb{Q}^*$. Also, $\mathbb{Q}_0$ clearly forces $\exists y \varphi(y, a)$ and is a Příkrý-type forcing with stems in ω_1 since this is true about $\mathbb{Q}^*$. Let $\mathbb{Q} = \pi``\mathbb{Q}_0$, where π is the Mostowski collapsing function of N. Then $\mathbb{Q} \subseteq \mathscr{H}(\kappa)$ by the choice of N and $\mathbb{Q}$ is still a Příkrý-type forcing with stems in ω_1. Finally, $\mathbb{Q}$ is of course isomorphic to $\mathbb{Q}_0$ and therefore it is proper and forces $\exists y \varphi(a, y)$. $\square$

Lemma 2.8 completes the proof of Theorem 2.2.

§ 2.2. **Some applications of CH-Pr_{ω_1}-BPFA.** We will finish the paper by showing that CH-Pr_{ω_1}-BPFA implies a number of interesting combinatorial consequences of PFA. We will focus on Baumgartner's Axiom for $\aleph_1$-dense sets of reals, Todorčević's Open Colouring Axiom for sets of size $\aleph_1$, Moore's Measuring principle, Baumgartner's Thinning-out Principle, and Todorčević's P-ideal Dichotomy for $\aleph_1$-generated ideals on ω_1. Hence, by Theorem 2.2, all these statements are simultaneously compatible with $2^{\aleph_0} > \aleph_2$.

A set A of reals is $\aleph_1$-dense if $A \cap (x, y)$ has cardinality $\aleph_1$ for all reals $x < y$, where (x, y) denotes the open interval $\{z \in \mathbb{R} : x < z < y\}$. *Baumgartner's axiom for $\aleph_1$-dense sets of reals*, BA, is the statement that all $\aleph_1$-dense sets of reals are order-isomorphic.

By a well-known theorem of Baumgartner ([7]), BA can be forced by a proper forcing. In fact, the following holds.

Lemma 2.9. ([7]) *Assume CH holds and suppose A and B are $\aleph_1$-dense sets of reals. Then there is a c.c.c. partial order $\mathbb{Q}$ of cardinality $\aleph_1$ and adding an order-isomorphism $\pi : A \to B$.*

We will also mention that [1] proves the consistency of BA with $2^{\aleph_0} > \aleph_2$.

Given a set X, Id_X, the identity on X, is $\{(x, x) : x \in X\}$. A colouring of a set of reals X is a partition (K_0, K_1) of $X \setminus \mathrm{Id}_X$ such that for all distinct x, $y \in X$, $(x, y) \in K_0$ if and only if $(y, x) \in K_0$. We say that (K_0, K_1) is open if K_0 is an open subset of $X \setminus \mathrm{Id}_X$ with the product topology.

Given a colouring (K_0, K_1) of X and $i \in \{0, 1\}$, we say that $H \subseteq X$ is i-homogeneous if $(x, y) \in K_i$ for all distinct x, $y \in H$.

Todorčević's Open Colouring Axiom ([27]), which we will denote by OCA, is the statement that if (K_0, K_1) is an open colouring of a set X of reals, then one of the following holds:

(1) There is an uncountable 0-homogeneous subset of X.

(2) There is a sequence $(X_n)_{n<\omega}$ such that $X = \bigcup_{n<\omega} X_n$ and each X_n is 1-homogeneous.

OCA follows from PFA ([27]). In fact we have the following (s. [28]).

Lemma 2.10. *Let X be a set of reals and suppose (K_0, K_1) is an open colouring of X. Suppose K_0 is not a union of $<2^{\aleph_0}$-many 1-homogeneous sets. Then there is $Y \subseteq X$ of size $2^{\aleph_0}$ such that the poset of finite 0-homogeneous subsets of Y, ordered by reverse inclusion, has the $2^{\aleph_0}$-chain condition.*

$\mathsf{OCA}(\aleph_1)$ will denote the restriction of OCA to colourings of sets $X \subseteq \mathbb{R}$ of cardinality $\aleph_1$. $\mathsf{OCA}(\aleph_1)$ is a useful fragment of OCA; for example, it is easy to see that $\mathsf{OCA}(\aleph_1)$ implies that every uncountable subset of $\mathcal{P}(\omega)$ contains an uncountable chain or an uncountable antichain under inclusion (s. [28]). Farah proved that $\mathsf{OCA}(\aleph_1)$ is consistent together with $2^{\aleph_0}$ large.

Measuring, defined by Moore (s. [11]), is the statement that for every club-sequence $\vec{C} = \langle C_\delta : \delta \in \omega_1 \rangle$[8] there is a club $C \subseteq \omega_1$ with the property that for every $\delta \in C$ there is some $\alpha < \delta$ such that either

- $(C \cap \delta) \setminus \alpha \subseteq C_\delta$, or
- $(C \setminus \alpha) \cap C_\delta = \emptyset$.

In the above situation, we say that C measures $\vec{C}$.

Measuring follows from PFA and can be regarded as a strong failure of Club Guessing at ω_1; in fact it is easy to see that it implies $\neg\,\mathsf{WCG}$.

Given a club-sequence $\vec{C} = \langle C_\delta : \delta \in \omega_1 \rangle$, there is a natural proper forcing $\mathbb{Q}_{\vec{C}}$ for adding a club of ω_1 measuring $\vec{C}$ (s. [11]). Conditions in $\mathbb{Q}_{\vec{C}}$ are pairs (x, C) such that:

(1) x is a closed countable subset of ω_1;

(2) for every $\delta \in \mathrm{Lim}(\omega_1) \cap x$ there is some $\alpha < \delta$ such that either

  - $(x \cap \delta) \setminus \alpha \subseteq C_\delta$, or

[8]I.e., each C_δ is a club of δ.

- $(x \setminus \alpha) \cap C_\delta = \emptyset$.

(3) C is a club of ω_1.

Given $\mathbb{Q}_{\vec{C}}$-conditions (x_0, C_0), (x_1, C_1), we let $(x_1, C_1) \leq_{\mathbb{Q}_{\vec{C}}} (x_0, C_0)$ iff

(1) x_1 is an end-extension of x_0 (i.e., $x_0 \subseteq x_1$ and $x_1 \cap (\max(x_0) + 1) = x_0$),

(2) $C_1 \subseteq C_0$, and

(3) $x_1 \setminus x_0 \subseteq C_0$.

The second assertion in the following lemma is clear from the definition,[9] and the first (the properness of $\mathbb{Q}_{\vec{C}}$) is a standard fact (s. e.g. [11]).

Lemma 2.11. *$\mathbb{Q}_{\vec{C}}$ is a proper Příkrý-type forcing notions with stems in $\mathscr{H}(\aleph_1)$.*

We finish our present discussion of Measuring by mentioning that the consistency of this principle with large continuum has already been shown in [14].

The Thinning-out Principle (TOP) is the following statement, defined by Baumgartner in [8]: Suppose $A \subseteq \omega_1$, $B \subseteq \omega_1$, and $(B_\alpha : \alpha \in B)$ is such that $B_\alpha \subseteq \alpha$ for each α. Suppose for every uncountable $X \subseteq A$ there is some $\beta < \omega_1$ such that

$$\{X\} \cup \{B_\alpha : \alpha \in B \setminus \beta\}$$

has the finite intersection property. Then there is an uncountable $X \subseteq A$ such that $(X \cap \alpha) \setminus B_\alpha$ is finite for every $\alpha \in B$.

The conjunction $MA_{\aleph_1}$ + TOP has several interesting consequences; for example, it implies the non-existence of S-spaces, the partition relation $\omega_1 \to (\omega_1, \alpha)^2$ for each $\alpha < \omega_1$, and that if $(D, \leq_D)$ is a directed set of size $\aleph_1$ and every uncountable subset of D contains a countable unbounded set, then there is an uncountable subset X of D such that every infinite subset of X is unbounded (s. [8]).

[9]In fact, the main reason we isolated the property of being a Příkrý-type forcing notions with stems in ω_1 was to accommodate $\mathbb{Q}_{\vec{C}}$ (in the presence of CH).

Lemma 2.12. *Suppose A and $\vec{B} = (B_\alpha : \alpha \in B)$ are as in the statement of TOP. Then there is a forcing notion $\mathbb{Q}_{A,\vec{B}}$ adding X as in the conclusion of TOP for A and $\vec{B}$ and such that $\mathbb{Q}_{A,\vec{B}}$ is of the form $\mathbb{Q}_0 * \underset{\sim}{\mathbb{Q}}_1 * \underset{\sim}{\mathbb{Q}}_2$, where*

1. *$\mathbb{Q}_0$ is $\mathrm{Add}(\omega, \omega_1)$, i.e., the standard forcing for adding $\aleph_1$-many Cohen reals,*
2. *$\underset{\sim}{\mathbb{Q}}_1$ is an $\mathrm{Add}(\omega, \omega_1)$-name for the standard forcing for adding a club diagonalizing the club filter on ω_1 in $\mathbf{V}[\dot{G}_{\mathbb{Q}_0}]$, and*
3. *$\underset{\sim}{\mathbb{Q}}_2$ is a $\mathbb{Q}_0 * \underset{\sim}{\mathbb{Q}}_1$-name for a c.c.c. forcing of size $\aleph_1$.*

In particular, $\mathbb{Q}_{A,\vec{B}}$ has a dense subset which is a proper Příkrý-type forcing notion with stems in $\mathscr{H}(\aleph_1)$.

Proof. The first assertion of the lemma is proved in [8]. The properness of $\mathbb{Q}_{A,\vec{B}}$, and therefore of every dense suborder of $\mathbb{Q}_{A,\vec{B}}$, is immediate from the fact that it is a forcing iteration of three proper posets. Finally, to see that $\mathbb{Q}_{A,\vec{B}}$ has a dense subset in in the class of Příkrý-type forcing notions with stems in $\mathscr{H}(\aleph_1)$, we first note that the standard forcing for adding a club diagonalizing the club filter on ω_1 is in this class: this is the partial order of pairs (x, C), where

- x is a closed countable subset of ω_1 and
- C is a club of ω_1,

and where (x_1, C_1) extends (x_0, C_0) if

- x_1 is an end-extension of x_0,
- $C_1 \subseteq C_0$, and
- $x_1 \setminus x_0 \subseteq C_0$.

Then we observe that since $\mathbb{Q}_0$ is a c.c.c. partial order of size $\aleph_1$ and $\underset{\sim}{\mathbb{Q}}_1$ is forced to be in the class and such that $\mathrm{Res}(\underset{\sim}{\mathbb{Q}}_1)$ is the set of clubs of ω_1 in the extension, it follows that $\mathbb{Q}_0 * \underset{\sim}{\mathbb{Q}}_1$ is also in the class. Finally, since $\underset{\sim}{\mathbb{Q}}_2$ is forced to be a forcing of size $\aleph_1$ and

$\mathbb{Q}_0 * \underset{\sim}{\mathbb{Q}}_1$ is in the class, it is easy to see that $(\mathbb{Q}_0 * \underset{\sim}{\mathbb{Q}}_1) * \underset{\sim}{\mathbb{Q}}_2 = \mathbb{Q}_{A,\vec{B}}$ has a dense subset in the class. $\square$

The last combinatorial principle we will consider in this subsection concerns ideals $\mathcal{I} \subseteq [S]^{\leq\aleph_0}$ on some set S consisting of countable sets and containing all finite subsets of S. $\mathcal{I}$ is said to be a P-ideal in case for every sequence $(X_n)_{n<\omega}$ of members of $\mathcal{I}$ there is some $Y \in \mathcal{I}$ such that $X_n \setminus Y$ is finite for each n.

Todorčević's *P-ideal Dichotomy* is the statement that for every set S and every P-ideal $\mathcal{I} \subseteq [S]^{\leq\aleph_0}$ on S, either

(1) there is an uncountable $X \subseteq S$ such that $[X]^{\aleph_0} \subseteq \mathcal{I}$, or
(2) $S = \bigcup_{n\in\omega} X_n$ for some sequence $(X_n)_{n\in\omega}$ such that $X_n \cap I$ is finite for every $n < \omega$ and every $I \in \mathcal{I}$.

Given an ideal $\mathcal{I}$, $\mathcal{J} \subseteq \mathcal{I}$ is a *generating set of* $\mathcal{I}$ if $\mathcal{I} = \{X \,:\, X \subseteq Y \text{ for some } Y \in \mathcal{J}\}$. Also, we say that $\mathcal{J}$ *generates* $\mathcal{I}$. An ideal $\mathcal{I}$ on ω_1 is said to be $\aleph_1$-generated if there is $\mathcal{J}$, a generating set of $\mathcal{I}$, such that $|\mathcal{J}| = \aleph_1$.

Given sets X, Y, $X \subseteq^* Y$ means that $X \setminus Y$ is finite. An ω_1-*tower* is a sequence $\langle X_\alpha \,:\, \alpha < \omega\rangle$ of countable subsets of ω_1 such that $X_\alpha \subseteq^* X_\beta$ for all $\alpha < \beta$. It is clear that every $\aleph_1$-generated P-ideal is in fact generated by an ω_1-tower.

We have the following.

Lemma 2.13. *Suppose CH holds, $\mathcal{I} \subseteq [\omega_1]^{\leq\aleph_0}$ is a P-ideal, and $\vec{X} = \langle X_\alpha \,:\, \alpha < \omega_1\rangle$ is an ω_1-tower generating $\mathcal{I}$. Suppose ω_1 cannot be decomposed into countably many sets X such that $X \cap I$ is finite for each $I \in \mathcal{I}$. Let $\mathbb{Q}_{\vec{X}}$ be the poset consisting of pairs $p = (x_p, A_p)$ such that*

- *$x_p \in [\omega_1]^{\leq\aleph_0}$ and*
- *$A_p = [\omega_1]^{\aleph_1} \setminus B_p$ for some $B_p \in [[\omega_1]^{\aleph_1}]^{\leq\aleph_0}$,*

where $(x_q, A_q) \leq_{\mathbb{Q}_{\vec{X}}} (x_p, A_p)$ if and only if

- x_q *is an end-extension of* x_p *(i.e.,* $x_p \subseteq x_q$ *and* $x_q \cap \sup\{\alpha+1 : \alpha \in x_p\} = x_p$*),*
- $A_q \subseteq A_p$*, and*
- *for every* $X \in B_p$*,* $\{\xi \in X : x_q \setminus x_p \subseteq A_\xi\}$ *is countable and belongs to* B_q*.*

Then

(1) $\mathbb{Q}_{\vec{X}}$ *forces the existence of some* $X \in [\omega_1]^{\aleph_1}$ *such that* $[X]^{\aleph_0} \subseteq \mathcal{I}$*,*

(2) $\mathbb{Q}_{\vec{X}}$ *is proper, and*

(3) $\mathbb{Q}$ *is a Příkrý-type forcing with stems in* $\mathscr{H}(\aleph_1)$*.*

Proof. (1) and (2) are proved in [2] — albeit with the (complementary) presentation of the forcing given by (x_p, B_p) rather than (x_p, A_p). (3) is immediate by the presentation of $\mathbb{Q}_{\vec{X}}$. □

The following corollary is now a consequence from Lemmas 2.9, 2.10, 2.11, 2.12, and 2.13.

Corollary 2.14. *The following statements follow from* CH-Pr_{ω_1}-BPFA*.*

(1) *BA,*

(2) *OCA*$(\aleph_1)$*,*

(3) *Measuring,*

(4) *TOP,*

(5) *The P-ideal Dichotomy for* $\aleph_1$*-generated ideals on* ω_1*.*

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