"JUST THE MATHS"

SLIDES NUMBER

9.10

MATRICES 10
(Symmetric matrices & quadratic forms)

by

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UNIT 9.10 - MATRICES 10

SYMMETRIC MATRICES AND QUADRATIC FORMS

9.10.1 SYMMETRIC MATRICES

We state the following without proof:

(i) All of the eigenvalues of a symmetric matrix are real and, hence, so are the eigenvectors.

(ii) A symmetric matrix of order $n \times n$ always has n linearly independent eigenvectors.

(iii) For a symmetric matrix, suppose that X_i and X_j are linearly independent eigenvectors associated with different eigenvectors; then

$$\mathrm{X}_i\mathrm{X}_j^{\mathrm{T}} \equiv x_i x_j + y_i y_j + z_i z_j = 0.$$

We say that X_i and X_j are **"mutually orthogonal"**.

If a symmetric matrix has any repeated eigenvalues, it is still possible to determine a full set of mutually orthogonal eigenvectors, but not every full set of eignvectors will have the orthogonality property.

(iv) A symmetric matrix always has a modal matrix whose columns are mutually orthogonal. When the eigenvalues are distinct, this is true for <u>every</u> modal matrix.

(v) A modal matrix, N, of normalised eigenvectors is an orthogonal matrix.

ILLUSTRATIONS

1. If N is of order 3×3, we have

$$\mathrm{N}^{\mathrm{T}}.N = \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix} . \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

2. It was shown, in Unit 9.6, that the matrix

$$\mathrm{A} = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}$$

has eigenvalues $\lambda = 8$, and $\lambda = -1$ (repeated), with associated eigenvectors

$$\alpha \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \quad \text{and} \quad \beta \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + \gamma \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \equiv \begin{bmatrix} -\frac{1}{2}\beta - \gamma \\ \beta \\ \gamma \end{bmatrix}.$$

A set of **linearly independent** eigenvectors may therefore be given by

$$X_1 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \quad X_2 = \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad X_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

Clearly, X_1 is orthogonal to X_2 and X_3, but X_2 and X_3 are not orthogonal to each other.

However, we may find β and γ such that

$$\beta \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + \gamma \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \text{is orthogonal to} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

We simply require that

$$\frac{1}{2}\beta + 2\gamma = 0$$

or

$$\beta + 4\gamma = 0.$$

This will be so, for example, when $\beta = 4$ and $\gamma = -1$.

A new set of linearly independent mutually orthogonal eigenvectors can thus be given by

$$X_1 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \quad X_2 = \begin{bmatrix} -1 \\ 4 \\ -1 \end{bmatrix}, \quad \text{and} \quad X_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

9.10.2 QUADRATIC FORMS

An algebraic expression of the form

$$ax^2 + by^2 + cz^2 + 2fyz + 2yzx + 2hxy$$

is called a **"quadratic form"**.

In matrix notation, it may be written as

$$\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \equiv X^T A X,$$

and we note that the matrix, A, is symmetric.

In the scientific applications of quadratic forms, it is desirable to know whether such a form is

(a) always positive; (b) always negative;

(c) both positive and negative.

It may be shown that, if we change to new variables, (u, v, w), using a linear transformation,

$$\mathrm{X} = \mathrm{PU},$$

where P is some non-singular matrix, then the new quadratic form has the same properties aas the original, concerning its sign.

We now show that a good choice for P is a modal matrix, N, of normalised, linearly independent, mutually orthogonal eigenvectors for A.

Putting $\mathrm{X} = \mathrm{NU}$, the expression $\mathrm{X^TAX}$ becomes $\mathrm{U^TN^TANX}$.

But, since N is orthogonal when A is symmetric, $\mathrm{N^T} = \mathrm{N^{-1}}$ and hence $\mathrm{N^TAN}$ is the spectral matrix, S, for A.

The new quadratic form is therefore

$$\mathrm{U^TSU} \equiv \begin{bmatrix} u & v & w \end{bmatrix} . \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} . \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

$$\equiv \lambda_1 u^2 + \lambda_2 v^2 + \lambda_3 w^2.$$

Clearly, if all of the eigenvalues are positive, then the new quadratic form is always positive; and, if all of the eigenvalues are negative, then the new quadratic form is always negative.

The new quadratic form is called the **“canonical form under similarity”** of the original quadratic form.