

“JUST THE MATHS”

UNIT NUMBER

1.1

ALGEBRA 1
(Introduction to algebra)

by

A.J. Hobson

1.1.1 The Language of Algebra
1.1.2 The Laws of Algebra
1.1.3 Priorities in Calculations
1.1.4 Factors
1.1.5 Exercises
1.1.5 Answers to exercises

UNIT 1.1 - ALGEBRA 1 - INTRODUCTION TO ALGEBRA

DEFINITION

An “**Algebra**” is any Mathematical system which uses the concepts of Equality ($=$), Addition ($+$), Subtraction ($-$), Multiplication (\times or \cdot) and Division (\div).

Note:

The Algebra of Numbers is what we normally call “**Arithmetic**” and, as far as this unit is concerned, it is only the algebra of numbers which we shall be concerned with.

1.1.1 THE LANGUAGE OF ALGEBRA

Suppose we use the symbols a , b and c to denote numbers of arithmetic; then

(a) $a + b$ is called the “**sum of a and b** ”.

Note:

$a + a$ is usually abbreviated to $2a$,

$a + a + a$ is usually abbreviated to $3a$ and so on.

(b) $a - b$ is called the “**difference between a and b** ”.

(c) $a \times b$, $a \cdot b$ or even just ab is called the “**product**” of a and b .

Notes:

(i)

$a \cdot a$ is usually abbreviated to a^2 ,

$a \cdot a \cdot a$ is usually abbreviated to a^3 and so on.

(ii) $-1 \times a$ is usually abbreviated to $-a$ and is called the “**negation**” of a .

(d) $a \div b$ or $\frac{a}{b}$ is called the “**quotient**” or “**ratio**” of a and b .

(e) $\frac{1}{a}$, [also written a^{-1}], is called the “**reciprocal**” of a .

(f) $|a|$ is called the “**modulus**”, “**absolute value**” or “**numerical value**” of a . It can be defined by the two statements

$|a| = a$ when a is positive or zero;

$|a| = -a$ when a is negative or zero.

Note:

Further work on fractions (ratios) will appear later, but we state here for reference the rules for combining fractions together:

Rules for combining fractions together

1.

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

2.

$$\frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd}$$

3.

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}$$

4.

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{a \cdot d}{b \cdot c}$$

EXAMPLES

1. How much more than the difference of 127 and 59 is the sum of 127 and 59 ?

Solution

The difference of 127 and 59 is $127 - 59 = 68$ and the sum of 127 and 59 is $127 + 59 = 186$.
The sum exceeds the difference by $186 - 68 = 118$.

2. What is the reciprocal of the number which is 5 multiplied by the difference of 8 and 2 ?

Solution

We require the reciprocal of $5 \cdot (8 - 2)$; that is, the reciprocal of 30. The answer is therefore $\frac{1}{30}$.

3. Calculate the value of $4\frac{2}{3} - 5\frac{1}{9}$ expressing the answer as a fraction.

Solution

Converting both numbers to a single fraction, we require

$$\frac{14}{3} - \frac{46}{9} = \frac{126 - 138}{27} = -\frac{12}{27} = -\frac{4}{9}.$$

We could also have observed that the 'lowest common multiple' (see later) of the two denominators, 3 and 9, is 9; hence we could write the alternative solution

$$\frac{42}{9} - \frac{46}{9} = -\frac{4}{9}.$$

4. Remove the modulus signs from the expression $|a - 2|$ in the cases when (i) a is greater than (or equal to) 2 and (ii) a is less than 2.

Solution

- (i) If a is greater than or equal to 2,

$$|a - 2| = a - 2;$$

- (ii) If a is less than 2,

$$|a - 2| = -(a - 2) = 2 - a.$$

1.1.2 THE LAWS OF ALGEBRA

If the symbols a , b and c denote numbers of arithmetic, then the following Laws are obeyed by them:

- (a) The Commutative Law of Addition $a + b = b + a$
- (b) The Associative Law of Addition $a + (b + c) = (a + b) + c$
- (c) The Commutative Law of Multiplication $a.b = b.a$
- (d) The Associative Law of Multiplication $a.(b.c) = (a.b).c$
- (e) The Distributive Laws $a.(b + c) = a.b + a.c$ and $(a + b).c = a.c + b.c$

Notes:

- (i) A consequence of the Distributive Laws is the rule for multiplying together a pair of bracketted expressions. It will be encountered more formally later, but we state it here for reference:

$$(a + b).(c + d) = a.c + b.c + a.d + b.d$$

- (ii) The alphabetical letters so far used for numbers in arithmetic have been taken from the **beginning** of the alphabet. These tend to be reserved for fixed quantities called **constants**. Letters from the **end** of the alphabet, such as w , x , y , z are normally used for quantities which may take many values, and are called **variables**.

1.1.3 PRIORITIES IN CALCULATIONS

Suppose that we encountered the expression $5 \times 6 - 4$. It would seem to be ambiguous, meaning either $30 - 4 = 26$ or $5 \times 2 = 10$.

However, we may remove the ambiguity by using brackets where necessary, together with a rule for precedence between the use of the brackets and the symbols $+$, $-$, \times and \div .

The rule is summarised in the abbreviation

B.O.D.M.A.S.

which means that the order of precedence is

B	brackets	()	First Priority
O	of	\times	Joint Second Priority
D	division	\div	Joint Second Priority
M	multiplication	\times	Joint Second Priority
A	addition	$+$	Joint Third Priority
S	subtraction	$-$	Joint Third Priority

Thus, $5 \times (6 - 4) = 5 \times 2 = 10$
but $5 \times 6 - 4 = 30 - 4 = 26$.

Similarly, $12 \div 3 - 1 = 4 - 1 = 3$
whereas $12 \div (3 - 1) = 12 \div 2 = 6$.

1.1.4 FACTORS

If a number can be expressed as a product of other numbers, each of those other numbers is called a “**factor**” of the original number.

EXAMPLES

1. We may observe that

$$70 = 2 \times 7 \times 5$$

so that the number 70 has factors of 2, 7 and 5. These three cannot be broken down into factors themselves because they are what are known as “**prime**” numbers (numbers whose only factors are themselves and 1). Hence the only factors of 70, apart from 70 and 1, are 2, 7 and 5.

2. Show that the numbers 78 and 182 have two common factors which are prime numbers. The two factorisations are as follows:

$$78 = 2 \times 3 \times 13,$$

$$182 = 2 \times 7 \times 13.$$

The common factors are thus 2 and 13, both of which are prime numbers.

Notes:

(i) If two or more numbers have been expressed as a product of their prime factors, we may easily identify the prime factors which are common to all the numbers and hence obtain the “**highest common factor**”, h.c.f.

For example, $90 = 2 \times 3 \times 3 \times 5$ and $108 = 2 \times 2 \times 3 \times 3 \times 3$. Hence the h.c.f = $2 \times 3 \times 3 = 18$

(ii) If two or more numbers have been expressed as a product of their prime factors, we may also identify the “**lowest common multiple**”, l.c.m.

For example, $15 = 3 \times 5$ and $20 = 2 \times 2 \times 5$. Hence the smallest number into which both 15 and 20 will divide requires two factors of 2 (for 20), one factor of 5 (for both 15 and 20) and one factor of 3 (for 15). The l.c.m. is thus $2 \times 2 \times 3 \times 5 = 60$.

(iii) If the numerator and denominator of a fraction have factors in common, then such factors may be cancelled to leave the fraction in its “**lowest terms**”.

For example $\frac{15}{105} = \frac{3 \times 5}{3 \times 5 \times 7} = \frac{1}{7}$.

1.1.5 EXERCISES

- Find the sum and product of
 - 3 and 6; (b) 10 and 7; (c) 2, 3 and 6;
 - $\frac{3}{2}$ and $\frac{4}{11}$; (e) $1\frac{2}{5}$ and $7\frac{3}{4}$; (f) $2\frac{1}{7}$ and $5\frac{4}{21}$.
- Find the difference between and quotient of
 - 18 and 9; (b) 20 and 5; (c) 100 and 20;
 - $\frac{3}{5}$ and $\frac{7}{10}$; (e) $3\frac{1}{4}$ and $2\frac{2}{9}$; (f) $1\frac{2}{3}$ and $5\frac{5}{6}$.
- Evaluate the following expressions:
 - $6 - 2 \times 2$; (b) $(6 - 2) \times 2$;
 - $6 \div 2 - 2$; (d) $(6 \div 2) - 2$;
 - $6 - 2 + 3 \times 2$; (f) $6 - (2 + 3) \times 2$;
 - $(6 - 2) + 3 \times 2$; (h) $\frac{16}{-2}$; (i) $\frac{-24}{-3}$; (j) $(-6) \times (-2)$.

4. Place brackets in the following to make them correct:

- (a) $6 \times 12 - 3 + 1 = 55$; (b) $6 \times 12 - 3 + 1 = 68$;
(c) $6 \times 12 - 3 + 1 = 60$; (d) $5 \times 4 - 3 + 2 = 7$;
(e) $5 \times 4 - 3 + 2 = 15$; (f) $5 \times 4 - 3 + 2 = -5$.

5. Express the following as a product of prime factors:

- (a) 26; (b) 100; (c) 27; (d) 71;
(e) 64; (f) 87; (g) 437; (h) 899.

6. Find the h.c.f of

- (a) 12, 15 and 21; (b) 16, 24 and 40; (c) 28, 70, 120 and 160;
(d) 35, 38 and 42; (e) 96, 120 and 144.

7. Find the l.c.m of

- (a) 5, 6, and 8; (b) 20 and 30; (c) 7, 9 and 12;
(d) 100, 150 and 235; (e) 96, 120 and 144.

1.1.6 ANSWERS TO EXERCISES

1. (a) 9, 18; (b) 17,70; (c) 11,36; (d) $\frac{41}{22}, \frac{6}{11}$; (e) $\frac{183}{20}, \frac{217}{20}$; (f) $\frac{154}{21}, \frac{545}{49}$.
2. (a) 9,2; (b) 15,4; (c) 80,5; (d) $-\frac{1}{10}, \frac{6}{7}$; (e) $\frac{37}{36}, \frac{117}{80}$; (f) $-\frac{25}{6}, \frac{2}{7}$.
3. (a) 2; (b) 8; (c) 1; (d) 1; (e) 10;
(f) -4; (g) 10; (h) -8; (i) 8; (j) 12;
4. (a) $6 \times (12 - 3) + 1 = 55$; (b) $6 \times 12 - (3 + 1) = 68$;
(c) $6 \times (12 - 3 + 1) = 60$; (d) $5 \times (4 - 3) + 2 = 7$;
(e) $5 \times 4 - (3 + 2) = 15$; (f) $5 \times (4 - [3 + 2]) = -5$.
5. (a) 2×13 ; (b) $2 \times 2 \times 5 \times 5$; (c) $3 \times 3 \times 3$; (d) 71×1 ;
(e) $2 \times 2 \times 2 \times 2 \times 2 \times 2$; (f) 3×29 ; (g) 19×23 ; (h) 29×31 .
6. (a) 3; (b) 8; (c) 2; (d) 1; (e) 24.
7. (a) 120; (b) 60; (c) 252; (d) 14100; (e) 1440.