"JUST THE MATHS"

UNIT NUMBER

1.10

ALGEBRA 10
(Inequalities 1)

by

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UNIT 1.10 - ALGEBRA 10 - INEQUALITIES 1.

1.10.1 INTRODUCTION

If the symbols a and b denote numerical quantities, then the statement

$$a < b$$

is used to mean "a is less than b" while the statement

$$b > a$$

is used to mean "b is greater than a"

These are called **"strict inequalities"** because there is no allowance for the possibilty that a and b might be equal to each other. For example, if a is the number of days in a particular month and b is the number of hours in that month, then $b > a$.

Some inequalities do allow the possibility of a and b being equal to each other and are called **"weak inequalities"** written in one of the forms

$$a \leq b \quad b \leq a \quad a \geq b \quad b \geq a$$

For example, if a is the number of students who enrolled for a particular module in a university and b is the number of students who eventually passed that module, then $a \geq b$.

1.10.2 ALGEBRAIC RULES FOR INEQUALITIES

Given two different numbers, one of them must be strictly less than the other. Suppose a is the smaller of the two and b the larger; i.e.

$$a < b$$

Then
1. $a + c < b + c$ for any number c.

2. $ac < bc$ when c is positive but $ac > bc$ when c is negative.

3. $\frac{1}{a} > \frac{1}{b}$ provided a and b are **both positive.**

Note:
The only other situation in 3. which is consistent with $a < b$ will occur if a is negative and b is positive. In this case, $\frac{1}{a} < \frac{1}{b}$ because a negative number is always less than a positive number.

EXAMPLES

1. Simplify the inequality
$$2x + 3y > 5x - y + 7.$$
Solution
We simply deal with this in the same way as we would deal with an equation by adding appropriate quantities to both sides or subtracting appropriate quantities from both sides. We obtain
$$-3x + 4y > 7 \quad \text{or} \quad 3x - 4y + 7 < 0.$$

2. Solve the inequality
$$\frac{1}{x-1} < 2,$$
assuming that $x \neq 1$.
Solution
Here we must be careful in case $x-1$ is negative; the argument is therefore in two parts:
(a) If $x > 1$, i.e. $x - 1$ is positive, then the inequality can be rewritten as
$$1 < 2(x-1) \quad \text{or} \quad x - 1 > \frac{1}{2}.$$
Hence,
$$x > \frac{3}{2}.$$
(b) If $x < 1$, i.e. $x - 1$ is negative, then the inequality is automatically true since a negative number is bound to be less than a positive number.
Conclusion:
The given inequality is satisfied when $x < 1$ and when $x > \frac{3}{2}$.

3. Solve the inequality
$$2x - 7 \leq 3.$$
Solution
Adding 7 to both sides, then dividing both sides by 2 gives
$$x \leq 5.$$

4. Solve the inequality
$$\frac{x-1}{x-6} \geq 0.$$

Solution

We first observe that the fraction on the left of the inequality can equal zero only when $x = 1$.

Secondly, the only way in which a fraction can be positive is for both numerator and denominator to be positive or both numerator and denominator to be negative.

(a) Suppose $x - 1 > 0$ and $x - 6 > 0$; these two are covered by $x > 6$.

(b) Suppose $x - 1 < 0$ and $x - 6 < 0$; these two are covered by $x < 1$.

Note:
The value $x = 6$ is problematic because the given expression becomes infinite; in fact, as x passes through 6 from values below it to values above it, there is a sudden change from $-\infty$ to $+\infty$.

Conclusion:
The inequality is satisfied when either $x > 6$ or $x \leq 1$.

1.10.3 INTERVALS

In scientific calculations, a variable quantity x may be restricted to a certain range of values called an **"interval"** which may extend to ∞ or $-\infty$; but, in many cases, such intervals have an upper and a lower **"bound"**. The standard types of interval are as follows:

(a) $a < x < b$ denotes an **"open interval"** of all the values of x between a and b but excluding a and b themselves. The symbol (a, b) is also used to mean the same thing. For example, if x is a purely decimal quantity, it must lie in the open interval

$$-1 < x < 1.$$

(b) $a \leq x \leq b$ denotes a **"closed interval"** of all the values of x from a to b inclusive. The symbol $[a, b]$ is also used to mean the same thing. For example, the expression $\sqrt{1 - x^2}$ has real values only when

$$-1 \leq x \leq 1.$$

Note:
It is possible to encounter intervals which are closed at one end but open at the other; they may be called either **"half open"** or **"half closed"**. For example

$$a < x \leq b \quad \text{or} \quad a \leq x < b,$$

which can also be denoted respectively by $(a, b]$ and $[a, b)$.

(c) Intervals of the types

$$x > a \quad x \geq a \quad x < a \quad x \leq a$$

are called **"infinite intervals"**.

1.10.4 EXERCISES

1. Simplify the following inequalities:

   (a)
   $$x + y \leq 2x + y + 1;$$

   (b)
   $$2a - b > 1 + a - 2b - c.$$

2. Solve the following inequalities to find the range of values of x.

   (a)
   $$x + 3 < 6;$$

   (b)
   $$-2x \geq 10;$$

   (c)
   $$\frac{2}{x} > 18;$$

   (d)
   $$\frac{x+3}{2x-1} \leq 0.$$

3. Classify the following intervals as open, closed, or half-open/half-closed:

   (a)
   $$(5, 8);$$

   (b)
   $$(-3, -2);$$

(c)

$$[2, 4);$$

(d)

$$[8, 23];$$

(e)

$$(-\infty, \infty);$$

(f)

$$(0, \infty);$$

(g)

$$[0, \infty).$$

1.10.5 ANSWERS TO EXERCISES

1. (a) $x \geq -1$;
   (b) $a + b + c > 1$.
2. (a) $x < 3$;
   (b) $x \leq -5$;
   (c) $x < \frac{1}{9}$ since $x > 0$;
   (d) $-3 \leq x < \frac{1}{2}$.
3. (a) open;
   (b) open;
   (c) half-open/half-closed;
   (d) closed;
   (e) open;
   (f) open;
   (g) half-open/half-closed.