

Euler's mathematics of the infinite

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UEA

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Leonhard Euler (1707–1783)



Arithmetic

Adding, multiplying, or dividing finitely many numbers is easy...

usually

...but does it make sense to add up infinitely many numbers?

Some sums make no sense

An infinite sum might not make sense as a number, for example:

$$1 + 2 + 3 + 4 + \dots$$

or

$$1 - 2 + 3 - 4 + 5 - 6 + 7 - \dots$$

Four examples

1) The *harmonic series*:

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$$

2) The square root of six times the sum of *square reciprocals*:

$$\sqrt{6 \left(1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \dots \right)}$$

3) The *sum of cube reciprocals*:

$$1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \dots$$

4) The *sum of prime reciprocals*:

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \frac{1}{13} + \frac{1}{17} + \dots$$

Convergence of positive series

Assume that every term a_n is positive.

This means that the *partial sums*

$$a_1, a_1 + a_2, a_1 + a_2 + a_3, \dots$$

get bigger and bigger.

So for such a series there are two possibilities:

Convergence: there is some number M with the property that ALL the partial sums are smaller than M (that is, the partial sums are *bounded*).

Divergence: there is no such number; equivalently the partial sums eventually become larger than any chosen number (the partial sums are *unbounded*).

A geometric series

Consider the geometric series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$.

By working with partial sums, we can do simple arithmetic: for any N ,

$$S_N = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{2^N},$$

so

$$2S_N = 2 + 1 + \frac{1}{2} + \dots + \frac{1}{2^{N-1}} = 2 + S_N - \frac{1}{2^N},$$

and therefore

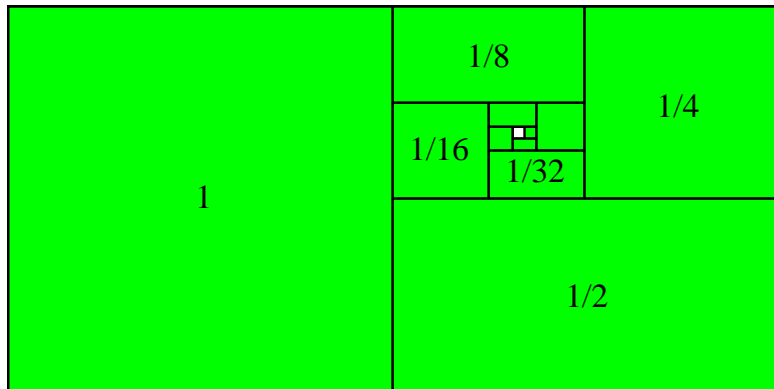
$$S_N = 2 - \frac{1}{2^N}.$$

We can write this as follows:

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots = 2.$$

The same proof in pictures

Sometimes we can 'see' why the partial sums are bounded.



The green area is

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \frac{1}{256} + \frac{1}{512} + \frac{1}{1024} \dots$$

The harmonic series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots$

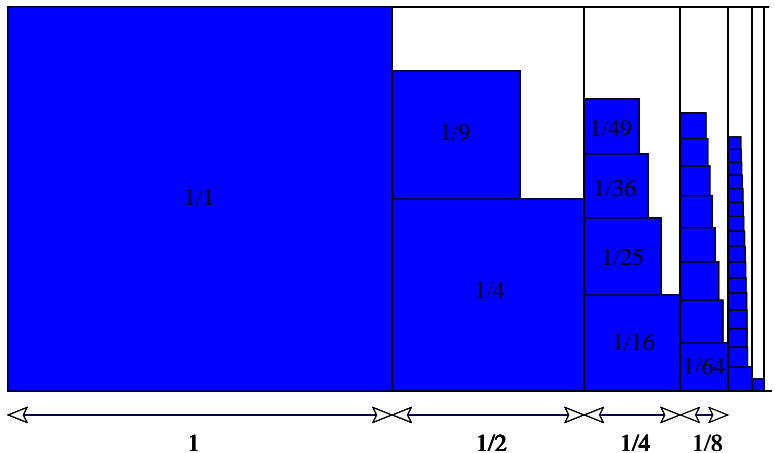
Nicole d'Oresme (1323–1382) noticed the following:

$$\begin{array}{rcl} & & 1 \geq \frac{1}{2} \\ & & \frac{1}{2} \geq \frac{1}{2} \\ & & \frac{1}{3} + \frac{1}{4} \geq \frac{1}{2} \text{ (2 terms as big as } \frac{1}{4} \text{)} \\ & & \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} \geq \frac{1}{2} \text{ (4 as big as } \frac{1}{8} \text{)} \\ \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16} & \geq & \frac{1}{2} \text{ (8 as big as } \frac{1}{16} \text{)} \\ & & \frac{1}{17} + \frac{1}{18} + \dots + \frac{1}{32} \geq \frac{1}{2} \text{ (16 as big as } \frac{1}{32} \text{)} \\ & & \frac{1}{33} + \frac{1}{34} + \dots + \frac{1}{64} \geq \frac{1}{2} \text{ (32 as big as } \frac{1}{64} \text{)} \end{array}$$

and so on.

In fact, we have $1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{2^k} \geq \frac{k}{2}$, so the harmonic series *diverges to infinity*.

We'll leave the sum of reciprocal primes (any suggestions?) and look at the sum of reciprocal squares.



This grouping of terms shows that $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots$ converges to some number smaller than 2.

The Basel problem (posed by Pietro Mengoli in 1644)

Can the sum $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots$ be described in terms of known constants?

Euler gave the following *non-rigorous* argument.

Imagine we have a polynomial which can be written in the form

$$\left(1 - \frac{x^2}{1}\right) \left(1 - \frac{x^2}{4}\right) \cdots \left(1 - \frac{x^2}{N^2}\right) = 1 - x^2 \left(\frac{1}{1} + \frac{1}{4} + \cdots + \frac{1}{N^2}\right) + x^4(\cdots)$$

Euler also knew the *Maclaurin expansion*

$$\frac{\sin \pi x}{\pi x} = 1 - \frac{\pi^2 x^2}{3!} + \frac{\pi^4 x^4}{5!} - \frac{\pi^6 x^6}{7!} + \cdots$$

and that the zeros of $\frac{\sin \pi x}{\pi x}$ occur at $\pm 1, \pm 2, \pm 3, \dots$ (What happens at $x = 0$?)

Euler's leap in the dark (1735)

Euler went on to argue as follows: **if** (a big if) we can make sense of writing the *infinite product*

$$\frac{\sin \pi x}{\pi x} = \left(1 - \frac{x^2}{1}\right) \left(1 - \frac{x^2}{4}\right) \left(1 - \frac{x^2}{9}\right) \dots$$

and **if** this still behaves like a polynomial, then perhaps the x^2 coefficients should match up:

$$\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \dots = \frac{\pi^2}{3!},$$

so

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots = \frac{\pi^2}{6}.$$

The bad news...

Euler's beautiful result

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \frac{\pi^2}{6}$$

is correct, but takes a **lot** of work to justify. The key step is to prove that the infinite product converges, and Euler did this nine years later, in 1744.

Sum of reciprocal cubes

One question is easy: since $n^3 \geq n^2$, we have (for any N)

$$1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \cdots + \frac{1}{N^3} \leq 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots + \frac{1}{N^2},$$

so the partial sums must be bounded, so the series converges.

Euler tried to express the sum

$$1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \dots$$

in terms of other constants and failed, and we still don't know much about it.

He did find a way to express the sum of reciprocal 4th, 6th, 8th,... powers in terms of π , but failed with the odd powers.

Roger Apéry proved that the sum is not a rational number (that is, cannot be written as a fraction $\frac{a}{b}$ with a and b whole numbers)...

(using methods Euler knew)

... in 1977!

It is still not known if the same is true for the sum of reciprocal 5th powers.

Some conclusions

An infinite sum $a_1 + a_2 + \dots$ does sometimes make sense.

If the *terms* a_n don't get small the sum makes no sense.

If the terms a_n become small fast enough (like $1/n^2$ for example) then the series converges to some number.

If the terms a_n become small, but not very fast (like $1/n$ for example) then the series can diverge to infinity even though the terms are going to zero.

Sum of reciprocal primes

We left the sum

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \frac{1}{13} + \dots$$

of reciprocal primes.

The n th prime is much bigger than n , so its reciprocal is much smaller, so perhaps this has a better chance of converging than the harmonic series (which does not converge)...

on the other hand, the n th prime is much smaller than n^2 , so perhaps it is less likely to converge than the sum of reciprocal squares...

It's a bit like having a bowl of porridge that is hotter than one that is too cold, and colder than one that is too hot...

so it might be too hot, or too cold, or just right.

We have learnt **nothing** from the two comments!

In fact the sum of reciprocal primes *diverges to infinity*... very slowly.

The sum of the reciprocals of the primes less than N is roughly $\log \log(N)$.

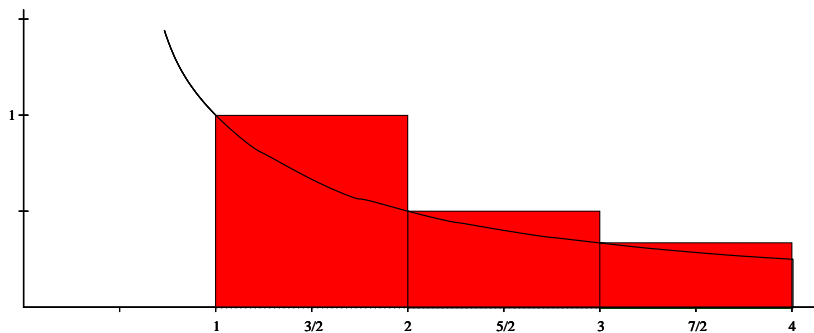
To get the sum bigger than 2.8, you need to add up the prime reciprocals for primes up to 314723...

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \cdots + \frac{1}{314723} = 2.800000376\dots$$

This is not an easy result – if anyone wants to see an account there will be one added to the slides for this talk on my webpage.

The harmonic series again

The graph represents $\int_1^4 \frac{1}{x} dx = \log 4$.



The green boxes show that

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} < \log 4,$$

while the red boxes show that

$$1 + \frac{1}{2} + \frac{1}{3} > \log 4.$$

This idea allows us to show that

$$\log(N + 1) < 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{N} < 1 + \log N.$$

That is, another proof that the harmonic series diverges to infinity, and some idea of how fast it does so.

A proof that the sum of reciprocal primes diverges

We use Σ to denote sums:

$$\sum_{n=1}^N a_n = a_1 + a_2 + \cdots + a_N$$

always makes sense.

We also write

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$$

to denote what happens as we let N go to infinity.

Write p_n for the n th prime, so the sum we are studying is

$$\sum_{n=1}^{\infty} \frac{1}{p_n}.$$

We argue by contradiction: assume that the sum converges.

Then the *tail* of the sum can be made as small as we please: there is some number k with

$$\sum_{n=k}^{\infty} \frac{1}{p_n} < \frac{1}{2}.$$

Now let $N(x)$ be the number of whole numbers smaller than x and only divisible by the primes p_1, \dots, p_k .

Any such number can be written as a product AB^2 where A is not divisible by a square. Since only k primes could divide A , there are only 2^k choices for A . On the other hand, there are no more than \sqrt{x} choices for B .

We conclude that $N(x) \leq 2^k \sqrt{x}$.

The number of whole numbers smaller than x and divisible by one of the primes p_1, \dots, p_k must be $x - N(x)$.

Finally the number of whole numbers smaller than x and divisible by a prime p is at most $\frac{x}{p}$.

It follows that

$$x - N(x) \leq \sum_{n=k}^{\infty} \frac{x}{p_n} \leq \frac{x}{2},$$

so

$$\frac{x}{2} \leq N(x) \leq 2^k \sqrt{x}$$

for all x .

This is impossible, so the assumption that the series converges must have been wrong.