# "JUST THE MATHS"

# UNIT NUMBER

# 10.7

# DIFFERENTIATION 7
# (Inverse hyperbolic functions)

# by



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## UNIT 10.7 - DIFFERENTIATION

## DERIVATIVES OF INVERSE HYPERBOLIC FUNCTIONS

### 10.7.1 SUMMARY OF RESULTS

The derivatives of inverse trigonometric and inverse hyperbolic functions should be considered as standard results. They will be stated here, first, before their proofs are discussed.

1.
$$\frac{\mathrm{d}}{\mathrm{d}x}[\sinh^{-1}x] = \frac{1}{\sqrt{1+x^2}},$$
where $-\infty < \sinh^{-1}x < \infty$.

2.
$$\frac{\mathrm{d}}{\mathrm{d}x}[\cosh^{-1}x] = \frac{1}{\sqrt{x^2-1}},$$
where $\cosh^{-1}x \geq 0$.

3.
$$\frac{\mathrm{d}}{\mathrm{d}x}[\tanh^{-1}x] = \frac{1}{1-x^2},$$
where $-\infty < \tanh^{-1}x < \infty$.

### 10.7.2 THE DERIVATIVE OF AN INVERSE HYPERBOLIC SINE

We shall consider the formula
$$y = \text{Sinh}^{-1}x$$
and determine an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$.

**Note:**
The use of the upper-case S in the formula is temporary; and the reason will be explained shortly.

The formula is equivalent to
$$x = \sinh y,$$
so we may say that
$$\frac{\mathrm{d}x}{\mathrm{d}y} = \cosh y \equiv \sqrt{1+\sin^2 y} \equiv \sqrt{1+x^2},$$

noting that $\cosh y$ is never negative.

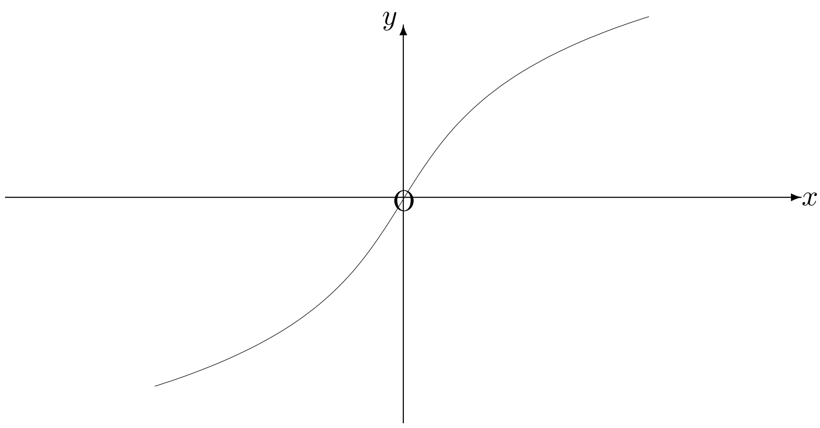
Thus,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sqrt{1+x^2}}.$$

Consider now the graph of the formula

$$y = \mathrm{Sinh}^{-1}x$$

which may be obtained from the graph of $y = \sinh x$ by reversing the roles of $x$ and $y$ and rearranging the new axes into the usual positions. We obtain:




**Observations**

(i) The variable $x$ may lie anywhere in the interval $-\infty < x < \infty$.

(ii) For each value of $x$, the variable $y$ has only one value.

(iii) For each value of $x$, there is only one possible value of $\frac{\mathrm{d}y}{\mathrm{d}x}$, which is positive.

(iv) In this case (unlike the case of an inverse sine, in Unit 10.6) there is no need to distinguish between a general value and a principal value of the inverse hyperbolic sine function. This is because there is only one value of both the function and its derivative.

However, it is customary to denote the inverse function by $\sinh^{-1}x$, using a lower-case s rather than an upper-case S.

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}x}[\sinh^{-1}x] = \frac{1}{\sqrt{1+x^2}}.$$

### 10.7.3 THE DERIVATIVE OF AN INVERSE HYPERBOLIC COSINE

We shall consider the formula

$$y = \mathrm{Cosh}^{-1}x$$

and determine an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$.

**Note:**
There is a special significance in using the upper-case C in the formula; the reason will be explained shortly.

The formula is equivalent to

$$x = \cosh y,$$

so we may say that

$$\frac{\mathrm{d}x}{\mathrm{d}y} = \sinh y \equiv \pm\sqrt{\cosh^2 y - 1} \equiv \pm\sqrt{x^2 - 1}.$$

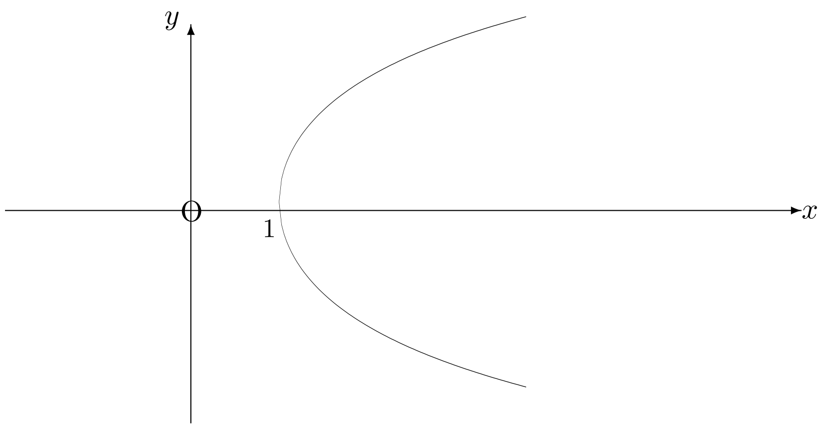
Thus,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sqrt{x^2 - 1}}.$$

Consider now the graph of the formula

$$y = \mathrm{Cosh}^{-1}x,$$

which may be obtained from the graph of $y = \cosh x$ by reversing the roles of $x$ and $y$ and rearranging the new axes into the usual positions. We obtain:



**Observations**

(i) The variable $x$ must lie in the interval $x \geq 1$.

(ii) For each value of $x$ in the interval $x > 1$, the variable $y$ has two values one of which is positive and the other negative.

(iii) For each value of $x$ in the interval $x > 1$, there are only two possible values of $\frac{\mathrm{d}y}{\mathrm{d}x}$, one of which is postive and the other negative.

(iv) By restricting the discussion to the part of the graph for which $y \geq 0$, there will be only one value of $y$ with one (positive) value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ for each value of $x$ in the interval $x \geq 1$.

The restricted part of the graph defines what is called the **"principal value"** of the inverse cosine function and is denoted by $\cosh^{-1}x$, using a lower-case c.

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}x}[\cosh^{-1}x] = \frac{1}{\sqrt{x^2-1}}.$$

### 10.7.4 THE DERIVATIVE OF AN INVERSE HYPERBOLIC TANGENT

We shall consider the formula

$$y = \mathrm{Tanh}^{-1}x$$

and determine an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$.

**Note:**
The use of the upper-case T in the formula is temporary; and the reason will be explained later.

The formula is equivalent to

$$x = \tanh y,$$

so we may say that

$$\frac{\mathrm{d}x}{\mathrm{d}y} = \mathrm{sech}^2 y \equiv 1 - \tanh^2 y \equiv 1 - x^2.$$

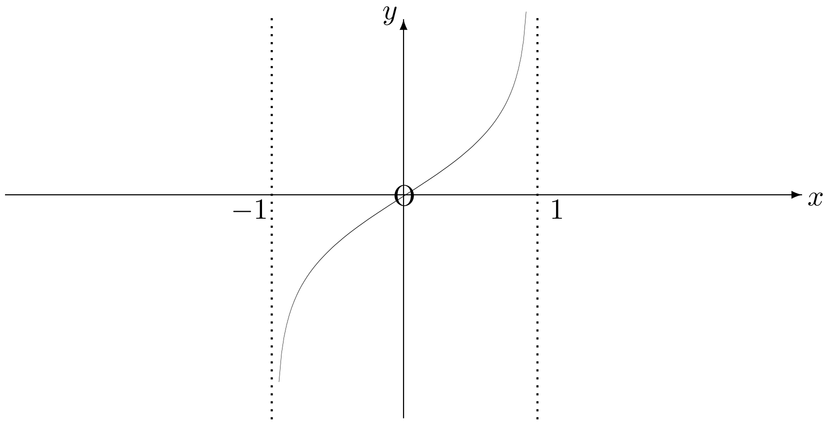
Thus,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{1-x^2}.$$

Consider now the graph of the formula

$$y = \mathrm{Tanh}^{-1}x,$$

which may be obtained from the graph of $y = \tanh x$ by reversing the roles of $x$ and $y$ and rearranging the new axes into the usual positions. We obtain:




**Observations**

(i) The variable $x$ must lie in the interval $-1 < x < 1$.

(ii) For each value of $x$ in the interval $-1 < x < 1$, the variable $y$ has just one value.

(iii) For each value of $x$ in the interval $-1 < x < 1$, there is only possible value of $\frac{dy}{dx}$, which is positive.

(iv) In this case (unlike the case of an inverse tangent in Unit 10.6) there is no need to distinguish between a general value and a principal value of the inverse hyperbolic tangent function. This is because there is only one value of both the function and its derivative.

However, it is customary to denote the inverse hyperbolic tangent by $\tan^{-1}x$ using a lower-case t rather than an upper-case T.

Hence,

$$\frac{d}{dx}[\tanh^{-1}x] = \frac{1}{1-x^2}.$$

**ILLUSTRATIONS**

1.

$$\frac{d}{dx}[\sin^{-1}(\tanh x)] = \frac{\text{sech}^2 x}{\sqrt{1-\tanh^2 x}} = \text{sech} x.$$

2.

$$\frac{\mathrm{d}}{\mathrm{d}x}[\cosh^{-1}(5x-4)] = \frac{5}{\sqrt{(5x-4)^2-1}},$$

assuming that $5x - 4 \geq 1$; that is, $x \geq 1$.

### 10.7.5 EXERCISES

Obtain an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ in the following cases:

1.

$$y = \cosh^{-1}(4+3x),$$

assuming that $4 + 3x \geq 1$; that is, $x \geq -1$.

2.

$$y = \cos^{-1}(\sinh x),$$

assuming that $-1 \leq \sinh x \leq 1$.

3.

$$y = \tanh^{-1}(\cos x).$$

4.

$$y = \cosh^{-1}(x^3),$$

assuming that $x \geq 1$.

5.

$$y = \tanh^{-1}\frac{2x}{1+x^2}.$$

### 10.7.6 ANSWERS TO EXERCISES

1.

$$\frac{3}{\sqrt{(4+3x)^2-1}}.$$

2.

$$-\frac{\cosh x}{\sqrt{1-\sinh^2 x}}.$$

3.

$$-\text{cosec}x.$$

4.

$$\frac{3x^2}{\sqrt{x^6-1}}.$$

5.

$$\frac{2}{1-x^2}.$$