"JUST THE MATHS"

SLIDES NUMBER

12.1

INTEGRATION 1
(Elementary indefinite integrals)

by

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UNIT 12.1 - INTEGRATION 1
ELEMENTARY INDEFINITE INTEGRALS

12.1.1 THE DEFINITION OF AN INTEGRAL

In Differential Calculus, we are given functions of x and asked to obtain their derivatives.

In Integral Calculus, we are given functions of x and asked what they are the derivatives of.

The process of answering this question is called **"integration"**.

Integration is the reverse of differentiation.

DEFINITION
Given a function $f(x)$, another function ,z, such that

$$\frac{\mathrm{d}z}{\mathrm{d}x} = f(x)$$

is called an integral of $f(x)$ with respect to x.

Notes:
(i) having found z, such that

$$\frac{\mathrm{d}z}{\mathrm{d}x} = f(x),$$

$z + C$ is also an integral for any constant value, C.

(ii) We call $z + C$ the **"indefinite integral of $f(x)$ with respect to x"** and we write

$$\int f(x)\mathrm{d}x = z + C.$$

(iii) C is an **arbitrary constant** called the **"constant of integration"**.

(iv) The symbol dx is a label, indicating the variable with respect to which we are integrating.

(v) In any integration problem, the function being integrated is called the **"integrand"**.

Result:
Two functions z_1 and z_2 are both integrals of the same function $f(x)$ if and only if they differ by a constant.

Proof:
(a) Suppose, firstly, that

$$z_1 - z_2 = C,$$

where C is a constant.

Then,

$$\frac{\mathrm{d}}{\mathrm{d}x}[z_1 - z_2] = 0.$$

That is,

$$\frac{\mathrm{d}z_1}{\mathrm{d}x} - \frac{\mathrm{d}z_2}{\mathrm{d}x} = 0,$$

or

$$\frac{\mathrm{d}z_1}{\mathrm{d}x} = \frac{\mathrm{d}z_2}{\mathrm{d}x}.$$

(b) Secondly, suppose that z_1 and z_2 are integrals of the same function.

Then,

$$\frac{\mathrm{d}z_1}{\mathrm{d}x} = \frac{\mathrm{d}z_2}{\mathrm{d}x}.$$

That is,

$$\frac{\mathrm{d}z_1}{\mathrm{d}x} - \frac{\mathrm{d}z_2}{\mathrm{d}x} = 0,$$

or

$$\frac{\mathrm{d}}{\mathrm{d}x}[z_1 - z_2] = 0.$$

Hence,

$$z_1 - z_2 = C$$

where C may be any constant.

Any result encountered in differentiation could be restated in reverse as a result on integration.

ILLUSTRATIONS

1.

$$\int 3x^2 \mathrm{d}x = x^3 + C.$$

2.

$$\int x^2 \mathrm{d}x = \frac{x^3}{3} + C.$$

3.

$$\int x^n \mathrm{d}x = \frac{x^{n+1}}{n+1} + C \quad \text{Provided} \quad n \neq -1.$$

4.

$$\int \frac{1}{x} \mathrm{d}x \quad \text{i.e.} \quad \int x^{-1} \mathrm{d}x = \ln x + C.$$

5.

$$\int e^x \mathrm{d}x = e^x + C.$$

6.

$$\int \cos x \mathrm{d}x = \sin x + C.$$

7.

$$\int \sin x \mathrm{d}x = -\cos x + C.$$

Note:
Basic integrals of the above kinds may be quoted from a table of standard integrals in a suitable formula booklet.

More advanced integrals are obtainable using the rules which follow.

12.1.2 ELEMENTARY TECHNIQUES OF INTEGRATION

(a) Linearity

Suppose $f(x)$ and $g(x)$ are two functions of x while A and B are constants.

Then

$$\int [Af(x) + Bg(x)]\mathrm{d}x = A\int f(x)\mathrm{d}x + B\int g(x)\mathrm{d}x.$$

The proof follows because differentiation is already linear.

The result is easily extended to linear combinations of three or more functions.

ILLUSTRATIONS

1.

$$\int (x^2 + 3x - 7)\mathrm{d}x = \frac{x^3}{3} + 3\frac{x^2}{2} - 7x + C.$$

2.

$$\int (3\cos x + 4\mathrm{sec}^2 x)\mathrm{d}x = 3\sin x + 4\tan x + C.$$

(b) Functions of a Linear Function

$$\int f(ax+b)\ \mathrm{d}x.$$

(i) Inspection Method

EXAMPLES

1. Determine the indefinite integral

$$\int (2x+3)^{12}\mathrm{d}x.$$

Solution

To arrive at $(2x+3)^{12}$ by differentiation, we must begin with a function related to $(2x+3)^{13}$.

In fact,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[(2x+3)^{13}\right] = 13(2x+3)^{12}.2 = 26(2x+3)^{12}.$$

This is 26 times the function we are trying to integrate.

Hence,

$$\int (2x+3)^{12} = \frac{(2x+3)^{13}}{26} + C.$$

2. Determine the indefinite integral

$$\int \cos(3-5x)\mathrm{d}x.$$

Solution

To arrive at $\cos(3-5x)$ by differentiation, we must begin with a function related to $\sin(3-5x)$.

In fact

$\frac{\mathrm{d}}{\mathrm{d}x}[\sin(3-5x)] = \cos(3-5x).-5 = -5\cos(3-5x).$

This is -5 times the function we are trying to integrate.

Hence,

$$\int \cos(3-5x) = -\frac{\sin(3-5x)}{5} + C.$$

3. Determine the indefinite integral

$$\int e^{4x+1}\mathrm{d}x.$$

Solution

To arrive at e^{4x+1} by differentiation, we must begin with a function related to e^{4x+1}.

In fact,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[e^{4x+1}\right] = e^{4x+1}.4$$

This is 4 times the function we are trying to integrate.

Hence,

$$\int e^{4x+1}\mathrm{d}x = \frac{e^{4x+1}}{4} + C.$$

4. Determine the indefinite integral

$$\int \frac{1}{7x+3} \mathrm{d}x.$$

Solution

To arrive at $\frac{1}{7x+3}$ by differentiation, we must begin with a function related to $\ln(7x+3)$.

In fact,

$$\frac{\mathrm{d}}{\mathrm{d}x}[\ln(7x+3)] = \frac{1}{7x+3}.7 = \frac{7}{7x+3}$$

This is 7 times the function we are trying to integrate.

Hence,

$$\int \frac{1}{7x+3} \mathrm{d}x = \frac{\ln(7x+3)}{7} + C.$$

Note:

We treat the linear function $ax+b$ like a single x, then divide the result by a.

(ii) Substitution Method

In the integral

$$\int f(ax+b)\mathrm{d}x,$$

we may substitute $u = ax + b$ as follows:

Suppose

$$z = \int f(ax+b)\mathrm{d}x.$$

Then,

$$\frac{\mathrm{d}z}{\mathrm{d}x} = f(ax+b).$$

That is,

$$\frac{\mathrm{d}z}{\mathrm{d}x} = f(u).$$

But

$$\frac{\mathrm{d}z}{\mathrm{d}u} = \frac{\mathrm{d}z}{\mathrm{d}x}.\frac{\mathrm{d}x}{\mathrm{d}u} = f(u).\frac{\mathrm{d}x}{\mathrm{d}u}.$$

Hence,

$$z = \int f(u)\frac{\mathrm{d}x}{\mathrm{d}u}\mathrm{d}u.$$

Note:

The secret is to replace $\mathrm{d}x$ with $\frac{\mathrm{d}x}{\mathrm{d}u}.\mathrm{d}u$.

EXAMPLES

1. Determine the indefinite integral

$$z = \int (2x+3)^{12}\mathrm{d}x.$$

Solution

Putting $u = 2x+3$ gives $\frac{\mathrm{d}u}{\mathrm{d}x} = 2$ and, hence, $\frac{\mathrm{d}x}{\mathrm{d}u} = \frac{1}{2}$.

Thus,

$$z = \int u^{12}.\frac{1}{2}\mathrm{d}u = \frac{u^{13}}{13} \times \frac{1}{2} + C.$$

That is,

$$z = \frac{(2x+3)^{13}}{26} + C$$

as before.

2. Determine the indefinite integral

$$z = \int \cos(3-5x)\mathrm{d}x.$$

Solution

Putting $u = 3-5x$ gives $\frac{\mathrm{d}u}{\mathrm{d}x} = -5$ and hence $\frac{\mathrm{d}x}{\mathrm{d}u} = -\frac{1}{5}$.

Thus,

$$z = \int \cos u. -\frac{1}{5}\mathrm{d}u = -\frac{1}{5}\sin u + C.$$

That is,

$$z = -\frac{1}{5}\sin(3-5x) + C,$$

as before.

3. Determine the indefinite integral

$$z = \int e^{4x+1} \mathrm{d}x.$$

Solution

Putting $u = 4x + 1$ gives $\frac{\mathrm{d}u}{\mathrm{d}x} = 4$ and, hence, $\frac{\mathrm{d}x}{\mathrm{d}u} = \frac{1}{4}$. Thus,

$$z = \int e^u . \frac{1}{4} \mathrm{d}u = \frac{e^u}{4} + C.$$

That is,

$$z = \frac{e^{4x+1}}{4} + C,$$

as before

4. Determine the indefinite integral

$$z = \int \frac{1}{7x+3} \mathrm{d}x.$$

Solution

Putting $u = 7x + 3$ gives $\frac{\mathrm{d}u}{\mathrm{d}x} = 7$ and, hence, $\frac{\mathrm{d}x}{\mathrm{d}u} = \frac{1}{7}$ Thus,

$$z = \int \frac{1}{u} . \frac{1}{7} \mathrm{d}u = \frac{1}{7} \ln u + C.$$

That is,

$$z = \frac{1}{7} \ln(7x + 3) + C.$$

as before.