

**“JUST THE MATHS”**

**UNIT NUMBER**

**4.2**

**HYPERBOLIC FUNCTIONS 2**  
**(Inverse hyperbolic functions)**

by

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## UNIT 4.2 - HYPERBOLIC FUNCTIONS 2

### INVERSE HYPERBOLIC FUNCTIONS

#### 4.2.1 - INTRODUCTION

The three basic inverse hyperbolic functions are  $\text{Cosh}^{-1}x$ ,  $\text{Sinh}^{-1}x$  and  $\text{Tanh}^{-1}x$ .

It may be shown that they are given by the following formulae:

(a) 
$$\text{Cosh}^{-1}x = \pm \ln(x + \sqrt{x^2 - 1});$$

(b) 
$$\text{Sinh}^{-1}x = \ln(x + \sqrt{x^2 + 1});$$

(c) 
$$\text{Tanh}^{-1}x = \frac{1}{2} \ln \frac{1+x}{1-x}.$$

#### Notes:

(i) The positive value of  $\text{Cosh}^{-1}x$  is called the ‘**principal value**’ and is denoted by  $\text{cosh}^{-1}x$  (using a lower-case c).

(ii)  $\text{Sinh}^{-1}x$  and  $\text{Tanh}^{-1}x$  have only **one** value but, for uniformity, we customarily denote them by  $\text{sinh}^{-1}x$  and  $\text{tanh}^{-1}x$ .

#### 4.2.2 THE PROOFS OF THE STANDARD FORMULAE

##### (a) Inverse Hyperbolic Cosine

If we let  $y = \text{Cosh}^{-1}x$ , then

$$x = \text{cosh } y = \frac{e^y + e^{-y}}{2}.$$

Hence,

$$2x = e^y + e^{-y}.$$

On rearrangement,

$$(e^y)^2 - 2xe^y + 1 = 0,$$

which is a quadratic equation in  $e^y$  having solutions, from the quadratic formula, given by

$$e^y = \frac{2x \pm \sqrt{4x^2 - 4}}{2} = x \pm \sqrt{x^2 - 1}.$$

Taking natural logarithms of both sides gives

$$y = \ln(x \pm \sqrt{x^2 - 1}).$$

However, the two values  $x + \sqrt{x^2 - 1}$  and  $x - \sqrt{x^2 - 1}$  are reciprocals of each other, since their product is the value, 1; and so

$$y = \pm \ln(x + \sqrt{x^2 - 1}).$$

### (b) Inverse Hyperbolic Sine

If we let  $y = \text{Sinh}^{-1}x$ , then

$$x = \sinh y = \frac{e^y - e^{-y}}{2}.$$

Hence,

$$2x = e^y - e^{-y},$$

or

$$(e^y)^2 - 2xe^y - 1 = 0,$$

which is a quadratic equation in  $e^y$  having solutions, from the quadratic formula, given by

$$e^y = \frac{2x \pm \sqrt{4x^2 + 4}}{2} = x \pm \sqrt{x^2 + 1}.$$

However,  $x - \sqrt{x^2 + 1}$  has a negative value and cannot, therefore, be equated to a power of  $e$ , which is positive. Hence, this part of the expression for  $e^y$  must be ignored.

Taking natural logarithms of both sides gives

$$y = \ln(x + \sqrt{x^2 + 1}).$$

### (c) Inverse Hyperbolic Tangent

If we let  $y = \text{Tanh}^{-1}x$ , then

$$x = \tanh y = \frac{e^y - e^{-y}}{e^y + e^{-y}} = \frac{e^{2y} - 1}{e^{2y} + 1}.$$

Hence,

$$x(e^{2y} + 1) = e^{2y} - 1,$$

giving

$$e^{2y} = \frac{1 + x}{1 - x}.$$

Taking natural logarithms of both sides,

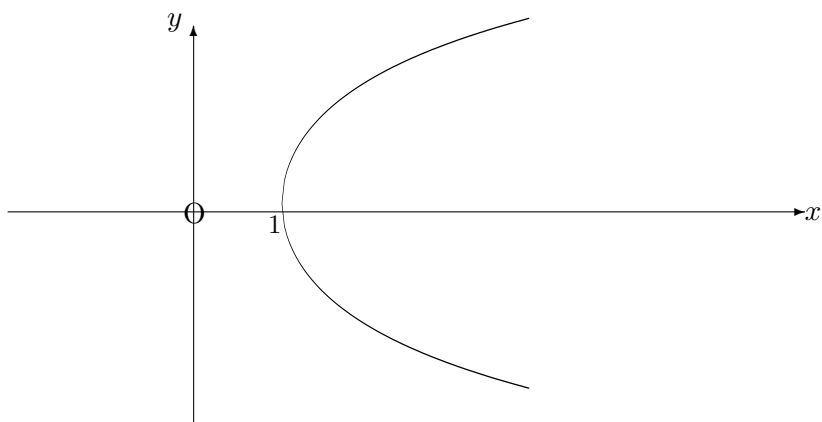
$$y = \frac{1}{2} \ln \frac{1 + x}{1 - x}.$$

#### **Note:**

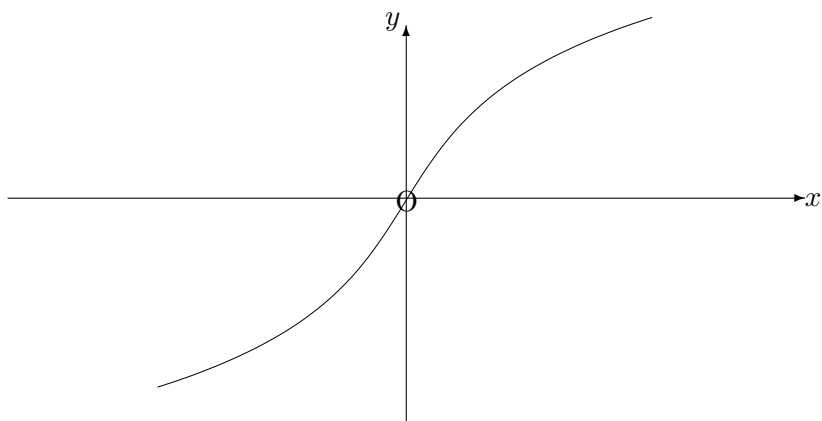
The graphs of inverse hyperbolic functions are discussed fully in Unit 10.7, but we include them here for the sake of completeness:

The graphs are as follows:

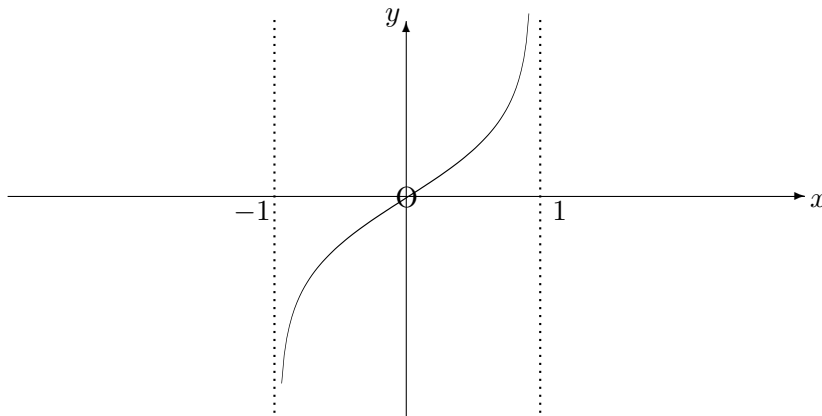
(a)  $y = \text{Cosh}^{-1}x$



(b)  $y = \text{Sinh}^{-1}x$



(c)  $y = \text{Tanh}^{-1}x$



### 4.2.3 EXERCISES

1. Use the standard formulae to evaluate (a)  $\sinh^{-1}7$  and (b)  $\cosh^{-1}9$ .
2. Express  $\cosh 2x$  and  $\sinh 2x$  in terms of exponentials and hence solve, for  $x$ , the equation

$$2 \cosh 2x - \sinh 2x = 2.$$

3. Obtain a formula which equates  $\text{cosech}^{-1}x$  to the natural logarithm of an expression in  $x$ , distinguishing between the two cases  $x > 0$  and  $x < 0$ .
4. If  $t = \tanh(x/2)$ , prove that

(a)

$$\sinh x = \frac{2t}{1-t^2}$$

and

(b)

$$\cosh x = \frac{1+t^2}{1-t^2}.$$

Hence solve, for  $x$ , the equation

$$7 \sinh x + 20 \cosh x = 24.$$

#### 4.2.4 ANSWERS TO EXERCISES

1. (a)

$$\ln(7 + \sqrt{49 + 1}) \simeq 2.644;$$

(b)

$$\ln(9 + \sqrt{81 - 1}) \simeq 2.887$$

2.

$$(e^{2x})^2 - 4e^{2x} + 3 = 0,$$

which gives  $e^{2x} = 1$  or  $3$  and hence  $x = 0$  or  $\frac{1}{2} \ln 3 \simeq 0.549$ .

3. If  $x > 0$ , then

$$\operatorname{cosech}^{-1}x = \ln \frac{1 + \sqrt{1 + x^2}}{x}.$$

If  $x < 0$ , then

$$\operatorname{cosech}^{-1}x = \ln \frac{1 - \sqrt{1 + x^2}}{x}.$$

4. Use

$$\sinh x \equiv \frac{2 \tanh(x/2)}{\operatorname{sech}^2(x/2)}$$

and

$$\cosh x \equiv \frac{(1 + \tanh^2(x/2))}{\operatorname{sech}^2(x/2)}.$$

This gives  $t = -\frac{1}{2}$  or  $t = \frac{2}{11}$  and hence  $x \simeq -1.099$  or  $0.368$  which agrees with the solution obtained using exponentials.