

“JUST THE MATHS”

UNIT NUMBER

9.9

MATRICES 9
(Modal & spectral matrices)

by

A.J.Hobson

<p>9.9.1 Assumptions and definitions 9.9.2 Diagonalisation of a matrix 9.9.3 Exercises 9.9.4 Answers to exercises</p>

UNIT 9.9 - MATRICES 9

MODAL AND SPECTRAL MATRICES

9.9.1 ASSUMPTIONS AND DEFINITIONS

For convenience, we shall make, here, the following assumptions:

(a) The n eigenvalues, $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n$, of an $n \times n$ matrix, A , are arranged in order of decreasing value.

(b) Corresponding to $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n$, respectively, A possesses a full set of eigenvectors $X_1, X_2, X_3, \dots, X_n$, which are linearly independent.

If two eigenvalues coincide, the order of writing down the corresponding pair of eigenvectors will be immaterial.

DEFINITION 1

The square matrix obtained by using as its columns any set of linearly independent eigenvectors of a matrix A is called a “**modal matrix**” of A , and may be denoted by M .

Notes:

(i) There are infinitely many modal matrices for a given matrix, A , since any multiple of an eigenvector is also an eigenvector.

(ii) It is sometimes convenient to use a set of normalised eigenvectors.

When using normalised eigenvectors, the modal matrix may be denoted by N and, for an $n \times n$ matrix, A , there are 2^n possibilities for N since each of the n columns has two possibilities.

DEFINITION 2

If $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n$ are the eigenvalues of an $n \times n$ matrix, A , then the diagonal matrix,

$$\begin{bmatrix} \lambda_1 & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & 0 & \dots & 0 \\ 0 & 0 & \lambda_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \lambda_n \end{bmatrix}$$

is called the “**spectral matrix**” of A , and may be denoted by S .

EXAMPLE

For the matrix,

$$A = \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix},$$

determine a modal matrix, a modal matrix of normalised eigenvectors and the spectral matrix.

Solution

The characteristic equation is

$$\begin{vmatrix} 1 - \lambda & 1 & -2 \\ -1 & 2 - \lambda & 1 \\ 0 & 1 & -1 - \lambda \end{vmatrix} = 0,$$

which may be shown to give

$$-(1 + \lambda)(1 - \lambda)(2 - \lambda) = 0.$$

Hence, the eigenvalues are $\lambda_1 = 2$, $\lambda_2 = 1$ and $\lambda_3 = -1$ in order of decreasing value.

Case 1. $\lambda = 2$

We solve the simultaneous equations

$$\begin{aligned} -x + y - 2z &= 0, \\ -x + 0y + z &= 0, \\ 0x + y - 3z &= 0, \end{aligned}$$

which give $x : y : z = 1 : 3 : 1$

Case 2. $\lambda = 1$

We solve the simultaneous equations

$$\begin{aligned}0x + y - 2z &= 0, \\ -x + y + z &= 0, \\ 0x + y - 2z &= 0,\end{aligned}$$

which give $x : y : z = 3 : 2 : 1$

Case 3. $\lambda = -1$

We solve the simultaneous equations

$$\begin{aligned}2x + y - 2z &= 0, \\ -x + 3y + z &= 0, \\ 0x + y + 0z &= 0,\end{aligned}$$

which give $x : y : z = 1 : 0 : 1$

A modal matrix for A may therefore be given by

$$M = \begin{bmatrix} 1 & 3 & 1 \\ 3 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

A modal matrix of normalised eigenvectors may be given by

$$N = \begin{bmatrix} \frac{1}{\sqrt{11}} & \frac{3}{\sqrt{14}} & \frac{1}{\sqrt{2}} \\ \frac{3}{\sqrt{11}} & \frac{2}{\sqrt{14}} & 0 \\ \frac{1}{\sqrt{11}} & \frac{1}{\sqrt{14}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

The spectral matrix is given by

$$S = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

9.9.2 DIAGONALISATION OF A MATRIX

Since the eigenvalues of a diagonal matrix are equal to its diagonal elements, it is clear that a matrix, A , and its spectral matrix, S , have the same eigenvalues.

From the Theorem in Unit 9.8, therefore, it seems reasonable that A and S could be similar matrices; and this is the content of the following result which will be illustrated rather than proven.

THEOREM

The matrix, A , is similar to its spectral matrix, S , the similarity transformation being

$$M^{-1}AM = S,$$

where M is a modal matrix for A .

ILLUSTRATION:

Suppose that X_1 , X_2 and X_3 are linearly independent eigenvectors of a 3×3 matrix, A , corresponding to eigenvalues λ_1 , λ_2 and λ_3 , respectively.

Then,

$$AX_1 = \lambda_1 X_1, \quad AX_2 = \lambda_2 X_2, \quad \text{and} \quad AX_3 = \lambda_3 X_3.$$

Also,

$$M = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}.$$

If M is premultiplied by A , we obtain a 3×3 matrix whose columns are AX_1 , AX_2 , and AX_3 .

That is,

$$AM = [AX_1 \quad AX_2 \quad AX_3] = [\lambda_1 X_1 \quad \lambda_2 X_2 \quad \lambda_3 X_3]$$

or

$$AM = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix} \cdot \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = MS.$$

We conclude that

$$M^{-1}AM = S.$$

Notes:

- (i) M^{-1} exists only because X_1 , X_2 and X_3 are linearly independent.
- (ii) The similarity transformation in the above theorem reduces the matrix, A , to “**diagonal form**” or “**canonical form**” and the process is often referred to as the “**diagonalisation**” of the matrix, A .

EXAMPLE

Verify the above Theorem for the matrix

$$A = \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}.$$

Solution

From an earlier example, a modal matrix for A may be given by

$$M = \begin{bmatrix} 1 & 3 & 1 \\ 3 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

The spectral matrix is given by

$$S = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

It may be shown that

$$M^{-1} = \frac{1}{6} \begin{bmatrix} -2 & 2 & 2 \\ 3 & 0 & -3 \\ -1 & -2 & 7 \end{bmatrix}$$

and, hence,

$$M^{-1}AM = \frac{1}{6} \begin{bmatrix} -2 & 2 & 2 \\ 3 & 0 & -3 \\ -1 & -2 & 7 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 3 & 1 \\ 3 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \frac{1}{6} \begin{bmatrix} -2 & 2 & 2 \\ 3 & 0 & -3 \\ -1 & -2 & 7 \end{bmatrix} \cdot \begin{bmatrix} 2 & 3 & -1 \\ 6 & 2 & 0 \\ 2 & 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = S.$$

9.9.3 EXERCISES

1. Determine a modal matrix, M , of linearly independent eigenvectors for the matrix

$$A = \begin{bmatrix} -1 & 6 & -12 \\ 0 & -13 & 30 \\ 0 & -9 & 20 \end{bmatrix}.$$

Verify that $M^{-1}AM = S$, where S is the spectral matrix of A .

2. Determine a modal matrix, M , of linearly independent eigenvectors for the matrix

$$B = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}.$$

Verify that $M^{-1}BM = S$, where S is the spectral matrix of B .

3. Determine a modal matrix, N , of linearly independent normalised eigenvectors for the matrix

$$C = \begin{bmatrix} 2 & -2 & 0 \\ 0 & 4 & 0 \\ 2 & 5 & 1 \end{bmatrix}.$$

Verify that $N^{-1}CN = S$, where S is the spectral matrix of C .

4. Show that the following matrices are not similar to a diagonal matrix:

$$(a) \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix}, \quad (b) \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{bmatrix}.$$

9.9.4 ANSWERS TO EXERCISES

1. The eigenvalues are 5, 2 and -1 , which gives

$$M = \begin{bmatrix} -1 & 0 & 1 \\ 5 & 2 & 0 \\ 3 & 1 & 0 \end{bmatrix}.$$

2. The eigenvalues are 2, 1 and -1 , which gives

$$M = \begin{bmatrix} 1 & 3 & 1 \\ 3 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

3. The eigenvalues are 4, 2 and 1, which gives

$$N = \begin{bmatrix} -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{5}} & 0 \\ \frac{1}{\sqrt{3}} & 0 & 0 \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{5}} & 1 \end{bmatrix}.$$

4. (a) The eigenvalues are 2 (repeated) and 1 but there are only two linearly independent eigenvectors, namely

$$\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}.$$

(b) There is only one eigenvalue, 1 (repeated), and only one linearly independent eigenvector, namely

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$