# "JUST THE MATHS"

# SLIDES NUMBER

# 1.9

# ALGEBRA 9
# (The theory of partial fractions)

## by


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**1.9.1 Introduction**
**1.9.2 Standard types of partial fraction problem**

# UNIT 1.9 - ALGEBRA 9
# THE THEORY OF PARTIAL FRACTIONS

## 1.9.1 INTRODUCTION

Applies chiefly to a **"Proper Rational Function"** (numerator powers lower than denominator powers)

**"Improper Rational Function"** = the sum of a polynomial and a proper rational function (by long division)

## ILLUSTRATION

$$\frac{1}{2x+3} + \frac{3}{x-1} \equiv \frac{7x+8}{(2x+3)(x-1)}.$$

## 1.9.2 STANDARD TYPES OF PARTIAL FRACTION PROBLEM

### (a) Denominator of the given rational function has all linear factors.

## EXAMPLE

$$\frac{7x+8}{(2x+3)(x-1)} \equiv \frac{A}{2x+3} + \frac{B}{x-1}.$$

### Solution

$$7x+8 \equiv A(x-1) + B(2x+3).$$

Substituting $x = 1$ gives

$$7 + 8 = B(2 + 3).$$

Hence,

$$B = \frac{7+8}{2+3} = \frac{15}{5} = 3.$$

Substituting $x = -\frac{3}{2}$ gives

$$7 \times -\frac{3}{2} + 8 = A(-\frac{3}{2} - 1).$$

Hence,

$$A = \frac{7 \times -\frac{3}{2} + 8}{-\frac{3}{2} - 1} = \frac{-\frac{5}{2}}{-\frac{5}{2}} = 1.$$

$$\frac{7x+8}{(2x+3)(x-1)} = \frac{1}{2x+3} + \frac{3}{x-1}.$$

Alternatively, use the "Cover-up" Rule

## (b) Denominator of the given rational function contains one linear and one quadratic factor

## EXAMPLE

$$\frac{3x^2+9}{(x-5)(x^2+2x+7)} \equiv \frac{A}{x-5} + \frac{Bx+C}{x^2+2x+7}.$$

**Solution**

$$3x^2 + 9 \equiv A(x^2 + 2x + 7) + (Bx + C)(x - 5).$$

$x = 5$ gives

$$3 \times 5^2 + 9 = A(5^2 + 2 \times 5 + 7);$$

So, $84 = 42A$ or $A = 2$.

Equating coefficients of $x^2$, $3 = A + B$ and hence $B = 1$.

Equating constant terms (the coefficients of $x^0$), $9 = 7A - 5C = 14 - 5C$ and hence $C = 1$.

Therefore,

$$\frac{3x^2 + 9}{(x - 5)(x^2 + 2x + 7)} \equiv \frac{2}{x - 5} + \frac{x + 1}{x^2 + 2x + 7}.$$

**Note:** $A$ may be found by the cover-up rule, $B$ and $C$ by inspection.

## (c) Denominator of the given rational function contains a repeated linear factor

## EXAMPLE

$$\frac{9}{(x+1)^2(x-2)}.$$

## Solution

First observe that

$$\frac{Ax+B}{(x+1)^2}.$$

may be written

$$\frac{A(x+1)+B-A}{(x+1)^2} \equiv \frac{A}{x+1}+\frac{B-A}{(x+1)^2} \equiv \frac{A}{x+1}+\frac{C}{(x+1)^2}.$$

Therefore, write

$$\frac{9}{(x+1)^2(x-2)} \equiv \frac{A}{x+1} + \frac{C}{(x+1)^2} + \frac{D}{x-2}.$$

$$9 \equiv A(x+1)(x-2) + C(x-2) + D(x+1)^2.$$

$x = -1$ gives $9 = -3C$ so that $C = -3$.

$x = 2$ gives $9 = 9D$ so that $D = 1$.

Equating coefficients of $x^2$ gives $0 = A + D$ so that $A = -1$.

Hence,

$$\frac{9}{(x+1)^2(x-2)} \equiv -\frac{1}{x+1} - \frac{3}{(x+1)^2} + \frac{1}{x-2}.$$

**Note:** $D$ could have been obtained by the cover-up rule.

## (d) Keily's Method (uses Cover-up Rule)

## EXAMPLE

$$\frac{9}{(x+1)^2(x-2)}.$$

## Solution

$$\frac{9}{(x+1)^2(x-2)} \equiv \frac{1}{x+1}\left[\frac{9}{(x+1)(x-2)}\right].$$

$$\equiv \frac{1}{x+1}\left[\frac{-3}{x+1} + \frac{3}{x-2}\right].$$

$$\equiv -\frac{3}{(x+1)^2} + \frac{3}{(x+1)(x-2)}.$$

$$\equiv -\frac{3}{(x+1)^2} - \frac{1}{x+1} + \frac{1}{x-2}$$

as before.

**Warning**

$$\frac{9x^2}{(x+1)^2(x-2)}$$

leads to an improper rational function

$$\frac{1}{x+1}\left[\frac{9x^2}{(x+1)(x-2)}\right].$$