

**“JUST THE MATHS”**

**SLIDES NUMBER**

**14.1**

**PARTIAL DIFFERENTIATION 1  
(Partial derivatives of the first order)**

**by**

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**14.1.1 Functions of several variables**

**14.1.2 The definition of a partial derivative**

## UNIT 14.1

### PARTIAL DIFFERENTIATION 1

#### PARTIAL DERIVATIVES OF THE FIRST ORDER

##### 14.1.1 FUNCTIONS OF SEVERAL VARIABLES

In most scientific problems, it is likely that a variable quantity under investigation will depend (for its values), not only on **one** other variable quantity, but on **several** other variable quantities.

The type of notation used may be indicated by examples such as the following:

1.

$$z = f(x, y).$$

2.

$$w = F(x, y, z).$$

Normally, the variables on the right-hand side are called the “**independent variables**”.

The variable on the left-hand side is called the “**dependent variable**”.

## Notes:

(i) Some relationships between several variables are not stated as an **explicit** formula for one of the variables in terms of the others.

## ILLUSTRATION

$$x^2 + y^2 + z^2 = 16.$$

In such cases, it may be necessary to specify separately which is the dependent variable.

(ii) The variables on the right-hand side of an explicit formula, may not always be independent of one another

## ILLUSTRATION

In the formula

$$z = xy^2 + \sin(x - y),$$

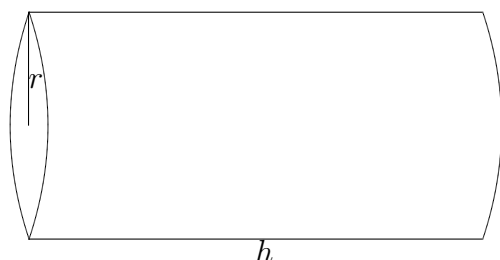
suppose that  $x = t - 1$  and  $y = 3t + 2$ ;

Then the variables,  $x$  and  $y$ , are not independent of each other.

In fact,  $y = 3(x + 1) + 2 = 3x + 5$ .

## 14.1.2 THE DEFINITION OF A PARTIAL DERIVATIVE

### ILLUSTRATION



The volume,  $V$ , and the surface area,  $S$ , of a solid right-circular cylinder with radius  $r$  and height  $h$  are given by

$$V = \pi r^2 h \quad \text{and} \quad S = 2\pi r^2 + 2\pi r h.$$

$V$  and  $S$  are functions of  $r$  and  $h$ .

Suppose  $r$  is held constant while  $h$  is allowed to vary.

Then,

$$\left[ \frac{dV}{dh} \right]_{r \text{ const.}} = \pi r^2$$

and

$$\left[ \frac{dS}{dh} \right]_{r \text{ const.}} = 2\pi r.$$

These are the “**partial derivatives of  $V$  and  $S$  with respect to  $h$** ”.

Similarly, suppose  $h$  is held constant while  $r$  is allowed to vary.

Then,

$$\left[ \frac{dV}{dr} \right]_{h \text{ const.}} = 2\pi r h$$

and

$$\left[ \frac{dS}{dr} \right]_{h \text{ const.}} = 4\pi r + 2\pi h.$$

These are the “**partial derivatives of  $V$  and  $S$  with respect to  $r$** ”.

## **THE NOTATION FOR PARTIAL DERIVATIVES**

This is indicated by

$$\frac{\partial V}{\partial h} = \pi r^2, \quad \frac{\partial S}{\partial h} = 2\pi r$$

and

$$\frac{\partial V}{\partial r} = 2\pi r h, \quad \frac{\partial S}{\partial r} = 4\pi r + 2\pi h.$$

## EXAMPLES

1. Determine the partial derivatives of the following functions with respect to each of the independent variables:

(a)

$$z = (x^2 + 3y)^5.$$

**Solution**

$$\frac{\partial z}{\partial x} = 5(x^2 + 3y)^4 \cdot 2x = 10x(x^2 + 3y)^4$$

and

$$\frac{\partial z}{\partial y} = 5(x^2 + 3y)^4 \cdot 3 = 15(x^2 + 3y)^4.$$

(b)

$$w = ze^{3x-7y}.$$

**Solution**

$$\frac{\partial w}{\partial x} = 3ze^{3x-7y},$$

$$\frac{\partial w}{\partial y} = -7ze^{3x-7y},$$

and

$$\frac{\partial w}{\partial z} = e^{3x-7y}.$$

(c)

$$z = x \sin(2x^2 + 5y).$$

**Solution**

$$\frac{\partial z}{\partial x} = \sin(2x^2 + 5y) + 4x^2 \cos(2x^2 + 5y)$$

and

$$\frac{\partial z}{\partial y} = 5x \cos(2x^2 + 5y).$$

2. If

$$z = f(x^2 + y^2),$$

show that

$$x \frac{\partial z}{\partial y} - y \frac{\partial z}{\partial x} = 0.$$

**Solution**

$$\frac{\partial z}{\partial x} = 2x f'(x^2 + y^2)$$

and

$$\frac{\partial z}{\partial y} = 2y f'(x^2 + y^2).$$

Hence,

$$x \frac{\partial z}{\partial y} - y \frac{\partial z}{\partial x} = 0.$$

3. If

$$\cos(x + 2z) + 3y^2 + 2xyz = 0$$

(where  $z$  is the dependent variable), determine expressions for  $\frac{\partial z}{\partial x}$  and  $\frac{\partial z}{\partial y}$  in terms of  $x$ ,  $y$  and  $z$ .

**Solution**

$$-\sin(x + 2z) \cdot \left(1 + 2\frac{\partial z}{\partial x}\right) + 2y \left(x\frac{\partial z}{\partial x} + y\right) = 0$$

and

$$-\sin(x + 2z) \cdot 2\frac{\partial z}{\partial y} + 6y + 2x \left(y\frac{\partial z}{\partial y} + z\right) = 0,$$

respectively.

Thus,

$$\frac{\partial z}{\partial x} = \frac{\sin(x + 2z) - 2y^2}{2yx - 2\sin(x + 2z)}$$

and

$$\frac{\partial z}{\partial y} = \frac{2xz + 6y}{2\sin(x + 2z) - 2xy} = \frac{xz + 3y}{\sin(x + 2z) - xy}.$$