Conversations between ergodic theory and number theory

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What are the links, and what would a more
effective formulation on either side mean/require?

Tom Ward (UEA) 30/10/09, QMUL

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Often X or T has additional structure (homogeneous space, compact group, metric space, Cantor set, continuous map, group rotation,...) and there are distinguished measures (uniquely invariant, maximizing, algebraic, absolutely continuous,...).

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Theorem 1 (Birkhoff 1931): Assume that T is ergodic (has no non-trivial measurable invariant sets). Then, given any representative f of an element of L^1_μ, there is a null set $N(f)$ for which

$$\frac{1}{N}\sum_{n=1}^{N-1} f(T^n x) \longrightarrow \int f \, \mathsf{d}\mu$$

for all $x \notin N(f)$.

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- Borel's 1909 paper is "characterized by convenient neglect of error terms in asymptotics, incorrect reasoning, and correct results" (Doob)

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Th.5 $\Rightarrow$ Th.3 as it is easy to check that Lebesgue measure is uniquely invariant. Th.4 looks more subtle, but turns out to be a consequence of the same phenomena.

Theorem 6 (Furstenberg 1961): If $T : X \to X$ is uniquely ergodic and $c : X \to G$ is continuous (G a compact group), then for the skew-product map

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- Th. 6 is a consequence of making the right definition of "generic point": $x \in X$ is generic for T and μ if the orbit of x is equidistributed with respect to μ. Lemma: for a continuous map T on a compact metric space, μ-a.e. point is generic for T and μ.

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- Th. 8 involves an approach to decomposing measure-preserving maps, and a correspondence principle to translate between ergodic theory and infinite combinatorics.

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Theorem 9 (Schlickewei 1990/van der Poorten & Schlickewei 1991): Let K be a field of characteristic zero, $G \subset K \setminus \{0\}$ a finitely-generated multiplicative subgroup, and $a_1, \ldots, a_n \in K \setminus \{0\}$. Then

$$a_1 x_1 + \cdots + a_n x_n = 1$$

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- There is an analogous correspondence between mixing on disconnected groups and unit theorems in positive characteristic, due to Masser (2004) but it is different: in particular Th. 10 is false on disconnected groups.

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Conjecture 12 (Margulis, special case): Let A be the group of positive diagonal matrices in $\mathrm{SL}_k(\mathbb{R})$, $k \geq 3$, acting on the space $\mathrm{SL}_k(\mathbb{R})/\mathrm{SL}_k(\mathbb{Z})$. If μ is an A-invariant ergodic probability measure on this space, is there a closed connected group $L > A$ for which μ is the unique L-invariant measure on a single closed L-orbit (that is, is μ automatically algebraic)?

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Conj.12 $\Rightarrow$ Conj.11, but more importantly a partial version of Conj. 12 gives a partial Conj. 11.

Theorem 13 (Einsiedler–Katok–Lindenstrauss 2006): Conjecture 12 holds under the additional hypothesis that μ gives positive entropy to some one-parameter subgroup of A. Hence, the set of counterexamples to Conjecture 11 lies inside a countable union of sets of box dimension zero.

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This is an instance of "abelian" measure rigidity, in contrast to the rigidity associated to Raghunathan's conjecture, in which individual elements of the action exhibit rigidity.

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Theorem 14: The set of topological entropies of group automorphisms is the closure of $\{m(f) \mid f \in \mathbb{Z}[x]\}$, where $m(f) = \int_0^1 \log|f(\mathrm{e}^{2\pi \mathrm{i} s})|\,\mathrm{d}s$ and $m(0) := \infty$.

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- Lehmer's problem has been solved for "non-reciprocal" polynomials and for bounded degree, but remains open.

(Nonsensical) Question: what are the dynamical properties of a “typical” compact group automorphism?

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Sample subquestion: Choose a subset S of the primes by throwing a fair coin. What is

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Easy arguments say this is $\geqslant \frac{1}{2}\log 2$ (Ward 1998) – but it clearly should be $\log 2$ almost surely.

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Uniform Boundedness Conjecture (Morton–Silverman 1994): Given $d \geq 2$, $N \geq 1$, $D \geq 1$ there is a constant $C(d, N, D)$ such that for any number field K with $[K : \mathbb{Q}] \leq D$ and any (finite) morphism $\phi : \mathbb{P}^N \to \mathbb{P}^N$ of degree d defined over K, the number of pre-periodic points of ϕ in $\mathbb{P}^N(K)$ is bounded above by $C(d, N, D)$.

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- For $N = 1$ and degree 4, it implies Merel's theorem (1996) that the size of the torsion subgroup of an elliptic curve over a number field is bounded in terms of the degree of number field only.

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Uniform Boundedness Conjecture (Morton–Silverman 1994): Given $d \geq 2$, $N \geq 1$, $D \geq 1$ there is a constant $C(d, N, D)$ such that for any number field K with $[K : \mathbb{Q}] \leq D$ and any (finite) morphism $\phi : \mathbb{P}^N \to \mathbb{P}^N$ of degree d defined over K, the number of pre-periodic points of ϕ in $\mathbb{P}^N(K)$ is bounded above by $C(d, N, D)$.

- Northcott gives a non-uniform bound.
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- For $N = 1$ and degree 4, it implies Merel's theorem (1996) that the size of the torsion subgroup of an elliptic curve over a number field is bounded in terms of the degree of number field only.
- Fakhruddin (2001) showed that UBC implies there is a constant $C(N, D)$ so that if $[K : \mathbb{Q}] \leqslant D$ and A is an abelian variety defined over K of dimension N, then $|A(K)_{\text{tors}}| \leqslant C(N, D)$.

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- If UBC holds for $\mathbb{Q}$, then it holds for number fields.

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