

**“JUST THE MATHS”**

**SLIDES NUMBER**

**16.10**

**Z-TRANSFORMS 3**

**(Solution of linear difference equations)**

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**16.10.1 First order linear difference equations**

**16.10.2 Second order linear difference equations**

## UNIT 16.10 - Z TRANSFORMS 3

### THE SOLUTION OF LINEAR DIFFERENCE EQUATIONS

Linear Difference Equations may be solved by constructing the Z-Transform of both sides of the equation.

#### 16.10.1 FIRST ORDER LINEAR DIFFERENCE EQUATIONS

#### EXAMPLES

1. Solve the linear difference equation,

$$u_{n+1} - 2u_n = (3)^{-n},$$

given that  $u_0 = 2/5$ .

#### **Solution**

Using the second shifting theorem,

$$Z\{u_{n+1}\} = z.Z\{u_n\} - z.\frac{2}{5}.$$

Taking the Z-Transform of the difference equation,

$$z.Z\{u_n\} - \frac{2}{5}.z - 2Z\{u_n\} = \frac{z}{z - \frac{1}{3}}.$$

On rearrangement,

$$\begin{aligned}Z\{u_n\} &= \frac{2}{5} \cdot \frac{z}{z-2} + \frac{z}{(z-\frac{1}{3})(z-2)} \\ &\equiv \frac{2}{5} \cdot \frac{z}{z-2} + z \cdot \left[ \frac{-\frac{3}{5}}{z-\frac{1}{3}} + \frac{\frac{3}{5}}{z-2} \right] \\ &\equiv \frac{z}{z-2} - \frac{3}{5} \cdot \frac{z}{z-\frac{1}{3}}.\end{aligned}$$

Taking the inverse Z-Transform of this function of  $z$ ,

$$\{u_n\} \equiv \left\{ (2)^n - \frac{3}{5}(3)^{-n} \right\}.$$

2. Solve the linear difference equation,

$$u_{n+1} + u_n = f(n),$$

given that

$$f(n) \equiv \begin{cases} 1 & \text{when } n = 0; \\ 0 & \text{when } n > 0. \end{cases}$$

and  $u_0 = 5$ .

**Solution**

Using the second shifting theorem,

$$Z\{u_{n+1}\} = z \cdot Z\{u_n\} - z \cdot 5$$

Taking the Z-Transform of the difference equation,

$$z.Z\{u_n\} - 5z + Z\{u_n\} = 1.$$

On rearrangement,

$$Z\{u_n\} = \frac{1}{z+1} + \frac{5z}{z+1}.$$

Hence,

$$\{u_n\} = \begin{cases} 5 & \text{when } n = 0; \\ (-1)^{n-1} + 5(-1)^n \equiv 4(-1)^n & \text{when } n > 0. \end{cases}$$

## 16.10.2 SECOND ORDER LINEAR DIFFERENCE EQUATIONS

### EXAMPLES

1. Solve the linear difference equation,

$$u_{n+2} = u_{n+1} + u_n,$$

given that  $u_0 = 0$  and  $u_1 = 1$ .

#### **Solution**

Using the second shifting theorem,

$$Z\{u_{n+1}\} = z.Z\{u_n - z.0\} \equiv z.Z\{u_n\}.$$

and

$$Z\{u_{n+2}\} = z^2 Z\{u_n\} - z.1 \equiv z^2 Z\{u_n\} - z.$$

Taking the Z-Transform of the difference equation,

$$z^2.Z\{u_n\} - z = z.Z\{u_n\} + Z\{u_n\}.$$

On rearrangement,

$$Z\{u_n\} = \frac{z}{z^2 - z - 1}.$$

This may be written

$$Z\{u_n\} = \frac{z}{(z - \alpha)(z - \beta)}.$$

From the quadratic formula,

$$\alpha, \beta = \frac{1 \pm \sqrt{5}}{2}.$$

Using partial fractions,

$$Z\{u_n\} = \frac{1}{\alpha - \beta} \left[ \frac{z}{z - \alpha} - \frac{z}{z - \beta} \right].$$

Taking the inverse Z-Transform of this function of  $z$  gives

$$\{u_n\} \equiv \left\{ \frac{1}{\alpha - \beta} [(\alpha)^n - (\beta)^n] \right\}.$$

2. Solve the linear difference equation,

$$u_{n+2} - 7u_{n+1} + 10u_n = 16n,$$

given that  $u_0 = 6$  and  $u_1 = 2$ .

### **Solution**

Using the second shifting theorem,

$$Z\{u_{n+1}\} = z.Z\{u_n\} - 6z$$

and

$$Z\{u_{n+2}\} = z^2.Z\{u_n\} - 6z^2 - 2z.$$

Taking the Z-Transform of the difference equation,

$$z^2.Z\{u_n\} - 6z^2 - 2z - 7[z.Z\{u_n\} - 6z] + 10Z\{u_n\} = \frac{16z}{(z-1)^2}.$$

On rearrangement,

$$Z\{u_n\}[z^2 - 7z + 10] - 6z^2 + 40z = \frac{16z}{(z-1)^2}.$$

Hence,

$$Z\{u_n\} = \frac{16z}{(z-1)^2(z-5)(z-2)} + \frac{6z^2 - 40z}{(z-5)(z-2)}.$$

Using partial fractions,

$$Z\{u_n\} = z \cdot \left[ \frac{4}{z-2} - \frac{3}{z-5} + \frac{4}{(z-1)^2} + \frac{5}{z-1} \right].$$

The solution to the difference equation is therefore

$$\{u_n\} \equiv \{4(2)^n - 3(5)^n + 4n + 5\}.$$

3. Solve the linear difference equation,

$$u_{n+2} + 2u_n = 0,$$

given that  $u_0 = 1$  and  $u_1 = \sqrt{2}$ .

### **Solution**

Using the second shifting theorem,

$$Z\{u_{n+2}\} = z^2 Z\{u_n\} - z^2 - z\sqrt{2}.$$

Taking the Z-Transform of the difference equation,

$$z^2 Z\{u_n\} - z^2 - z\sqrt{2} + 2Z\{u_n\} = 0.$$

On rearrangement,

$$Z\{u_n\} = \frac{z^2 + z\sqrt{2}}{z^2 + 2} \equiv z \cdot \frac{z + \sqrt{2}}{z^2 + 2} \equiv z \cdot \frac{z + \sqrt{2}}{(z + j\sqrt{2})(z - j\sqrt{2})}.$$

Using partial fractions,

$$Z\{u_n\} = z \left[ \frac{\sqrt{2}(1+j)}{j2\sqrt{2}(z-j\sqrt{2})} + \frac{\sqrt{2}(1-j)}{-j2\sqrt{2}(z+j\sqrt{2})} \right]$$

or

$$Z\{u_n\} \equiv z \cdot \left[ \frac{(1-j)}{2(z-j\sqrt{2})} + \frac{(1+j)}{2(z+j\sqrt{2})} \right].$$

Hence,

$$\begin{aligned} \{u_n\} &\equiv \left\{ \frac{1}{2}(1-j)(j\sqrt{2})^n + \frac{1}{2}(1+j)(-j\sqrt{2})^n \right\} \\ &\equiv \left\{ \frac{1}{2}(\sqrt{2})^n [(1-j)(j)^n + (1+j)(-j)^n] \right\} \\ &\equiv \left\{ \frac{1}{2}(\sqrt{2})^n \left[ \sqrt{2}e^{-j\frac{\pi}{4}} \cdot e^{j\frac{n\pi}{2}} + \sqrt{2}e^{j\frac{\pi}{4}} \cdot e^{-j\frac{n\pi}{2}} \right] \right\} \\ &\equiv \left\{ \frac{1}{2}(\sqrt{2})^{n+1} \left[ e^{j\frac{(2n-1)\pi}{4}} + e^{-j\frac{(2n-1)\pi}{4}} \right] \right\} \\ &\equiv \left\{ \frac{1}{2}(\sqrt{2})^{n+1} \cdot 2 \cos \frac{(2n-1)\pi}{4} \right\} \\ &\equiv \left\{ (\sqrt{2})^{n+1} \cos \frac{(2n-1)\pi}{4} \right\}. \end{aligned}$$