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Periodic points detect subdynamics in rank one

(joint with Richard Miles)

Let $\beta : \mathbb{Z}^d \rightarrow \text{Homeo}(X, \rho)$ be a \mathbb{Z}^d -action on a compact metric space, so $\beta^{\mathbf{m}} \circ \beta^{\mathbf{n}} = \beta^{\mathbf{m}+\mathbf{n}}$ for all $\mathbf{m}, \mathbf{n} \in \mathbb{Z}^d$.

The action is *expansive* if there is some $\delta > 0$ with

$$\sup_{\mathbf{n} \in \mathbb{Z}^d} \rho(\beta^{\mathbf{n}}x, \beta^{\mathbf{n}}y) \leq \delta \implies x = y.$$

Subdynamics: (Boyle & Lind) $A \subset \mathbb{R}^d$ is *expansive for β* , or β is *expansive along A* , if there exist constants $\delta > 0$ and $t > 0$ with

$$\sup_{\mathbf{n}, d(\mathbf{n}, A) < t} \rho(\beta^{\mathbf{n}}x, \beta^{\mathbf{n}}y) \leq \delta \implies x = y \text{ for all } x, y \in X$$

where $d(\mathbf{n}, A)$ denotes the distance from the point \mathbf{n} to the set A in the Euclidean metric on \mathbb{R}^d .

Special case: Write G_k for the Grassmannian of k -dimensional subspaces of \mathbb{R}^d ; this is a compact $k(d - k)$ -dimensional manifold in the usual topology (subspaces are close if their intersections with the unit $(d - 1)$ -sphere S_{d-1} are close in the Hausdorff topology). Write

$$N_k(\beta) = \{V \in G_k \mid V \text{ is not expansive for } \beta\}.$$

Boyle & Lind (1997):

- $N_{d-1}(\beta)$ is a non-empty compact set;
- any element of $N_k(\beta)$ ($k < d$) is a subspace of some element of $N_{d-1}(\beta)$.

So $N_{d-1}(\beta)$ is a fundamental geometrical invariant of a topological \mathbb{Z}^d -action.

Einsiedler, Lind, Miles, Ward (2001):

Complete description of $N_{d-1}(\beta)$ for β algebraic in terms of 'adelic amoebas'.

The $\times 2, \times 3$ example

Let α be the \mathbb{Z}^2 -action obtained as the invertible extension of the commuting maps $x \mapsto 2x, x \mapsto 3x$ on the circle.

Equivalent formulation: α is the shift action of \mathbb{Z}^2 on

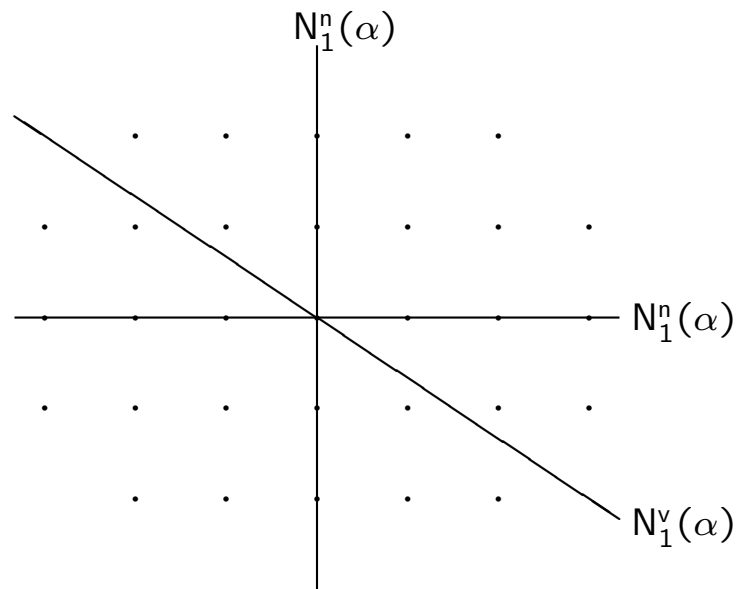
$$X = \{x \in \mathbb{T}^{\mathbb{Z}^2} \mid x_{\mathbf{n}+\mathbf{e}_1} = 2x_{\mathbf{n}}, x_{\mathbf{n}+\mathbf{e}_2} = 3x_{\mathbf{n}} \text{ for all } \mathbf{n} \in \mathbb{Z}^2\}.$$

There are two different ways a line can be non-expansive:

A 'variety' condition: \mathfrak{n} with $2^{n_1}3^{n_2} \sim 1$ means that $\alpha^{\mathfrak{n}}$ behaves almost like an isometry (in a real eigen-direction), so we expect $2^x 3^y = 1$ to be in $N_1(\alpha)$. Morally this is a vanishing of real Lyapunov exponents. These lines make up $N_1^v(\alpha)$.

An 'algebraic' condition: a map like $x \mapsto 2x$ has small invariant subgroups in X corresponding to powers of 3. So we expect $2^x = 1$ and $3^y = 1$ to be in $N_1(\alpha)$. These lines make up $N_1^n(\alpha)$.

The set $N_1(\alpha)$ for $\times 2, \times 3$:



Ledrappier's example

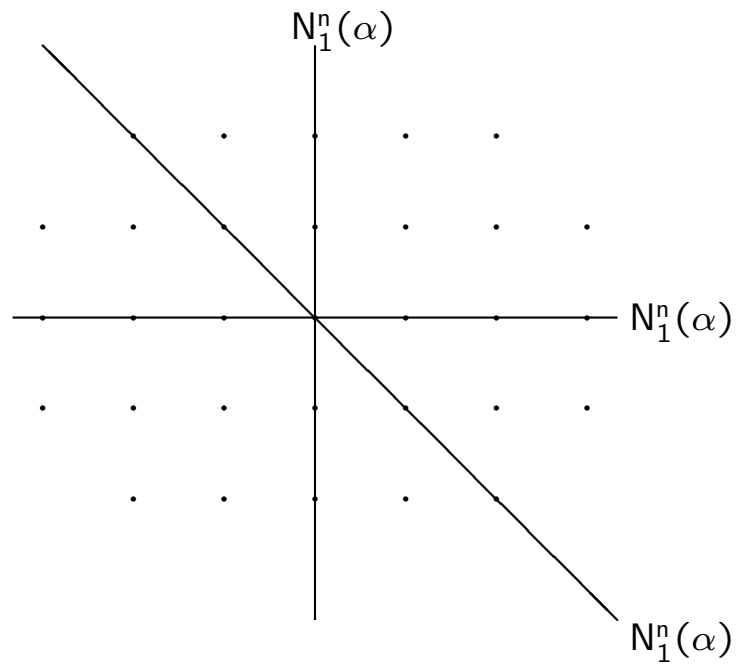
This is a totally disconnected analogue of $\times 2, \times 3$, and it may be described as the shift action of \mathbb{Z}^2 on the compact group

$$X = \{x \in \{0, 1\}^{\mathbb{Z}^2} \mid x_{\mathbf{n}+\mathbf{e}_1} + x_{\mathbf{n}} + x_{\mathbf{n}+\mathbf{e}_2} = 0 \pmod{2} \text{ for all } \mathbf{n} \in \mathbb{Z}^2\}.$$

This has no real Lyapunov exponents so $N_1^v(\alpha)$ is empty.

The line $x = 0$ is in $N_1(\alpha)$ since a point x with $x_{\mathbf{n}} = 0$ for all \mathbf{n} to the right of $x = -K$ for some large K can be non-zero (but close to zero no matter how it is moved in the vertical direction).

The set $N_1(\alpha)$ for Ledrappier's example:



Both $\times 2, \times 3$ and Ledrappier's example are 'rank one' in that they have individual elements α^n with finite positive entropy.

Einsiedler & Lind have shown that any such system can be described in terms of products of local fields.

Periodic points: Write $F_n(\beta) = \{x \in X \mid \beta^n x = x\}$ for the set of points fixed by the homeomorphism β^n . The combinatorial data of all these numbers may be thought of as a map

$$\mathbf{n} \mapsto |F_n(\beta)| \in \mathbb{N} \cup \{\infty\}.$$

There is a 'formula' to compute this, which subsumes a range of things including:

- there are $|\det(A^n - I)|$ points of period n for a toral endomorphism defined by a matrix A ;
- a full shift on a symbols has a^n points of period n ;
- $x \mapsto 2x$ (the map $\alpha^{(1,0)}$) in $\times 2, \times 3$ has $|2^n - 1| \times |2^n - 1|_3$ points of period n ;
- the map $\alpha^{(1,0)}$ in Ledrappier has $2^{n-2^{\text{ord}_2(n)}}$ points of period n ;
- and finally an algebraic miracle that occurs in rank one which makes periodic point counts multiply nicely (Miles).

Periodic points for $\times 2, \times 3$:

211	227	235	239	241	121	485	971	1943	3887	7775
49	65	73	77	79	5	161	323	647	1295	2591
5	11	19	23	25	13	53	107	215	431	863
23	7	1	5	7	1	17	35	71	143	287
29	13	5	1	1	1	5	11	23	47	95
31	5	7	1	1	∞	1	1	7	5	31

Periodic points for Ledrappier's example:

16	32	32	32	32	16	64	128	256	512	1024
32	1	16	16	16	1	32	64	128	256	512
32	16	4	8	8	4	16	32	64	128	256
32	16	8	1	4	1	8	16	32	64	128
32	16	8	4	1	1	4	8	16	32	64
16	1	4	1	1	∞	1	1	4	1	16

Is the *geometry* of the subdynamics (the three special directions in the two examples) visible in the *combinatorics* of the periodic points (the arrays of integers)?

Theorem: Yes!

The connection goes via the dynamical zeta function:

$$\zeta_T(z) = \exp \sum_{n=1}^{\infty} \frac{z^n}{n} |F_n(T)|.$$

If $|F_n(T)| < \infty$ for all $n \geq 1$ and grows at most exponentially, then this defines a complex function in some disc.

In our setting there is a fixed \mathbb{Z}^d -action α , so write $\zeta_{\mathbf{n}}$ for the zeta function of the map $\alpha^{\mathbf{n}}$. Define $Q(\alpha)$ to be the set of $\mathbf{n} \in \mathbb{Z}^d$ for which $\zeta_{\mathbf{n}}$ is a rational function.

Notice that any $\mathbf{n} \in \mathbb{Z}^d$ with the property that $F_j(\alpha^{\mathbf{n}})$ is infinite for some $j \geq 1$ is not a member of $Q(\alpha)$.

There are more exotic absences: in the $\times 2, \times 3$ example $\zeta_{(1,0)}$ has a natural boundary, for example.

Example: In $\times 2, \times 3$,

$$Q(\alpha) = \{\mathbf{n} \in \mathbb{Z}^d \mid n_1 n_2 \neq 0\}.$$

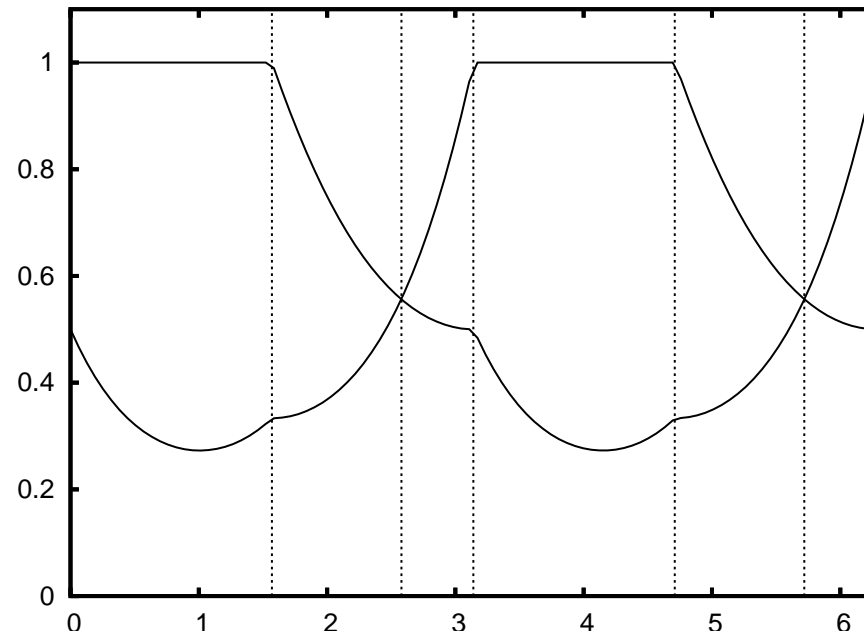
The issue here is to show that the zeta functions $\zeta_{(1,0)}$ and $\zeta_{(0,1)}$ in the two rational non-expansive lines are not rational, and the point is that $Q(\alpha)$ is not able to detect all of N_1 .

Given a rational function $h \in \mathbb{C}(z)$, denote the set of poles and zeros of h by $PZ(h) \subset \mathbb{C}$. For α an algebraic \mathbb{Z}^d -action of entropy rank one, define

$$\Omega_\alpha = \left\{ (\hat{\mathbf{n}}, |z|^{1/\|\mathbf{n}\|}) \mid z \in PZ(\zeta_{\mathbf{n}}), \mathbf{n} \in Q(\alpha) \right\} \subset S_{d-1} \times \mathbb{R}$$

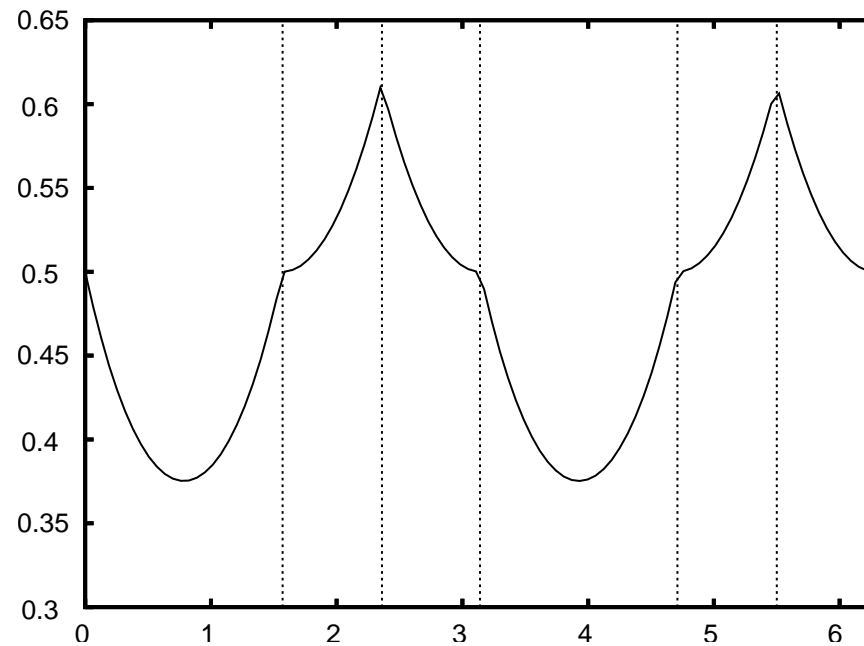
where $\hat{\mathbf{n}}$ denotes the unit vector in the direction of \mathbf{n} .

For the $\times 2, \times 3$ example the set $\overline{\Omega}_\alpha$ looks like this:



The non-expansive directions are shown with dashed lines.

For Ledrappier's example the set $\overline{\Omega}_\alpha$ looks like this:



Again the non-expansive directions are shown with dashed lines.

Theorem: Let α be an ergodic algebraic \mathbb{Z}^d -action with rank one. Then there exist continuous functions $f_1, \dots, f_r \in \mathcal{C}(S_{d-1}, \mathbb{R})$ such that

$$\overline{\Omega}_\alpha = \bigcup_{k=1}^r \{(\mathbf{v}, f_k(\mathbf{v})) \mid \mathbf{v} \in S_{d-1}\}.$$

Moreover,

- $N_1^v(\alpha) \subset \{\mathbf{v} \in S_{d-1} \mid f_j(\mathbf{v}) = f_k(\mathbf{v}) \text{ for some } j \neq k\},$

with equality if $\dim(X) \leq 1$, and

- $N_1^n(\alpha) = \{\mathbf{v} \in S_{d-1} \mid f_k \text{ is not smooth at } \mathbf{v} \text{ for some } k\}.$

Steps in the proof: Use a structure theorem for rank one systems due to Einsiedler & Lind and the algebraic ‘miracle’ due to Miles to express the functions f_j in a more or less explicit way.

On the other side, use a description of N_1 in terms of valuations due to Miles.

The proof has no inner meaning: it somehow amounts to computing both sides.

Strange rank-one phenomena on higher-dimensional groups:

A feature of \mathbb{Z}^d -actions with $d \geq 2$ is the possibility of *genuinely partially hyperbolic systems*.

Damjanović and Katok show that for $r \geq 6$ there are such actions on the r -torus: Let

$$f(x) = x^6 - 2x^5 - 5x^4 - 3x^3 - 5x^2 - 2x + 1$$

and write $k = \mathbb{Q}(a)$, where a is a complex root of f .

Then a and

$$b = 2a^5 - 6a^4 - 3a^3 - 6a^2 - 6a$$

are fundamental units for the ring of integers in k , and $M = \mathbb{Z}[a]$ carries a \mathbb{Z}^2 action of multiplication by a and by b .

- $\{1, a, a^2, a^3, a^4, a^5\}$ is an integral basis for M so the dual action α is a \mathbb{Z}^2 -action on \mathbb{T}^6 , the 6-torus;
- α_M has entropy rank one;
- all the $\alpha^{\mathbf{n}}$ share a two-dimensional eigenspace on which they act like rotations;
- a and b are multiplicatively independent, so $\alpha^{\mathbf{n}}$ is ergodic for every $\mathbf{n} \in \mathbb{Z}^2$;
- hence $\zeta_{\mathbf{n}}$ is rational for every $\mathbf{n} \in \mathbb{Z}^2$, and the whole action is not expansive, so the results above do not apply – in particular, $N_1(\alpha)$ consists of all directions.

The crossing set and the non-smooth set for $\overline{\Omega}_\alpha$ may be found (in principle – there are many functions involved).

These points correspond to directions $\mathbf{n} \in \mathbb{R}^2$ where the product of some subset of the eigenvalues of the integer matrix defining the automorphism $\widehat{\alpha}_M^{\mathbf{n}}$ is 1.

Thus the portrait obtained from $\overline{\Omega}_\alpha$ contains those directions in which the action is non-expansive transverse to the two-dimensional stable foliation arising from the common two-dimensional eigenspace on which the actions behaves like a rotation.

An infinite-dimensional (non)-example

Subdynamics makes sense for actions of countable abelian groups. The simplest example in this setting shows that the result does not extend without modification.

Consider the natural action α of $\mathbb{Q}_{>0}^{\times}$ on $X = \widehat{\mathbb{Q}}$ dual to the action

$$x \xrightarrow{\times r} rx$$

of $r \in \mathbb{Q}_{>0}^{\times}$ on \mathbb{Q} .

Algebraically, this is associated to a cyclic (hence Noetherian) module over the integral group ring $\mathbb{Z}[\mathbb{Q}_{>0}^{\times}]$, but that is deceptive since the ring itself is not Noetherian.

A neighborhood of the identity in X is isometric to an open set in the adèle ring $\mathbb{Q}_{\mathbb{A}}$; in this picture any $x \in X \setminus \{0\}$ must have some real or p -adic coordinate not equal to zero.

Multiplication by a large real rational, or a large negative power of that prime p , will move such a point far from the identity, showing the whole action to be expansive.

Just as before any non-expansive subgroup lies in a non-expansive subgroup Γ with $\mathbb{Q}_{>0}^{\times}/\Gamma \cong \mathbb{Z}$ (that is, lies in ' $N_{\infty-1}(\alpha)$ ').

Any such subgroup acts non-expansively, since it must omit some prime p , and will therefore act like an isometry on that direction in X .

On the other hand, $\overline{\Omega}_\alpha$ does not detect this rich non-expansive structure:

Since \widehat{X} is a field, every map α^r for $r \in \mathbb{Q}_{>0}^\times$ has $\zeta_{\alpha^r}(z) = \frac{1}{1-z}$, so the set $\overline{\Omega}_\alpha$ comprises the graph of the constant function 1.