"JUST THE MATHS"

UNIT NUMBER

6.2

COMPLEX NUMBERS 2
(The Argand Diagram)

by

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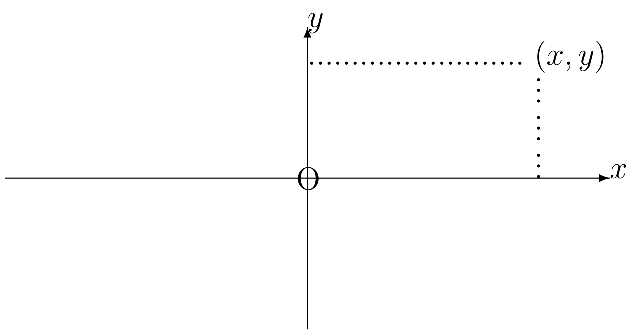
UNIT 6.2 - COMPLEX NUMBERS 2

THE ARGAND DIAGRAM

6.2.1 INTRODUCTION

It may be observed that a complex number $x + jy$ is completely specified if we know the values of x and y in the correct order. But the same is true for the cartesian co-ordinates, (x, y), of a point in two dimesions. There is therefore a **"one-to-one correspondence"** between the complex number $x + jy$ and the point with co-ordinates (x, y).

Hence it is possible to represent the complex number $x+jy$ by the point (x, y) in a geometrical diagram called the Argand Diagram:



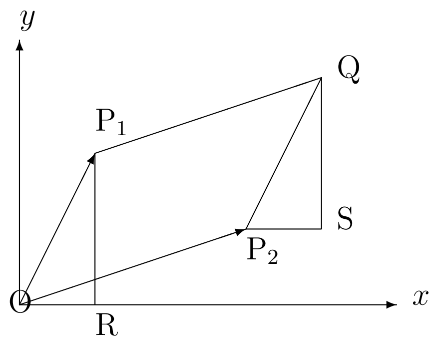
DEFINITIONS:

1. The x-axis is called the **"real axis"** since the points on it represent real numbers.
2. The y-axis is called the **"imaginary axis"** since the points on it represent purely imaginary numbers.

6.2.2 GRAPHICAL ADDITON AND SUBTRACTION

If two complex numbers, $z_1 = x_1 + jy_1$ and $z_2 = x_2 + jy_2$, are represented in the Argand Diagram by the points $\mathrm{P}_1(x_1, y_1)$ and $\mathrm{P}_2(x_2, y_2)$ respectively, then the sum, $z_1 + z_2$, of the complex numbers will be represented by the point $\mathrm{Q}(x_1 + x_2, y_1 + y_2)$.

If O is the origin, it is possible to show that Q is the fourth vertex of the parallelogram having OP_1 and OP_2 as adjacent sides.



In the diagram, the triangle ORP_1 has exactly the same shape as the triangle P_2SQ. Hence, the co-ordinates of Q must be $(x_1 + x_2, y_1 + y_2)$.

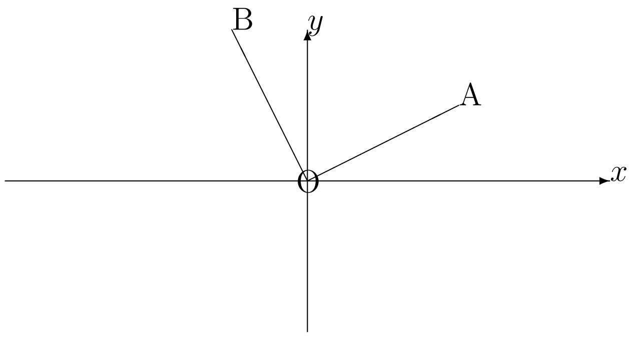
Note:
The difference $z_1 - z_2$ of the two complex numbers may similarly be found by completing the parallelogram of which two adjacent sides are the straight line segments joining the origin to the points with co-ordinates (x_1, y_1) and $(-x_2, -y_2)$.

6.2.3 MULTIPLICATION BY j OF A COMPLEX NUMBER

Given any complex number $z = x + jy$, we observe that

$$jz = j(x + jy) = -y + jx.$$

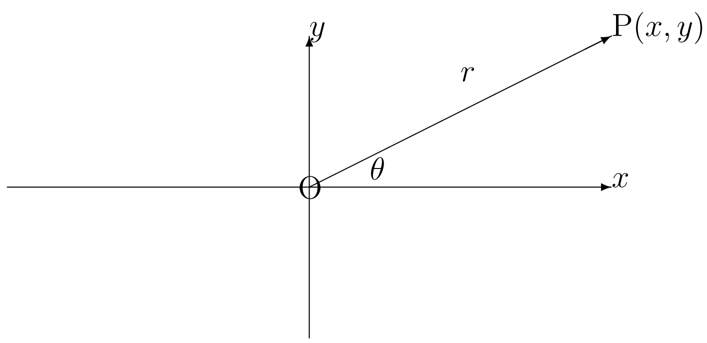
Thus, if z is represented in the Argand Diagram by the point with co-ordinates $A(x, y)$, then jz is represented by the point with co-ordinates $B(-y, x)$.



But OB is in the position which would be occupied by OA if it were rotated through 90° in a counter-clockwise direction.

We conclude that, in the Argand Diagram, multiplication by j of a complex number rotates, through 90° in a counter-clockwise direction, the straight line segment joining the origin to the point representing the complex number.

6.2.4 MODULUS AND ARGUMENT



(a) Modulus

If a complex number, $z = x + jy$ is represented in the Argand Diagram by the point, P, with cartesian co-ordinates (x, y) then the distance, r, of P from the origin is called the **"modulus"** of z and is denoted by either $|z|$ or $|x + jy|$.

Using the theorem of Pythagoras in the diagram, we conclude that

$$r = |z| = |x + jy| = \sqrt{x^2 + y^2}.$$

Note:
This definition of modulus is consistent with the definition of modulus for real numbers (which are included in the system of complex numbers). For any real number x, we may say that

$$|x| = |x + j0| = \sqrt{x^2 + 0^2} = \sqrt{x^2},$$

giving the usual numerical value of x.

ILLUSTRATIONS

1.
$$|3 - j4| = \sqrt{3^2 + (-4)^2} = \sqrt{25} = 5.$$

2.
$$|1 + j| = \sqrt{1^2 + 1^2} = \sqrt{2}.$$

3.
$$|j7| = |0 + j7| = \sqrt{0^2 + 7^2} = \sqrt{49} = 7.$$

Note:
The result of the last example above is obvious from the Argand Diagram since the point on the y-axis representing $j7$ is a distance of exactly 7 units from the origin. In the same way, a real number is represented by a point on the x-axis whose distance from the origin is the numerical value of the real number.

(b) Argument

The **"argument"** (or **"amplitude"**) of a complex number, z, is defined to be the angle, θ, which the straight line segment OP makes with the positive real axis (measuring θ positively from this axis in a counter-clockwise sense).

In the diagram,

$$\tan\theta = \frac{y}{x}; \quad \text{that is,} \quad \theta = \tan^{-1}\frac{y}{x}.$$

Note:
For a given complex number, there will be infinitely many possible values of the argument, any two of which will differ by a whole multiple of 360°. The complete set of possible values is denoted by Arg$z$, using an upper-case A.

The particular value of the argument which lies in the interval $-180° < \theta \leq 180°$ is called the **"principal value"** of the argument and is denoted by $\arg z$ using a lower-case a. The particular value, 180°, in preference to $-180°$, represents the principal value of the argument of a negative real number.

ILLUSTRATIONS

1.

$$\text{Arg}(\sqrt{3} + j) = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = 30° + k360°,$$

where k may be any integer. But we note that

$$\arg(\sqrt{3} + j) = 30° \text{ only.}$$

2.

$$\text{Arg}(-1 + j) = \tan^{-1}(-1) = 135° + k360°$$

but **not** $-45° + k360°$, since the complex number $-1 + j$ is represented by a point in the second quadrant of the Argand Diagram.

We note also that

$$\arg(-1 + j) = 135° \text{ only.}$$

3.

$$\text{Arg}(-1 - j) = \tan^{-1}(1) = 225° + k360° \text{ or } -135° + k360°$$

but **not** $45° + k360°$ since the complex number $-1 - j$ is represented by a point in the third quadrant of the Argand Diagram.

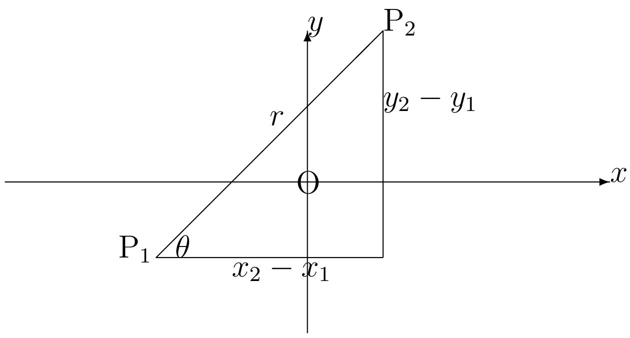
We note also that

$$\arg(-1 - j) = -135° \text{ only.}$$

Note:
It is worth mentioning here that, in the Argand Diagram, the directed straight line segment described from the point P_1 (representing the complex number $z_1 = x_1 + jy_1$) to the point P_2 (representing the complex number $z_2 = x_2 + jy_2$) has length, r, equal to $|z_2 - z_1|$, and is inclined to the positive direction of the real axis at an angle, θ, equal to $\arg(z_2 - z_1)$. This follows from the relationship

$$z_2 - z_1 = (x_2 - x_1) + j(y_2 - y_1)$$

in which $x_2 - x_1$ and $y_2 - y_1$ are the distances separating the two points, parallel to the real axis and the imaginary axis respectively.



6.2.5 EXERCISES

1. Determine the modulus (in decimals, where appropriate, correct to three significant figures) and the principal value of the argument (in degrees, correct to the nearest degree) of the following complex numbers:

   (a)
$$1 - j;$$

   (b)
$$-3 + j4;$$

   (c)
$$-\sqrt{2} - j\sqrt{2};$$

(d)

$$\frac{1}{2} - j\frac{\sqrt{3}}{2};$$

(e)

$$-7 - j9.$$

2. If $z = 4 - j5$, verify that jz has the same modulus as z but that the principal value of the argument of jz is greater, by 90° than the principal value of the argument of z.

3. Illustrate the following statements in the Argand Diagram:

   (a)

$$(6 - j11) + (5 + j3) = 11 - j8;$$

   (b)

$$(6 - j11) - (5 + j3) = -1 - j14.$$

6.2.6 ANSWERS TO EXERCISES

1. (a) 1.41 and $-45°$;
   (b) 5 and 127°;
   (c) 2 and $-135°$;
   (d) 1 and $-60°$;
   (e) 11.4 and $-128°$.

2. $4 - j5$ has modulus $\sqrt{41}$ and argument $-51°$;
   $j(4 - j5) = 5 + j4$ has modulus $\sqrt{41}$ and argument $39° = -51° + 90°$.

3. Construct the graphical sum and difference of the two complex numbers.