"JUST THE MATHS"

SLIDES NUMBER

15.10

ORDINARY DIFFERENTIAL EQUATIONS 10 (Simultaneous equations (C))

by

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UNIT 15.10 - ORDINARY DIFFERENTIAL EQUATIONS 10

SIMULTANEOUS EQUATIONS (C)

15.10.1 MATRIX METHODS FOR NON-HOMOGENEOUS SYSTEMS

The general solution of a single linear differential equation with constant coefficients is made up of a particular integral and a complementary function.

The complementary function is the general solution of the corresponding homogeneous differential equation.

A similar principle is now applied to a pair of simultaneous non-homogeneous differential equations of the form

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = ax_1 + bx_2 + f(t),$$
$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = cx_1 + dx_2 + g(t).$$

The method will be illustrated by an example.

EXAMPLE

Determine the general solution of the simultaneous differential equations

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = x_2, ------------(1)$$
$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = -4x_1 - 5x_2 + g(t), ---- (2)$$

where $g(t)$ is (a) t, (b) e^{2t} (c) $\sin t$, (d) e^{-t}.

Solutions

(i) First, we write the differential equations in matrix form as

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} g(t).$$

This may be interpreted as

$$\frac{\mathrm{dX}}{\mathrm{dt}} = \mathrm{MX} + \mathrm{N}g(t),$$

where

$$\mathrm{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \mathrm{M} = \begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix} \text{ and } \mathrm{N} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

(ii) Secondly, we consider the corresponding homogeneous system

$$\frac{\mathrm{dX}}{\mathrm{dt}} = \mathrm{MX}.$$

The characteristic equation is

$$\begin{vmatrix} 0-\lambda & 1 \\ -4 & -5-\lambda \end{vmatrix} = 0$$

and gives

$$\lambda(5+\lambda)+4=0,$$

or

$$\lambda^2+5\lambda+4=0,$$

or

$$(\lambda+1)(\lambda+4)=0.$$

Hence, $\lambda = -1$ or $\lambda = -4$.

(iii) The eigenvectors of M are obtained from the homogeneous equations

$$\begin{aligned} -\lambda k_1 + k_2 &= 0, \\ -4k_1 - (5+\lambda)k_2 &= 0. \end{aligned}$$

Hence, when $\lambda = -1$, we solve

$$\begin{aligned} k_1 + k_2 &= 0, \\ -4k_1 - 4k_2 &= 0. \end{aligned}$$

These are satisfied by any two numbers in the ratio $k_1 : k_2 = 1 : -1$.

Also, when $\lambda = -4$, we solve

$$\begin{aligned} 4k_1 + k_2 &= 0, \\ -4k_1 - k_2 &= 0. \end{aligned}$$

These are satisfied by any two numbers in the ratio $k_1 : k_2 = 1 : -4$.

The complementary function may now be written in the form

$$A\begin{bmatrix}1\\-1\end{bmatrix}e^{-t}+B\begin{bmatrix}1\\-4\end{bmatrix}e^{-4t},$$

where A and B are arbitrary constants.

(iv) In order to obtain a particular integral for the equation

$$\frac{\mathrm{dX}}{\mathrm{dt}}=\mathrm{MX}+\mathrm{N}g(t),$$

we note the second term on the right hand side and investigate a trial solution of a similar form.

The three cases in this example are as follows:

(a) $g(t)\equiv t$

$$\text{Trial solution}\ \ \mathrm{X}=\mathrm{P}+\mathrm{Q}t,$$

where P and Q are constant matrices of order 2×1.

We require that

$$\mathrm{Q}=\mathrm{M}(\mathrm{P}+\mathrm{Q}t)+\mathrm{N}t.$$

Equating the matrix coefficients of t and the constant matrices,

$$\mathrm{MQ} + \mathrm{N} = \mathbf{0} \quad \text{and} \quad \mathrm{Q} = \mathrm{MP}.$$

Thus,

$$\mathrm{Q} = -\mathrm{M}^{-1}\mathrm{N} \quad \text{and} \quad \mathrm{P} = \mathrm{M}^{-1}\mathrm{Q}.$$

Using

$$\mathrm{M}^{-1} = \frac{1}{4}\begin{bmatrix} -5 & -1 \\ 4 & 0 \end{bmatrix},$$

we obtain

$$\mathrm{Q} = -\frac{1}{4}\begin{bmatrix} -5 & -1 \\ 4 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.25 \\ 0 \end{bmatrix}$$

and

$$\mathrm{P} = \frac{1}{4}\begin{bmatrix} -5 & -1 \\ 4 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0.25 \\ 0 \end{bmatrix} = \begin{bmatrix} -0.3125 \\ 0.25 \end{bmatrix}.$$

The general solution, in this case, is

$$\mathrm{X} = A\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + B\begin{bmatrix} 1 \\ -4 \end{bmatrix} e^{-4t} + \begin{bmatrix} -0.3125 \\ 0.25 \end{bmatrix} + \begin{bmatrix} 0.25 \\ 0 \end{bmatrix} t.$$

(b) $g(t) \equiv e^{2t}$

$$\text{Trial solution } X = Pe^{2t}.$$

We require that

$$2Pe^{2t} = MPe^{2t} + Ne^{2t}.$$

That is,

$$2P = MP + N.$$

The matrix P may now be determined from the formula

$$(2I - M)P = N.$$

In more detail,

$$\begin{bmatrix} 2 & -1 \\ 4 & 7 \end{bmatrix}.P = N.$$

Hence,

$$P = \frac{1}{18}\begin{bmatrix} 7 & 1 \\ -4 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{18}\begin{bmatrix} 7 \\ -4 \end{bmatrix}.$$

The general solution, in this case, is

$$X = A\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + B\begin{bmatrix} 1 \\ -4 \end{bmatrix} e^{-4t} + \frac{1}{18}\begin{bmatrix} 7 \\ -4 \end{bmatrix} e^{2t}.$$

(c) $g(t) \equiv \sin t$

$$\text{Trial solution} \quad X = P\sin t + Q\cos t.$$

We require that

$$P\cos t - Q\sin t = M(P\sin t + Q\cos t) + N\sin t.$$

Equating the matrix coefficients of $\cos t$ and $\sin t$,

$$P = MQ \quad \text{and} \quad -Q = MP + N.$$

This means that

$$-Q = M^2Q + N \quad \text{or} \quad (M^2 + I)Q = -N.$$

Thus,

$$Q = -(M^2 + I)^{-1}N.$$

$$M^2 + I = \begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -3 & -5 \\ 20 & 22 \end{bmatrix}$$

Hence,

$$Q = -\frac{1}{34}\begin{bmatrix} 22 & 5 \\ -20 & -3 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{34}\begin{bmatrix} -5 \\ 3 \end{bmatrix}$$

Also,

$$P = MQ = \frac{1}{34}\begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix} \cdot \begin{bmatrix} -5 \\ 3 \end{bmatrix} = \frac{1}{34}\begin{bmatrix} 3 \\ 5 \end{bmatrix}.$$

The general solution, in this case, is

$$X = A\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + B\begin{bmatrix} 1 \\ -4 \end{bmatrix} e^{-4t} + \frac{1}{34}\begin{bmatrix} 3 \\ 5 \end{bmatrix} \sin t + \frac{1}{34}\begin{bmatrix} -5 \\ 3 \end{bmatrix} \cos t.$$

(d) $g(t) \equiv e^{-t}$

In this case, the function, $g(t)$, is already included in the complementary function and it becomes necessary to assume a particular integral of the form

$$X = (P + Qt)e^{-t},$$

where P and Q are constant matrices of order 2×1.

We require that

$$Qe^{-t} - (P + Qt)e^{-t} = M(P + Qt)e^{-t} + Ne^{-t}.$$

Equating the matrix coefficients of te^{-t} and e^{-t}, we obtain

$$-\mathrm{Q} = \mathrm{MQ} \quad \text{and} \quad \mathrm{Q} - \mathrm{P} = \mathrm{MP} + \mathrm{N}.$$

The first of these conditions shows that Q is an eigenvector of the matrix M corresponding the eigenvalue -1.

From earlier work,

$$\mathrm{Q} = k \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

for any constant k.

Also,

$$(\mathrm{M} + \mathrm{I})\mathrm{P} = \mathrm{Q} - \mathrm{N};$$

or, in more detail,

$$\begin{bmatrix} 1 & 1 \\ -4 & -4 \end{bmatrix} \cdot \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = k \begin{bmatrix} 1 \\ -1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Hence,

$$\begin{aligned} p_1 + p_2 &= k, \\ -4p_1 - 4p_2 &= -k - 1. \end{aligned}$$

Thus, $k = \frac{k+1}{4}$ giving $k = \frac{1}{3}$; and the matrix P is given by

$$\mathrm{P} = \begin{bmatrix} l \\ \frac{1}{3} - l \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 0 \\ 1 \end{bmatrix} + l\begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

for any number, l.

Taking $l = 0$ for simplicity, a particular integral is, therefore,

$$\mathrm{X} = \frac{1}{3}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} t\right\} e^{-t}.$$

The general solution is

$$\mathrm{X} = A\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + B\begin{bmatrix} 1 \\ -4 \end{bmatrix} e^{-4t} + \frac{1}{3}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} t\right\} e^{-t}.$$

Note:

In examples for which neither $f(t)$ nor $g(t)$ is identically equal to zero, the particular integral may be found by adding together the separate forms of particular integral for $f(t)$ and $g(t)$ and writing the system of differential equations in the form

$$\frac{\mathrm{dX}}{\mathrm{dt}} = \mathrm{MX} + \mathrm{N}_1 f(t) + \mathrm{N}_2 g(t),$$

where

$$\mathrm{N}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } \mathrm{N}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

For instance, if $f(t) \equiv t$ and $g(t) \equiv e^{2t}$, the particular integral would take the form

$$\mathrm{X} = \mathrm{P} + \mathrm{Q}t + \mathrm{R}e^{2t},$$

where P, Q and R are matrices of order 2×1.