3

Continued Fractions

The continued fraction decomposition of real numbers grows naturally from the Euclidean algorithm, and continued fractions have been used in some form for thousands of years. One goal of this volume is to show how they relate to a natural action on a homogeneous space. To start there would be to willfully reverse their historical development: We start instead with their basic properties[38] from an elementary point of view in Section 3.1, then show how continued fractions are related to an explicit measure-preserving transformation in Section 3.2. In Chapter 9 we will see how the continued fraction map fits into the more general framework of actions on homogeneous spaces.

Let us mention one result proved in this chapter. We will show that for every irrational $x \in \mathbb{R}$ there is a sequence of 'best rational approximations' $\frac{p_n(x)}{q_n(x)} \in \mathbb{Q}$, defined by the continued fraction expansion of x. Moreover, for almost every x we have

$$\lim_{n\to\infty} \frac{1}{n} \log \left| x - \frac{p_n(x)}{q_n(x)} \right| \longrightarrow -\frac{\pi^2}{6\log 2},$$

which gives a precise description of the expected speed of approximation along this sequence.

3.1 Elementary Properties

A (simple) *continued fraction* is a formal expression of the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cdots}}}} \tag{3.1}$$

which we will also denote by

$$[a_0; a_1, a_2, a_3, \dots]$$

with $a_n \in \mathbb{N}$ for $n \geqslant 1$ and $a_0 \in \mathbb{N}_0$. Also write

$$[a_0; a_1, a_2, \dots, a_n]$$

for the finite fraction

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \dots + \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}}.$$

Thus, for example

$$[a_0; a_1, a_2, \dots, a_n] = a_0 + \frac{1}{[a_1; a_2, \dots, a_n]}.$$

We will see later that the expression in equation (3.1) – when suitably interpreted – converges, and therefore defines a real number. The numbers a_n are the *partial quotients* of the continued fraction. The following simple lemma is crucial for many of the basic properties of the continued fraction expansion.

Lemma 3.1. *Fix a sequence $(a_n)_{n \geqslant 0}$ with $a_0 \in \mathbb{N}_0$ and $a_n \in \mathbb{N}$ for $n \geqslant 1$. Then the rational numbers*

$$\frac{p_n}{q_n} = [a_0; a_1, a_2, \dots, a_n] \tag{3.2}$$

for $n \geqslant 0$ with coprime numerator $p_n \geqslant 1$ and denominator $q_n \geqslant 1$ can be found recursively from the relation

$$\begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix} = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} \text{ for } n \geqslant 0. \tag{3.3}$$

In particular, we set $p_{-1} = 1, q_{-1} = 0, p_0 = a_0,$ and $q_0 = 1$.

Proof. Notice first that the sequence $(a_n)_{n \geqslant 0}$ defines the sequences $(p_n)_{n \geqslant -1}$ and $(q_n)_{n \geqslant -1}$. The claim of the lemma is proved by induction on n. Assume that equation (3.3) holds for $0 \leqslant n \leqslant k-1$ and p_n, q_n as defined by equation (3.2) for any sequence $(a_0, a_1, \dots)$. This is clear for $n = 0$. Thus, on replacing the first k terms of the sequence $(a_n)_{n \geqslant 0}$ with the first k terms of the sequence $(a_n)_{n \geqslant 1}$, we have

$$\frac{x}{y} = [a_1; a_2, \dots, a_k]$$

as a fraction in lowest terms where x and y are defined by

$$\begin{pmatrix} x & x' \\ y & y' \end{pmatrix} = \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_k & 1 \\ 1 & 0 \end{pmatrix}.$$

Then

$$\begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x & x' \\ y & y' \end{pmatrix} = \begin{pmatrix} p_k & p_{k-1} \\ q_k & q_{k-1} \end{pmatrix} = \begin{pmatrix} a_0x + y & a_0x' + y' \\ x & x' \end{pmatrix},$$

so

$$\frac{p_k}{q_k} = \frac{a_0x + y}{x} = a_0 + \frac{y}{x} = a_0 + \frac{1}{[a_1; a_2, \dots, a_k]} = [a_0; a_1, \dots, a_k],$$

which shows that equation (3.2) holds for $n = k$ also. □

An immediate consequence of Lemma 3.1 is a pair of recursive formulas

$$p_{n+1} = a_{n+1}p_n + p_{n-1}$$

and

$$q_{n+1} = a_{n+1}q_n + q_{n-1} \tag{3.4}$$

for any $n \geqslant 1$, since

$$\begin{pmatrix} p_{n+1} & p_n \\ q_{n+1} & q_n \end{pmatrix} = \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix} \begin{pmatrix} a_{n+1} & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a_{n+1}p_n + p_{n-1} & p_n \\ a_{n+1}q_n + q_{n-1} & q_n \end{pmatrix}.$$

It follows that

$$1 = q_0 \leqslant q_1 < q_2 < \cdots \tag{3.5}$$

since $a_n \geqslant 1$ for all $n \geqslant 1$; by induction

$$q_n \geqslant 2^{(n-2)/2} \tag{3.6}$$

and similarly

$$p_n \geqslant 2^{(n-2)/2} \tag{3.7}$$

for all $n \geqslant 1$. Taking determinants in equation (3.3) shows that

$$p_nq_{n-1} - p_{n-1}q_n = (-1)^{n+1} \tag{3.8}$$

and hence $\frac{p_1}{q_1} = a_0 + \frac{1}{q_0q_1}$, $\frac{p_2}{q_2} = \frac{p_1}{q_1} - \frac{1}{q_1q_2} = a_0 + \frac{1}{q_0q_1} - \frac{1}{q_1q_2}$ and

$$\begin{aligned} \frac{p_n}{q_n} &= \frac{p_{n-1}}{q_{n-1}} + (-1)^{n+1}\frac{1}{q_{n-1}q_n} \\ &= a_0 + \frac{1}{q_0q_1} - \frac{1}{q_1q_2} + \cdots + (-1)^{n+1}\frac{1}{q_{n-1}q_n} \end{aligned} \tag{3.9}$$

for all $n \geqslant 1$ by induction.

This shows that an infinite continued fraction is not just a formal object, it in fact converges to a real number. Namely,

$$\begin{aligned} u = [a_0; a_1, a_2, \dots] &= \lim_{n\to\infty} [a_0; a_1, \dots, a_n] \\ &= \lim_{n\to\infty} \frac{p_n}{q_n} = a_0 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{q_{n-1}q_n}, \end{aligned} \tag{3.10}$$

is always convergent (indeed, is absolutely convergent) by the inequality (3.6). Moreover, an immediate consequence of equation (3.10) and equation (3.5) is a sequence of inequalities describing how the continued fraction converges: if $a_n \in \mathbb{N}$ for $n \geqslant 1$ then

$$\frac{p_0}{q_0} < \frac{p_2}{q_2} < \dots < \frac{p_{2n}}{q_{2n}} < \dots < u < \dots < \frac{p_{2m+1}}{q_{2m+1}} < \dots < \frac{p_3}{q_3} < \frac{p_1}{q_1}. \tag{3.11}$$

We say that $[a_0; a_1, \dots]$ is the *continued fraction expansion* for u. The name suggests that the expansion is (almost) unique and that it always exists. We will see that in fact any irrational number u has a continued fraction expansion, and that it is unique (Lemmas 3.6 and 3.4).

The rational numbers $\frac{p_n}{q_n}$ are called the *convergents* of the continued fraction for u and they provide very rapid rational approximations to u. Indeed,

$$u - \frac{p_n}{q_n} = (-1)^n \left[\frac{1}{q_n q_{n+1}} - \frac{1}{q_{n+1}q_{n+2}} + \cdots \right] \tag{3.12}$$

so by equation (3.5) we have[39]

$$\left| u - \frac{p_n}{q_n} \right| < \frac{1}{q_n q_{n+1}}. \tag{3.13}$$

By equation (3.4) we deduce that

$$\left| u - \frac{p_n}{q_n} \right| < \frac{1}{a_{n+1} q_n^2} \leqslant \frac{1}{q_n^2}. \tag{3.14}$$

Recall from Section 1.5 that we write

$$\langle t \rangle = \min_{q\in\mathbb{Z}} |t - q|$$

for the distance from t to the nearest integer. The inequality (3.14) gives one explanation* for the comment made on p. 12: using the fact that any irrational has a continued fraction expansion, it follows that for any real number u, there is a sequence (q_n) with $q_n \to \infty$ such that $q_n \langle q_n u \rangle < 1$.

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* This can also be seen more directly as a consequence of the Dirichlet principle (see Exercise 3.1.3).

Lemma 3.2. *Let $a_n \in \mathbb{N}$ for all $n \geqslant 0$. Then the limit in equation* (3.10) *is irrational.*

Proof. Suppose that $u = \frac{a}{b} \in \mathbb{Q}$. Then, by equation (3.14),

$$|q_n a - bp_n| < \frac{b}{a_{n+1}q_n} \leqslant \frac{b}{q_n}.$$

Since $q_n \to \infty$ by the inequality (3.6) and $q_n a - bp_n \in \mathbb{Z}$ we see that

$$q_n a - bp_n = 0$$

and hence $u = \frac{a}{b} = \frac{p_n}{q_n}$ for large enough n. However, by Lemma 3.1 p_n and q_n are coprime, so this contradicts the fact that $q_n \to \infty$ as $n \to \infty$. Thus u is irrational. □

The continued fraction convergents to a given irrational not only provide good rational approximants. In fact, they provide *optimal* rational approximants in the following sense (see Exercise 3.1.4).

Proposition 3.3. *Let $u = [a_0; a_1, \dots] \in \mathbb{R} \smallsetminus \mathbb{Q}$ as in equation* (3.10). *For any $n > 1$ and p, q with $0 < q \leqslant q_n$, if $\frac{p}{q} \neq \frac{p_n}{q_n}$, then*

$$|p_n - q_n u| < |p - qu|.$$

In particular,

$$\left|\frac{p_n}{q_n} - u\right| < \left|\frac{p}{q} - u\right|.$$

Proof. Note that $|p_n - q_n u| < |p - qu|$ and $0 < q \leqslant q_n$ together imply that

$$\frac{1}{q}\left|\frac{p_n}{q_n} - u\right| < \frac{1}{q_n}\left|\frac{p}{q} - u\right| \leqslant \frac{1}{q}\left|\frac{p}{q} - u\right|,$$

giving the second statement of the proposition. It is enough therefore to prove the first inequality. Recall from equation (3.13) that

$$\left|u - \frac{p_n}{q_n}\right| < \frac{1}{q_n q_{n+1}}$$

and

$$\left|u - \frac{p_{n+1}}{q_{n+1}}\right| < \frac{1}{q_{n+1}q_{n+2}}.$$

By the alternating behavior of the convergents in equation (3.11), each of the three bracketed expressions in the identity

$$\left(u - \frac{p_n}{q_n}\right) = \left(\frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n}\right) - \left(\frac{p_{n+1}}{q_{n+1}} - u\right)$$

is positive (if n is even) or negative (if n is odd). It follows that

$$\left|u - \frac{p_n}{q_n}\right| = \left|\frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n}\right| - \left|\frac{p_{n+1}}{q_{n+1}} - u\right|,$$

so

$$\left|u - \frac{p_n}{q_n}\right| > \frac{1}{q_n q_{n+1}} - \frac{1}{q_{n+1} q_{n+2}} = \frac{q_{n+2} - q_n}{q_n q_{n+1} q_{n+2}} = \frac{a_{n+2}}{q_n q_{n+2}}$$

by equations (3.4) and (3.14). It follows that

$$\frac{1}{q_{n+2}} < |p_n - q_n u| < \frac{1}{q_{n+1}} \tag{3.15}$$

for $n \geqslant 1$.

By the inequalities (3.15),

$$|q_n u - p_n| < \frac{1}{q_{n+1}} < |q_{n-1} u - p_{n-1}|$$

so we may assume that $q_{n-1} < q \leqslant q_n$ (if not, use downwards induction on n).

If $q = q_n$, then $|\frac{p_n}{q_n} - \frac{p}{q}| \geqslant \frac{1}{q_n}$, while

$$\left|\frac{p_n}{q_n} - u\right| < \frac{1}{q_n q_{n+1}} \leqslant \frac{1}{2q_n},$$

since $q_{n+1} \geqslant 2$ for all $n \geqslant 1$. Therefore,

$$\left|\frac{p}{q} - u\right| \geqslant \frac{1}{2q_n} = \frac{1}{2q}$$

and so $|q_n u - p_n| < |qu - p|$.

Assume now that $q_{n-1} < q < q_n$ and write

$$\begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix},$$

so that $a, b \in \mathbb{Z}$ by equation (3.8). Clearly $ab \neq 0$ since otherwise $q = q_{n-1}$ or $q = q_n$. Now $q = aq_n + bq_{n-1} < q_n$, so $ab < 0$; by equation (3.11) we also know that $p_n - q_n u$ and $p_{n-1} - q_{n-1} u$ are of opposite signs. It follows that $a(p_n - q_n u)$ and $b(p_{n-1} - q_{n-1} u)$ are of the same sign, so the fact that

$$p - qu = a(p_n - q_n u) + b(p_{n-1} - q_{n-1} u)$$

implies that

$$|p - qu| > |p_{n-1} - q_{n-1} u| > |p_n - q_n u|$$

as required. □

We end this section with the uniqueness of the continued fraction expansion.

Lemma 3.4. *The map that sends the sequence*

$$(a_0, a_1, \dots) \in \mathbb{N}_0 \times \mathbb{N}^{\mathbb{N}}$$

to the limit in equation (3.10) *is injective.*

PROOF. Let $u = (a_0, a_1, \dots) \in \mathbb{N}_0 \times \mathbb{N}^{\mathbb{N}}$ be given. Then it is clear that

$$u = [a_0; a_1, \dots]$$

is positive. Applying this to $(a_1, a_2, \dots)$ and the inductive relation

$$u = a_0 + \frac{1}{[a_1; a_2, \dots]}$$

we see that

$$u \in (a_0, a_0 + \tfrac{1}{a_1}) \subseteq (a_0, a_0 + 1).$$

It follows that u uniquely determines a_0. Using the inductive relation again, we have

$$[a_1; a_2, \dots] = \frac{1}{u - a_0},$$

which by the argument above shows that u uniquely determines a_1. Iterating the procedure shows that all the terms in the continued fraction can be reconstructed from u. □

The argument used in the proof of Lemma 3.4 also suggests a way to find the continued fraction expansion of a given irrational number $u \in \mathbb{R} \smallsetminus \mathbb{Q}$. This will be pursued further in the next section.

Exercises for Section 3.1

Exercise 3.1.1. Show that any positive rational number has exactly two continued fraction expansions, both of which are finite.

Exercise 3.1.2. Show that a continued fraction in which some of the digits are allowed to be zero (but that is not allowed to end with infinitely many zeros) can always be rewritten with digits in $\mathbb{N}$.

Exercise 3.1.3. [Dirichlet principle] For a given $u \in \mathbb{R}$ and $n \geqslant 1$ consider the points $0, u, 2u, \dots, nu$ (mod 1) as elements of the circle $\mathbb{T}$. Show that for some k, $0 < k < n$ we have $\langle ku \rangle \leqslant \frac{1}{n}$, and deduce that there exists a sequence $q_n \to \infty$ with $q_n \langle q_n u \rangle < 1$.

Exercise 3.1.4. Extend Proposition 3.3 in the following way. Given u as in equation (3.10), and the nth convergent $\frac{p_n}{q_n}$, the $(n+1)$th convergent $\frac{p_{n+1}}{q_{n+1}}$ is characterized by being the ratio of the unique pair of positive integers (p_{n+1}, q_{n+1}) for which $|p_{n+1} - q_{n+1} u| < |p_n - q_n u|$ and for which $q_{n+1} > q_n$

is minimal. Notice that the same cannot be said when using the expression $\left|u - \frac{p_n}{q_n}\right|$, as becomes clear in the case where $u > \frac{1}{3}$ is very close to $\frac{1}{3}$, in which case the first approximation is not $\frac{1}{2}$.

Exercise 3.1.5. Let $u = [a_0; a_1, \dots]$ with convergents $\frac{p_n}{q_n}$. Show that

$$\frac{1}{2q_{n+1}} \leqslant |p_n - q_n u| < \frac{1}{q_{n+1}}.$$

3.2 The Continued Fraction Map and the Gauss Measure

Let $Y = [0,1] \smallsetminus \mathbb{Q}$, and define a map $T : Y \to Y$ by

$$T(x) = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor,$$

where $\lfloor t \rfloor$ denotes the greatest integer less than or equal to t. Thus $T(x)$ is the fractional part $\left\{\frac{1}{x}\right\}$ of $\frac{1}{x}$. The graph of this so-called *continued fraction* or *Gauss map* is shown in Figure 3.1.

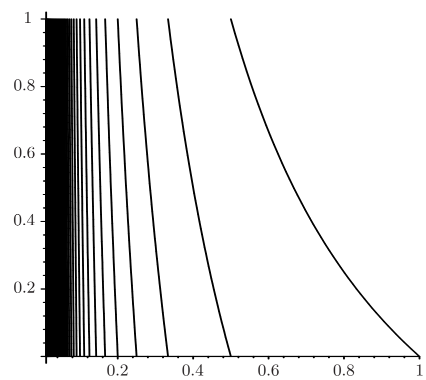

Fig. 3.1. The Gauss map.

Gauss observed in 1845 that T preserves[40] the probability measure given by

$$\mu(A) = \frac{1}{\log 2} \int_A \frac{1}{1+x} \, \mathrm{d}x,$$

by showing that the Lebesgue measure of $T^{-n} I$ converges to $\mu(I)$ for each interval I.

This map will be studied via a geometric model (for its invertible extension) in Chapter 9; in this section we assemble some basic facts from an elementary point of view, showing that the Gauss measure is T-invariant and ergodic. Since the measure defined in Lemma 3.5 is non-atomic, we may extend the map to include the points 0 and 1 in any way without affecting the measurable structure of the system.

Lemma 3.5. *The continued fraction map $T(x) = \left\{\frac{1}{x}\right\}$ on $(0,1)$ preserves the Gauss measure μ given by*

$$\mu(A) = \frac{1}{\log 2}\int_A \frac{1}{1+x}\,\mathrm{d}x$$

for any Borel measurable set $A \subseteq [0,1]$.

A geometric and less formal proof of this will be given on page 97 using basic properties of the invertible extension of the continued fraction map in Proposition 3.15.

Proof of Lemma 3.5. It is sufficient to show that $\mu\left(T^{-1}[0,s]\right) = \mu\left([0,s]\right)$ for every $s > 0$. Clearly

$$T^{-1}[0,s] = \{x \mid 0 \leqslant T(x) \leqslant s\} = \bigsqcup_{n=1}^{\infty}\left[\frac{1}{s+n}, \frac{1}{n}\right]$$

is a disjoint union. It follows that

$$\begin{aligned}
\mu\left(T^{-1}[0,s]\right) &= \frac{1}{\log 2}\sum_{n=1}^{\infty}\int_{1/(s+n)}^{1/n}\frac{1}{1+x}\,\mathrm{d}x \\
&= \frac{1}{\log 2}\sum_{n=1}^{\infty}\Big(\log(1+\tfrac{1}{n}) - \log(1+\tfrac{1}{s+n})\Big) \\
&= \frac{1}{\log 2}\sum_{n=1}^{\infty}\Big(\log(1+\tfrac{s}{n}) - \log(1+\tfrac{s}{n+1})\Big) \qquad (3.16) \\
&= \frac{1}{\log 2}\sum_{n=1}^{\infty}\int_{s/(n+1)}^{s/n}\frac{1}{1+x}\,\mathrm{d}x \\
&= \mu\left([0,s]\right),
\end{aligned}$$

completing the proof. The identity used in equation (3.16) amounts to

$$\frac{1+\frac{s}{n}}{1+\frac{s}{n+1}} = \frac{1+\frac{1}{n}}{1+\frac{1}{s+n}},$$

which may be seen by multiplying numerator and denominator of the left-hand side by $\frac{n+1}{n+s}$, and the interchange of integral and sum is justified by absolute convergence. □

Thus Lemma 3.5 shows that $\left([0,1],\mathscr{B}_{[0,1]},\mu,T\right)$ is a measure-preserving system.

Define for $x\in Y=[0,1]\smallsetminus\mathbb{Q}$ and $n\geqslant 1$ the sequence of natural numbers $(a_n)=(a_n(x))$ by

$$\frac{1}{1+a_n}<T^{n-1}(x)<\frac{1}{a_n}, \tag{3.17}$$

or equivalently by

$$a_n(x)=\left\lfloor\frac{1}{T^{n-1}x}\right\rfloor\in\mathbb{N}. \tag{3.18}$$

For any sequence $(a_n)_{n\geqslant 1}$ of natural numbers we define the continued fraction $[a_1,a_2,\dots]$ just as in equation (3.1) with $a_0=0$.

Lemma 3.6. *For any irrational $x\in[0,1]\smallsetminus\mathbb{Q}$ the sequence $(a_n(x))$ defined in equation* (3.18) *gives the digits of the continued fraction expansion to x. That is,*

$$x=[a_1(x),a_2(x),\dots].$$

Proof. Define $a_n=a_n(x)$ and let $u=[a_1,a_2,\dots]$ be the limit as in equation (3.10) with $a_0=0$. By equation (3.11) we have

$$\frac{p_{2n}}{q_{2n}}<u<\frac{p_{2n+1}}{q_{2n+1}}$$

and by equation (3.8) and the inequality (3.6) we have

$$\frac{p_{2n+1}}{q_{2n+1}}-\frac{p_{2n}}{q_{2n}}=\frac{1}{q_{2n}q_{2n-1}}\leqslant\frac{1}{2^{2n-2}}.$$

We now show by induction that

$$[a_1,\dots,a_{2n}]=\frac{p_{2n}}{q_{2n}}<x<\frac{p_{2n+1}}{q_{2n+1}}=[a_1,\dots,a_{2n+1}], \tag{3.19}$$

which together with the above shows that $u=x$.

Recall that $\frac{p_0}{q_0}=0$ and $\frac{p_1}{q_1}=\frac{1}{a_1}$, so equation (3.19) holds for $n=0$ because of the definition of a_1 in equation (3.18). Now assume that the inequality (3.19) holds for a given n and all $x\in[0,1]$. In particular, we may apply it to $T(x)$ to get

$$[a_2,\dots,a_{2n+1}]<T(x)<[a_2,\dots,a_{2n+2}].$$

Since $T(x)=\frac{1}{x}-a_1$ we get

$$a_1+[a_2,\dots,a_{2n+1}]<\frac{1}{x}<a_1+[a_2,\dots,a_{2n+2}]$$

and therefore

$$[a_1, \dots, a_{2n+2}] = \frac{1}{a_1 + [a_2, \dots, a_{2n+2}]} < x,$$

$$x < \frac{1}{a_1 + [a_2, \dots, a_{2n+1}]} = [a_1, \dots, a_{2n+1}]$$

as required. □

This gives a description of the continued fraction map as a shift map: the list of digits in the continued fraction expansion of $x \in [0,1] \smallsetminus \mathbb{Q}$ defines a unique element of $\mathbb{N}^{\mathbb{N}}$, and the diagram

$$\begin{array}{ccc} \mathbb{N}^{\mathbb{N}} & \xrightarrow{\ \sigma\ } & \mathbb{N}^{\mathbb{N}} \\ \downarrow & & \downarrow \\ (0,1) & \xrightarrow[T]{} & (0,1) \end{array}$$

commutes, where σ is the left shift and the vertical map sends a sequence of digits $(a_n)_{n \geqslant 1}$ to the real irrational number defined by the continued fraction expansion.

In Corollary 3.8 we will draw some easy consequences[41] of ergodicity for the Gauss measure μ in terms of properties of the continued fraction expansion for almost every real number. Given a continued fraction expansion, recall that the *convergents* are the terms of the sequence of rationals $\frac{p_n(x)}{q_n(x)}$ in lowest terms defined by

$$\frac{p_n(x)}{q_n(x)} = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots + \cfrac{1}{a_n}}}}.$$

Theorem 3.7. *The continued fraction map* $T(x) = \{\frac{1}{x}\}$ *on* $(0,1)$ *is ergodic with respect to the Gauss measure* μ.

Before proving this[42] we develop some more of the basic identities for continued fractions. Given a continued fraction expansion $u = [a_0; a_1, \dots]$ of an irrational number u, we write $u_n = [a_n; a_{n+1}, \dots]$ for the nth tail of the expansion. By Lemma 3.1 applied twice, we have

$$\begin{aligned} \begin{pmatrix} p_{n+k} \\ q_{n+k} \end{pmatrix} &= \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_{n+k} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix} \begin{pmatrix} a_{n+1} & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_{n+k} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \end{aligned}$$

Writing $p_k(u_{n+1})$ and $q_k(u_{n+1})$ for the numerator and denominator of the kth convergents to u_{n+1}, we can apply Lemma 3.1 again to deduce that

$$\begin{pmatrix} p_{n+k} \\ q_{n+k} \end{pmatrix} = \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix} \begin{pmatrix} p_{k-1}(u_{n+1}) & p_{k-2}(u_{n+1}) \\ q_{k-1}(u_{n+1}) & q_{k-2}(u_{n+1}) \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

so

$$\frac{p_{n+k}}{q_{n+k}} = \frac{p_n \frac{p_{k-1}(u_{n+1})}{q_{k-1}(u_{n+1})} + p_{n-1}}{q_n \frac{p_{k-1}(u_{n+1})}{q_{k-1}(u_{n+1})} + q_{n-1}},$$

which gives

$$u = \frac{p_n u_{n+1} + p_{n-1}}{q_n u_{n+1} + q_{n-1}} \tag{3.20}$$

in the limit as $k \to \infty$. Notice that the above formulas are derived for a general positive irrational number u. If $u = [a_1, \dots] \in (0,1)$, then $u_{n+1} = (T^n(u))^{-1}$ so that

$$u = \frac{p_n + p_{n-1}T^n(u)}{q_n + q_{n-1}T^n(u)}. \tag{3.21}$$

Proof of Theorem 3.7. The description of the continued fraction map as a shift on the space $\mathbb{N}^{\mathbb{N}}$ described above suggests the method of proof: the measure μ corresponds to a rather complicated measure on the shift space, but if we can control the measure of cylinder sets (and their intersections) well enough then we may prove ergodicity along the lines of the proof of ergodicity for Bernoulli shifts in Proposition 2.15. For two expressions f, g we write $f \asymp g$ to mean that there are absolute constants $C_1, C_2 > 0$ such that

$$C_1 f \leqslant g \leqslant C_2 f.$$

Given a vector $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{N}^n$ of length $|\mathbf{a}| = n$, define a set

$$I(\mathbf{a}) = \{[x_1, x_2, \dots] \mid x_i = a_i \text{ for } 1 \leqslant i \leqslant n\}$$

(which may be thought of as an interval in $(0,1)$, or as a cylinder set in $\mathbb{N}^{\mathbb{N}}$).

The main step towards the proof of the theorem is to show that

$$\mu\left(T^{-n}A \cap I(\mathbf{a})\right) \asymp \mu(A)\mu(I(\mathbf{a})) \tag{3.22}$$

for any measurable set A. Notice that for the proof of equation (3.22) it is sufficient to show it for any interval $A = [d, e]$; the case of a general Borel set then follows by a standard approximation argument (the set of Borel sets satisfying equation (3.22) with a fixed choice of constants is easily seen to be a monotone class, so Theorem A.4 may be applied.)

Now define $\frac{p_n}{q_n} = [a_1, \dots, a_n]$ and $\frac{p_{n-1}}{q_{n-1}} = [a_1, \dots, a_{n-1}]$. Then $u \in I(\mathbf{a})$ if and only if $u = [a_1, \dots, a_n, a_{n+1}(u), \dots]$, and so $u \in I(\mathbf{a}) \cap T^{-n}A$ if and only if u can be written as in equation (3.21), with $T^n(u) \in A = [d, e]$. As T^n restricted to $I(\mathbf{a})$ is continuous and monotone (increasing if n is even, and decreasing if n is odd), it follows that $I(\mathbf{a}) \cap T^{-n}A$ is an interval with endpoints given by

$$\frac{p_n + p_{n-1}d}{q_n + q_{n-1}d}$$

and

$$\frac{p_n + p_{n-1}e}{q_n + q_{n-1}e}.$$

Thus the Lebesgue measure of $I(\mathbf{a}) \cap T^{-n}A$,

$$\left| \frac{p_n + p_{n-1}d}{q_n + q_{n-1}d} - \frac{p_n + p_{n-1}e}{q_n + q_{n-1}e} \right|,$$

expands to

$$\begin{aligned}
&\left| \frac{(p_n + p_{n-1}d)(q_n + q_{n-1}e) - (p_n + p_{n-1}e)(q_n + q_{n-1}d)}{(q_n + q_{n-1}d)(q_n + q_{n-1}e)} \right| \\
&= \left| \frac{p_n q_{n-1} e + p_{n-1} q_n d - p_n q_{n-1} d - p_{n-1} q_n e}{(q_n + q_{n-1}d)(q_n + q_{n-1}e)} \right| \\
&= (e-d) \frac{|p_n q_{n-1} - p_{n-1} q_n|}{(q_n + q_{n-1}e)(q_n + q_{n-1}f)} = (e-d) \frac{1}{(q_n + q_{n-1}e)(q_n + q_{n-1}f)}
\end{aligned}$$

by equation (3.8). On the other hand, the Lebesgue measure of $I(\mathbf{a})$ is

$$\left| \frac{p_n}{q_n} - \frac{p_n + p_{n-1}}{q_n + q_{n-1}} \right| = \frac{|p_n q_{n-1} - p_{n-1} q_n|}{q_n(q_n + q_{n-1})} = \frac{1}{q_n(q_n + q_{n-1})} \tag{3.23}$$

again by equation (3.8), which implies that

$$\begin{aligned}
m(I(\mathbf{a}) \cap T^{-n}A) &= m(A) m(I(\mathbf{a})) \frac{q_n(q_n + q_{n-1})}{(q_n + q_{n-1}e)(q_n + q_{n-1}f)} \\
&\asymp m(A) m(I(\mathbf{a})),
\end{aligned} \tag{3.24}$$

where m denotes Lebesgue measure on $(0,1)$. Next notice that

$$\frac{m(B)}{2\log 2} \leqslant \mu(B) \leqslant \frac{m(B)}{\log 2}$$

for any Borel set $B \subseteq (0,1)$, which together with equation (3.24) gives equation (3.22).

Now assume that $A \subseteq (0,1)$ is a Borel set with $T^{-1}A = A$. For such a set, the estimate in equation (3.22) reads as

$$\mu(A \cap I(\mathbf{a})) \asymp \mu(A)\mu(I(\mathbf{a}))$$

for any interval $I(\mathbf{a})$ defined by $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{N}^n$ and any n. However, for a fixed n the intervals $I(\mathbf{a})$ partition $(0,1)$ (as $\mathbf{a}$ varies in $\mathbb{N}^n$), and by equation (3.23)

$$\begin{aligned}
\operatorname{diam}(I(\mathbf{a})) &= \frac{1}{q_n(q_n + q_{n-1})} \\
&\leqslant \frac{1}{2^{n-2}} \qquad \text{(by (3.6))},
\end{aligned}$$

so the lengths of the sets in this partition shrink to zero uniformly as $n \to \infty$. Therefore, the intervals $I(\mathbf{a})$ generate the Borel σ-algebra, and so

$$\mu(A \cap B) \asymp \mu(A)\mu(B)$$

for any Borel subset $B \subseteq (0,1)$ (again by Theorem A.4). We apply this to the set $B = (0,1) \smallsetminus A$ and obtain $0 \asymp \mu(A)\mu(B)$, which shows that either $\mu(A) = 0$ or $\mu((0,1) \smallsetminus A) = 0$, as needed. □

We will use the ergodicity of the Gauss map in Corollary 3.8 to deduce statements about the digits of the continued fraction expansion of a typical real number. Just as Borel's normal number theorem (Example 1.2) gives precise statistical information about the decimal expansion of almost every real number, ergodicity of the Gauss map gives precise statistical information about the continued fraction digits of almost every real number. Of course the form of the conclusion is necessarily different. For example, since there are infinitely many different digits in the continued fraction expansion, they cannot all occur with equal frequency, and equation (3.25) makes precise the way in which small digits occur more frequently than large ones. We also obtain information on the geometric and arithmetic mean of the digits a_n in equations (3.26) and (3.27), the growth rate of the denominators q_n in equation (3.28), and the rate at which the convergents $\frac{p_n}{q_n}$ approximate a typical real number in equation (3.29).

In particular, equations (3.28) and (3.29) together say that the digit a_{n+1} appearing in the estimate (3.14) does not affect the logarithmic rate of approximation of an irrational by the continued fraction partial quotients significantly.

Corollary 3.8. *For almost every real number $x = [a_1, a_2, \dots] \in (0,1)$, the digit j appears in the continued fraction with density*

$$\frac{2\log(1+j) - \log j - \log(2+j)}{\log 2}, \tag{3.25}$$

$$\lim_{n\to\infty} (a_1 a_2 \dots a_n)^{1/n} = \prod_{a=1}^{\infty} \left(\frac{(a+1)^2}{a(a+2)} \right)^{\log a/\log 2}, \tag{3.26}$$

$$\lim_{n\to\infty} \frac{1}{n}(a_1 + a_2 + \cdots + a_n) = \infty, \tag{3.27}$$

$$\lim_{n\to\infty} \frac{1}{n} \log q_n(x) = \frac{\pi^2}{12\log 2}, \tag{3.28}$$

and

$$\lim_{n\to\infty} \frac{1}{n} \log \left| x - \frac{p_n(x)}{q_n(x)} \right| \longrightarrow -\frac{\pi^2}{6\log 2}. \tag{3.29}$$

Proof. The digit j appears in the first N digits with frequency

$$\begin{aligned}\frac{1}{N}|\{i \mid i \leqslant N, a_i = j\}| &= \frac{1}{N}|\{i \mid i \leqslant N, T^i x \in (\tfrac{1}{j+1}, \tfrac{1}{j})\}| \\ &\to \frac{1}{\log 2}\int_{1/(j+1)}^{1/j} \frac{1}{1+y}\,\mathrm{d}y \\ &= \frac{2\log(1+j) - \log j - \log(2+j)}{\log 2},\end{aligned}$$

which proves equation (3.25).

Define a function f on $(0,1)$ by $f(x) = \log a$ for $x \in \left(\frac{1}{a+1}, \frac{1}{a}\right)$. Then

$$\begin{aligned}\int_0^1 f(x)\,\mathrm{d}x &= \sum_{a=1}^{\infty}\left(\frac{1}{a} - \frac{1}{a+1}\right)\log a \\ &\leqslant \sum_{a=1}^{\infty}\frac{1}{a^2}\log a < \infty,\end{aligned}$$

so $\int_0^1 f\,\mathrm{d}\mu < \infty$ also, since the density $\frac{\mathrm{d}\mu}{\mathrm{d}x} = \frac{1}{(1+x)\log 2}$ is bounded on $[0,1]$. By the pointwise ergodic theorem (Theorem 2.30) we therefore have, for almost every x,

$$\frac{1}{n}\sum_{j=0}^{n-1}\log a_j = \frac{1}{n}\sum_{j=0}^{n-1} f(T^j x) \longrightarrow \int f(x)\,\mathrm{d}\mu.$$

This shows equation (3.26) since

$$\int_0^1 f\,\mathrm{d}\mu = \sum_{a=1}^{\infty}\frac{\log a}{\log 2}\int_{1/(1+a)}^{1/a}\frac{1}{1+x}\,\mathrm{d}x.$$

Now consider the function $g(x) = \mathrm{e}^{f(x)}$ (so $g(x) = a_1$ is the first digit in the continued fraction expansion of x). We have

$$\frac{1}{n}(a_1 + \cdots + a_n) = \frac{1}{n}\sum_{j=0}^{n-1} g(T^j x),$$

but the pointwise ergodic theorem cannot be applied to g since $\int_0^1 g\,\mathrm{d}\mu = \infty$ (the result needed is Exercise 2.6.5(2); the argument here shows how to do this exercise). However, for any fixed N the truncated function

$$g_N(x) = \begin{cases} g(x) & \text{if } g(x) \leqslant N; \\ 0 & \text{if not}\end{cases}$$

is in L^1_μ since

$$\int g_N \,\mathrm{d}\mu = \frac{1}{\log 2}\sum_{a=1}^{N}\int_{1/(a+1)}^{1/a} a\,\mathrm{d}x = \frac{1}{\log 2}\sum_{a=1}^{N}\frac{1}{a+1}.$$

Notice that $\int_0^1 g_N\,\mathrm{d}\mu \to \infty$ as $N \to \infty$. By the ergodic theorem,

$$\begin{aligned}\liminf_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1} g(T^j x) &\geqslant \lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1} g_N(T^j x)\\ &= \int_0^1 g_N\,\mathrm{d}\mu \to \infty\end{aligned}$$

as $N \to \infty$, showing equation (3.27).

The proofs of (3.25) and (3.26) were straightforward applications of the ergodic theorem, and (3.27) only required a simple extension to measurable functions. Proving (3.28) and (3.29) takes a little more effort.

First notice that

$$\begin{aligned}\frac{p_n(x)}{q_n(x)} &= \frac{1}{a_1 + [a_2,\dots,a_n]}\\ &= \frac{1}{a_1 + \frac{p_{n-1}(Tx)}{q_{n-1}(Tx)}}\\ &= \frac{q_{n-1}(Tx)}{p_{n-1}(Tx) + q_{n-1}(Tx)a_1},\end{aligned}$$

so $p_n(x) = q_{n-1}(Tx)$ since the convergents are in lowest terms. Recall that we always have $p_1 = q_0 = 1$. It follows that

$$\frac{1}{q_n(x)} = \frac{p_n(x)}{q_n(x)}\cdot\frac{p_{n-1}(Tx)}{q_{n-1}(Tx)}\cdots\frac{p_1(T^{n-1}x)}{q_1(T^{n-1}x)},$$

so

$$-\frac{1}{n}\log q_n(x) = \frac{1}{n}\sum_{j=0}^{n-1}\log\left[\frac{p_{n-j}(T^j x)}{q_{n-j}(T^j x)}\right].$$

Let $h(x) = \log x$ (so $h \in L^1_\mu$). Then

$$-\frac{1}{n}\log q_n(x) = \underbrace{\frac{1}{n}\sum_{j=0}^{n-1} h(T^j x)}_{S_n} - \underbrace{\frac{1}{n}\sum_{j=0}^{n-1}\left[\log(T^j x) - \log\left(\frac{p_{n-j}(T^j x)}{q_{n-j}(T^j x)}\right)\right]}_{R_n}$$

gives a splitting of $-\frac{1}{n}\log q_n(x)$ into an ergodic average $S_n = \mathsf{A}_h^n$ and a remainder term R_n. By the ergodic theorem,

$$\lim_{n\to\infty}\frac{1}{n}S_n = \frac{1}{\log 2}\int_0^1 \frac{\log x}{1+x}\,\mathrm{d}x = -\frac{\pi^2}{12\log 2}.$$

To complete the proof of equation (3.28), we need to show that $\frac{1}{n}R_n \to 0$ as $n \to \infty$. This will follow from the observation that $\frac{p_{n-j}(T^j x)}{q_{n-j}(T^j x)}$ is a good approximation to $T^j x$ if $(n-j)$ is large enough. Recall from equations (3.7) and (3.6) that

$$p_k \geqslant 2^{(k-2)/2},\ q_k \geqslant 2^{(k-1)/2},$$

so, by using the inequality (3.13),

$$\left|\frac{x}{p_k/q_k} - 1\right| = \frac{q_k}{p_k}\left|x - \frac{p_k}{q_k}\right| \leqslant \frac{1}{p_k q_{k+1}} \leqslant \frac{1}{2^{k-1}}.$$

By using this together with the fact that $|\log u| \leqslant 2|u-1|$ whenever $u \in [\frac{1}{2}, \frac{3}{2}]$ (which applies in the sum below with $j \leqslant n-2$), we get

$$\begin{aligned}|R_n| &\leqslant \sum_{j=0}^{n-1}\left|\log \frac{T^j x}{p_{n-j}(T^j x)/q_{n-j}(T^j x)}\right| \\ &\leqslant \underbrace{2\sum_{j=0}^{n-2}\left|\frac{T^j x}{p_{n-j}(T^j x)/q_{n-j}(T^j x)} - 1\right|}_{T_n} + \underbrace{\left|\log \frac{T^{n-1}x}{p_1(T^{n-1}x)/q_1(T^{n-1}x)}\right|}_{U_n}.\end{aligned}$$

Now

$$T_n \leqslant \sum_{j=0}^{n-2} \frac{2}{2^{n-j-1}} \leqslant 2$$

for all n. For the second term, notice that

$$U_n = \left|\log\left[\left(T^{n-1}x\right) a_1\left(T^{n-1}x\right)\right]\right|,$$

and by the inequality (3.17) we have

$$1 \geqslant \left(T^{n-1}x\right) a_1\left(T^{n-1}x\right) \geqslant \frac{a_1\left(T^{n-1}x\right)}{1 + a_1\left(T^{n-1}x\right)} \geqslant \frac{1}{2}$$

since $a_1(T^{n-1}x) \geqslant 1$. Therefore,

$$|\log\left[\left(T^{n-1}x\right) a_1\left(T^{n-1}x\right)\right]| \leqslant \log 2,$$

which completes the proof that

$$\frac{1}{n}R_n \to 0$$

as $n \to \infty$, and hence shows equation (3.28).

Equation (3.29) follows from equation (3.28), since from the inequalities (3.13) and (3.15) we have

$$\log q_n + \log q_{n+1} \leqslant -\log\left|x - \frac{p_n}{q_n}\right| \leqslant \log q_n + \log q_{n+2}.$$

□

Exercises for Section 3.2

Exercise 3.2.1. Use the idea in the proof of equation (3.27) to extend the pointwise ergodic theorem (Theorem 2.30) to the case of a measurable function $f \geqslant 0$ with $\int_X f \, \mathrm{d}\mu = \infty$ without the assumption of ergodicity.

Exercise 3.2.2. Show that the map from $\mathbb{N}^{\mathbb{N}}$ to $[0,1] \smallsetminus \mathbb{Q}$ that sends $(a_1, a_2, \dots)$ to $[a_1, a_2, \dots]$ is a homeomorphism with respect to the discrete topology on $\mathbb{N}$ and the product topology on $\mathbb{N}^{\mathbb{N}}$.

Exercise 3.2.3. Let $\mathbf{p} = (p_1, p_2, \dots)$ be an infinite probability vector (this means that $p_i \geqslant 0$ for all i, and $\sum_i p_i = 1$). Show that $\mathbf{p}$ gives rise to a σ-invariant and ergodic probability measure $\mathbf{p}^{\mathbb{N}}$ on $\mathbb{N}^{\mathbb{N}}$.

Exercise 3.2.4. Let $\phi : \mathbb{N}^{\mathbb{N}} \to (0,1) \smallsetminus \mathbb{Q}$ be the map discussed on page 83, and let μ be the Gauss measure on $[0,1]$. Show that $\phi_*^{-1}\mu$ is not of the form $\mathbf{p}^{\mathbb{N}}$ for any infinite probability vector $\mathbf{p}$.

3.3 Badly Approximable Numbers

While Corollary 3.8 gives precise information about the behavior of typical real numbers, it does not say anything about the behavior of all real numbers. In this section we discuss a special class of real numbers that behave very differently to typical real numbers.

Definition 3.9. *A real number $u = [a_1, a_2, \dots] \in (0,1)$ is called* badly approximable *if there is some bound M with the property that $a_n \leqslant M$ for all $n \geqslant 1$.*

Clearly a badly approximable number cannot satisfy equation (3.27). It follows that the set of all badly approximable numbers in $(0,1)$ is a null set with respect to the Gauss measure, and hence is a null set with respect to Lebesgue measure(43). The next result explains the terminology: badly approximable numbers cannot be approximated very well by rationals.

Proposition 3.10. *A number $u \in (0,1)$ is badly approximable if and only if there exists some $\varepsilon > 0$ with the property that*

$$\left| u - \frac{p}{q} \right| \geqslant \frac{\varepsilon}{q^2}$$

for all rational numbers $\frac{p}{q}$.

Proof. If u is badly approximable, then equation (3.4) shows that

$$q_{n+1} \leqslant (M+1) q_n$$

for all $n \geqslant 0$. For any q there is some n with $q \in (q_{n-1}, q_n]$, and by Proposition 3.3 and equation (3.15) we therefore have

$$\left|\frac{p}{q} - u\right| > \left|\frac{p_n}{q_n} - u\right| > \frac{1}{q_n q_{n+2}} > \frac{1}{(M+1)^4 q^2}$$

as required.

Conversely, if

$$\left|u - \frac{p}{q}\right| \geqslant \frac{\varepsilon}{q^2}$$

for all rational numbers $\frac{p}{q}$ then, in particular,

$$\frac{\varepsilon}{q_n^2} \leqslant \left|u - \frac{p_n}{q_n}\right| < \frac{1}{q_n q_{n+1}}$$

by equation (3.13). This implies that

$$a_{n+1} q_n < a_{n+1} q_n + q_{n-1} = q_{n+1} < \frac{1}{\varepsilon} q_n,$$

so $a_{n+1} \leqslant \frac{1}{\varepsilon}$ for all $n \geqslant 1$. □

Example 3.11. Notice that $\frac{2}{\sqrt{5}-1} = \frac{\sqrt{5}+1}{2} \in (1, 2)$ and $\frac{\sqrt{5}+1}{2} - 1 = \frac{\sqrt{5}-1}{2}$. It follows that if

$$\frac{\sqrt{5}-1}{2} = [a_1, a_2, \dots]$$

then $a_1 + [a_2, a_3, \dots] \in (1, 2)$, so $a_1 = 1$, and hence

$$[a_2, a_3, \dots] = \frac{\sqrt{5}+1}{2} - 1 = \frac{\sqrt{5}-1}{2} = [a_1, a_2, \dots].$$

We deduce by the uniqueness of the continued fraction digits that

$$\frac{\sqrt{5}-1}{2} = [1, 1, 1, \dots],$$

so $\frac{\sqrt{5}-1}{2}$ is badly approximable.

Indeed, the specific number in Example 3.11 is, in a precise sense, the most badly approximable real number in $(0, 1)$. In the next section we generalize this example to show that all quadratic irrationals are badly approximable.

3.3.1 Lagrange's Theorem

The periodicity of the continued fraction expansion seen in Example 3.11 is a general property of quadratics. A real number u is called a *quadratic irrational* if $u \notin \mathbb{Q}$ and there are integers a, b, c with $au^2 + bu + c = 0$. Notice that u is a quadratic irrational if and only if $\mathbb{Q}(u)$ is a subfield of $\mathbb{R}$ of degree 2 over $\mathbb{Q}$.

Definition 3.12. *A continued fraction* $[a_0; a_1, \dots]$ *is* eventually periodic *if there are numbers* $N \geqslant 0$ *and* $k \geqslant 1$ *with* $a_{n+k} = a_n$ *for all* $n \geqslant N$. *Such a continued fraction will be written*

$$[a_0; a_1, \dots, a_{N-1}, \overline{a_N, \dots, a_{N+k}}].$$

The main result describing the special properties of quadratic irrationals is Lagrange's Theorem [218, Sect. 34].

Theorem 3.13 (Lagrange). *Let* u *be an irrational positive real number. Then the continued fraction expansion of* u *is eventually periodic if and only if* u *is a quadratic irrational.*

Proof. Assume first that $u = [\overline{a_0; a_1, \dots, a_k}]$ has a strictly periodic continued fraction expansion, so that $u_{k+1} = u_0 = u$. Thus

$$u = \frac{up_k + p_{k-1}}{uq_k + q_{k-1}}$$

by equation (3.20), so

$$u^2 q_k + u(q_{k-1} - p_k) - p_{k-1} = 0$$

and u is a quadratic irrational (u cannot be rational, since it has an infinite continued fraction; alternatively notice that the quadratic equation satisfied by u has discriminant $(q_{k-1} - p_k)^2 + 4q_k p_{k-1} = (q_{k-1} + p_k)^2 - 4(-1)^k$ by equation (3.8), so cannot be a square).

Now assume that

$$u = [a_0; \dots, a_{N-1}, \overline{a_N, \dots, a_{N+k}}].$$

Then, by equation (3.20),

$$u = \frac{[\overline{a_N; a_{N+1}, \dots, a_{N+k}}]p_{N-1} + p_{N-2}}{[\overline{a_N; a_{N+1}, \dots, a_{N+k}}]q_{N-1} + q_{N-2}},$$

so $\mathbb{Q}(u) = \mathbb{Q}([\overline{a_N; a_{N+1}, \dots, a_{N+k}}])$, and therefore u is a quadratic irrational.

The converse is more involved[(44)]. Assume now that u is a quadratic irrational, with

$$f_0(u) = \alpha_0 u^2 + \beta_0 u + \gamma_0 = 0$$

for some $\alpha_0, \beta_0, \gamma_0 \in \mathbb{Z}$ and $\delta = \beta_0^2 - 4\alpha_0\gamma_0$ not a square. We claim that for each $n \geqslant 0$ there is a polynomial

$$f_n(x) = \alpha_n x^2 + \beta_n + \gamma_n$$

with

$$\beta_n^2 - 4\alpha_n\gamma_n = \delta$$

and with the property that $f_n(u_n) = 0$. This claim again follows from the fact that $\mathbb{Q}(u) = \mathbb{Q}(u_n)$, but we will need specific properties of the numbers $\alpha_n, \beta_n, \gamma_n$, so we proceed by induction.

Assume such a polynomial exists for some $n \geqslant 0$. Since $u_n = a_n + \frac{1}{u_{n+1}}$, we therefore have

$$u_{n+1}^2 f_n\left(a_n + \frac{1}{u_{n+1}}\right) = 0.$$

The resulting relation for u_{n+1} may be written in the form

$$f_{n+1}(x) = \alpha_{n+1}x^2 + \beta_{n+1}x + \gamma_{n+1}$$

where

$$\begin{aligned} \alpha_{n+1} &= a_n^2\alpha_n + a_n\beta_n + \gamma_n, \\ \beta_{n+1} &= 2a_n\alpha_n + \beta_n, & (3.30) \\ \gamma_{n+1} &= \alpha_n. & (3.31) \end{aligned}$$

It is clear that $\alpha_{n+1}, \beta_{n+1}, \gamma_{n+1} \in \mathbb{Z}$, and a simple calculation shows that

$$\beta_{n+1}^2 - 4\alpha_{n+1}\gamma_{n+1} = \beta_n^2 - 4\alpha_n\gamma_n,$$

proving the claim.

Notice that all the polynomials f_n have the same discriminant δ, which is not a square, so $\alpha_n \neq 0$ for $n \geqslant 0$. If there is some N with $\alpha_n > 0$ for all $n \geqslant N$, then equation (3.30) shows that the sequence $\beta_N, \beta_{N+1}, \dots$ is increasing since $a_n > 0$ for $n \geqslant 1$. Thus for large enough n, by equation (3.31), all three of α_n, β_n and γ_n are positive. This is impossible, since $f_n(u_n) = 0$ and $u_n > 0$. A similar argument shows that there is no N with $\alpha_n < 0$ for all $n \geqslant N$. We deduce that α_n must change in sign infinitely often, so in particular there is an infinite set $A \subseteq \mathbb{N}$ with the property that $\alpha_n\alpha_{n-1} < 0$ for all $n \in A$. By equation (3.31), it follows that $\alpha_n\gamma_n < 0$ for all $n \in A$. Now $\beta_n^2 - 4\alpha_n\gamma_n = \delta$, so for $n \in A$ we must have

$$\begin{aligned} |\alpha_n| &\leqslant \tfrac{1}{4}\delta, \\ |\beta_n| &< \sqrt{\delta}, \end{aligned}$$

and

$$|\gamma_n| \leqslant \tfrac{1}{4}\delta.$$

It follows that as n runs through the infinite set A there are only finitely many possibilities for the polynomials f_n, so there must be some $n_0 < n_1 < n_2$ with $f_{n_0} = f_{n_1} = f_{n_2}$. Since a quadratic polynomial has only two zeros, and $u_{n_0}, u_{n_1}, u_{n_2}$ are all zeros of the same polynomial, we see that two of them coincide so the continued fraction expansion of u is eventually periodic.

□

Corollary 3.14. *Any quadratic irrational is badly approximable.*

PROOF. This is an immediate consequence of Theorem 3.13 and Definition 3.9. □

It is not known if any other algebraic numbers are badly approximable.

Exercises for Section 3.3

Exercise 3.3.1. [45] Show that $\mathbb{Q}(\sqrt{5})$ contains infinitely many elements with a uniform bound on their partial quotients, by checking that the numbers $[\overline{1^{k+1},4,2,1^k,3}]$ for $k \geqslant 0$ all lie in $\mathbb{Q}(\sqrt{5})$ (here 1^k denotes the string $1,1,\dots,1$ of length k). Can you find a similar pattern in any real quadratic field $\mathbb{Q}(\sqrt{d})$?

Exercise 3.3.2. A number $u \in (0,1)$ is called *very well approximable* if there is some $\delta > 0$ with the property that there are infinitely many rational numbers $\frac{p}{q}$ with $\gcd(p,q) = 1$ for which

$$\left| u - \frac{p}{q} \right| \leqslant \frac{1}{q^{2+\delta}}.$$

(a) Show that u is very well approximable if and only if there is some $\varepsilon > 0$ with the property that $a_{n+1} \geqslant q_n^{\varepsilon}$ for infinitely many values of n.
(b) Show that for any very well approximable number the convergence in equation (3.28) fails.

Exercise 3.3.3. Prove Liouville's Theorem[46]: if u is a real algebraic number of degree $d \geqslant 2$, then there is some constant $c(u) > 0$ with the property that

$$\frac{c(u)}{q^d} < \left| u - \frac{p}{q} \right|$$

for any rational number $\frac{p}{q}$.

Exercise 3.3.4. Use Liouville's Theorem from Exercise 3.3.3 to show that the number

$$u = \sum_{n=1}^{\infty} 10^{-n!}$$

is transcendental (that is, u is not a zero of any integral polynomial)[47].

Exercise 3.3.5. Prove that the theorem of Margulis from p. 12 does not hold for quadratic forms in 2 variables.

3.4 Invertible Extension of the Continued Fraction Map

We are interested in finding a geometrically convenient invertible extension of the non-invertible map T, and in Section 9.6 will re-prove the ergodicity of the Gauss measure in that context.

Define a set

$$\overline{Y} = \{(y,z) \in [0,1)^2 \mid 0 \leqslant z \leqslant \frac{1}{1+y}\}$$

(this set is illustrated in Figure 3.2) and a map $\overline{T} : \overline{Y} \to \overline{Y}$ by

$$\overline{T}(y,z) = (Ty, y(1-yz)).$$

The map $\overline{T}$ will also be called the Gauss map.

Proposition 3.15. *The map $\overline{T} : \overline{Y} \to \overline{Y}$ is an area-preserving bijection off a null set. More precisely, there is a countable union N of lines and curves in $\overline{Y}$ with the property that $T|_{\overline{Y}\smallsetminus N} : \overline{Y}\smallsetminus N \to \overline{Y}\smallsetminus N$ is a bijection preserving the Lebesgue measure.*

PROOF. The derivative of the map $\overline{T}$ is

$$\begin{pmatrix} -\frac{1}{y^2} & 0 \\ 1-2yz & -y^2 \end{pmatrix},$$

with determinant 1. It follows that $\overline{T}$ preserves area locally. To see that the map is a bijection, define regions A_n and B_n in $\overline{Y}$ by

$$A_n = \{(y,z) \in \overline{Y} \mid \frac{1}{n+1} < y < \frac{1}{n}\}$$

and

$$B_n = \{(y,z) \in \overline{Y} \mid \frac{1}{n+1+y} < z < \frac{1}{n+y} \text{ and } y > 0\}.$$

These sets are shown in Figure 3.2. Both

$$\{A_n \mid n = 1,2,\dots\}$$

and

$$\{B_n \mid n = 1,2,\dots\}$$

define partitions of $\overline{Y}$ after removing countably many vertical lines (or curves in the case of $\{B_n\}$). Since this is a Lebesgue null set, it is enough to show that $\overline{T}|_{A_n} : A_n \to B_n$ is a bijection for each $n \geqslant 1$, for then

$$\overline{T}|_{\cup_{n\geqslant 1} A_n} : \bigcup_{n\geqslant 1} A_n \longrightarrow \bigcup_{n\geqslant 1} B_n$$

is also a bijection, and we can take for the null set N the set of all images and pre-images of

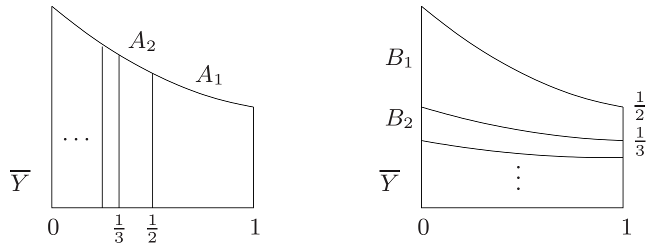

Fig. 3.2. The Gauss map is a bijection between $\overline{Y}$ and $\overline{Y}$, sending the subset $A_n \subseteq \overline{Y}$ to the subset $B_n \subseteq \overline{Y}$ for each $n \geqslant 1$.

$$\Big(\overline{Y} \smallsetminus \bigcup_{n \geqslant 1} A_n\Big) \cup \Big(\overline{Y} \smallsetminus \bigcup_{n \geqslant 1} B_n\Big).$$

Notice that $y > 0$ and $0 < z < \frac{1}{1+y}$ implies that

$$0 < yz < \frac{y}{1+y},$$

$$\frac{1}{1+y} < (1 - yz) < 1,$$

and

$$\frac{y}{1+y} < y(1 - yz) < y. \tag{3.32}$$

If now $(y, z) \in A_n$ for some $n \geqslant 1$ then $y = \frac{1}{n+y_1}$ for $\overline{T}(y, z) = (y_1, z_1)$ and the inequality (3.32) becomes

$$\frac{1}{n+1+y_1} = \frac{y}{1+y} < z_1 = y(1 - yz) < y = \frac{1}{n+y_1},$$

so that $(y_1, z_1) \in B_n$ and therefore $\overline{T}(A_n) \subseteq B_n$. To see that the restriction to A_n is a bijection, fix $(y_1, z_1) \in B_n$. Then $y = \frac{1}{n+y_1}$ is uniquely determined, and the equation $z_1 = y(1 - yz)$ then determines z uniquely. Clearly

$$y \in \left(\frac{1}{n+1}, \frac{1}{n}\right)$$

since $y_1 \in (0, 1)$, and by reversing the argument above (or by a straightforward calculation) we see that

$$\frac{y}{1+y} = \frac{1}{n+1+y_1} < z_1 < \frac{1}{n+y_1} = y$$

implies $0 < z < \frac{1}{1+y}$ so that $(y, z) \in A_n$. $\square$

Lemma 3.5 gives no indication of where the Gauss measure might have came from. The invertible extension, which preserves Lebesgue measure, gives an alternative proof that the Gauss measure is invariant, and gives one explanation of where it might come from.

SECOND PROOF OF LEMMA 3.5. Let $\pi : \overline{Y} \to Y$ be the projection

$$\pi(y, z) = y \tag{3.33}$$

onto Y. The Gauss measure μ on Y is the measure defined* by

$$\mu(B) = m(\pi^{-1}B)$$

where m is the normalized Lebesgue measure on $\overline{Y}$. Since $\overline{T} : \overline{Y} \to \overline{Y}$ preserves m by Proposition 3.15 and $\pi \circ \overline{T} = T \circ \pi$, the measure μ is T-invariant. □

The projection map $\pi : \overline{Y} \to Y$ defined in equation (3.33) shows that $\overline{T}$ on $\overline{Y}$ is an invertible extension of the non-invertible map T on Y.

Notes to Chapter 3

(38) (Page 73) The material in Section 3.1 may be found in many places; a convenient source for the path followed here using matrices is a note of van der Poorten [295].

(39) (Page 76) In particular, we have Dirichlet's theorem: for any $u \in \mathbb{R}$ and $Q \in \mathbb{N}$, there exists a rational number $\frac{p}{q}$ with $0 < q \leqslant Q$ and $|u - \frac{p}{q}| \leqslant \frac{1}{q(Q+1)}$, which can also be seen via the pigeon-hole principle.

(40) (Page 80) A broad overview of continued fractions from an ergodic perspective may be found in the monograph of Iosifescu and Kraaikamp [161]. Kraaikamp and others have suggested ways in which Gauss could have arrived at this measure; see also Keane [187]. Other approaches to the Gauss measure are described in the book of Khinchin [191]. The ergodic approach to continued fractions has a long history. Knopp [205] showed that the Gauss measure is ergodic (in different language); Kuz′min [217] found results on the rate of mixing of the Gauss measure; Doeblin [71] showed ergodicity; Ryll-Nardzewski [327] also showed this (that the Gauss measure is "indecomposable") and used the ergodic theorem to deduce results like equation (3.26). This had also been shown earlier by Khinchin [190]. Lévy [227] showed equation (3.25), an implicitly ergodic result, in 1936 (using the language of probability rather than ergodic theory).

(41) (Page 83) These results are indeed easily seen given both the ergodic theorem and the ergodicity of the Gauss map; their original proofs by other methods are not easy. For other results on the continued fraction expansion from the ergodic perspective, see Cornfeld, Fomin and Sinaĭ [60, Chap. 7] and from a number-theoretic perspective, see Khinchin [191]. The limit in equation (3.26), approximately 2.685, is known as Khinchin's constant; the problem of estimating it numerically is considered by Bailey, Borwein and Crandall [14]. Little is known about its arithmetical properties. The (exponential of the) constant appearing in equation (3.28) is usually called the Khinchin–Lévy constant. Just as in Example 2.31, it is a quite different

* This construction of μ from m is called the *push-forward* of m by π.

problem to exhibit any specific number that satisfies these almost everywhere results: Adler, Keane and Smorodinsky exhibit a normal number for the continued fraction map in [2].

(42)(Page 83) This is proved here directly, using estimates for conditional measures on cylinder sets; see Billingsley [31] for example. We will re-prove it in Proposition 9.25 on p. 318 using a geometrical argument.

(43)(Page 90) Most of this section is devoted to quadratic irrationals, but it is clear there are uncountably many badly approximable numbers; the survey of Shallit [341] describes some of the many settings in which these numbers appear, gives other families of such numbers, and has an extensive bibliography on these numbers (which are also called numbers of *constant type*). For example, Kmošek [203] and Shallit [340] showed that if $\sum_{n=0}^{\infty} k^{-2^n} = [a_1^{(k)}, a_2^{(k)}, \dots]$ then $\sup_{n\geqslant 1}\{a_n^{(2)}\} = 6$ and $\sup_{n\geqslant 1}\{a_n^{(k)}\} = n+2$ for $n \geqslant 3$.

(44)(Page 92) There are many ways to prove this; we follow the argument of Steinig [354] here.

(45)(Page 94) This remarkable uniformity in Definition 3.9 was shown by Woods [389] for $\mathbb{Q}(\sqrt{5})$ and by Wilson [385] in general, who showed that any real quadratic field $\mathbb{Q}(\sqrt{d})$ contains infinitely many numbers of the form $[\overline{a_1, a_2, \dots, a_k}]$ with $1 \leqslant a_n \leqslant M_d$ for all $n \geqslant 1$. McMullen [259] has explained these phenomena in terms of closed geodesics; the connection between continued fractions and closed geodesics will be developed in Chapter 9. Exercise 3.3.1 shows that we may take $M_5 = 4$, and the question is raised in [259] of whether there is a tighter bound allowing M_d to be taken equal to 2 for all d.

(46)(Page 94) Liouville's Theorem [234], [236] (on Diophantine approximation; there are several important results bearing his name) marked the start of an important series of advances in Diophantine approximation, attempting to sharpen the lower bound. These results may be summarized as follows. The statement that for any algebraic number u of degree d there is a constant $c(u)$ so that for all rationals p/q we have $|u-p/q| > c(u)/q^{\lambda(u)}$ holds: for $\lambda(u) = d$ (Liouville 1844); for any $\lambda(u) > \frac{1}{2}d+1$ (Thue [362], 1909); for any $\lambda(u) > 2\sqrt{d}$ (Siegel [344], 1921); for any $\lambda(u) > \sqrt{2d}$ (Dyson [77], 1947); finally, and definitively, for any $\lambda(u) > 2$ (Roth [320], 1955).

(47)(Page 94) This observation of Liouville [235] dates from 1844 and seems to be the earliest construction of a transcendental number; in 1874 Cantor [47] used set theory to show that the set of algebraic numbers is countable, deducing that there are uncountably many transcendental real numbers (as pointed out by Herstein and Kaplansky [150, p. 238], and despite what is often taught, Cantor's proof can be used to exhibit many explicit transcendental numbers). In a different direction, many important constants were shown to be transcendental. Examples include: e (Hermite [149], 1873); π (Lindemann [232], 1882); α^β for α algebraic and not equal to 0 or 1 and β algebraic and irrational (Gelfond [113] and Schneider [335], 1934).