Euler's mathematics of the infinite

Tom Ward

UEA

14 November 2007

Leonhard Euler (1707–1783)

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...but does it make sense to add up infinitely many numbers?

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4) The *sum of prime reciprocals*:

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Divergence: there is no such number; equivalently the partial sums eventually become larger than any chosen number (the partial sums are *unbounded*).

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We can write this as follows:

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 2.$$

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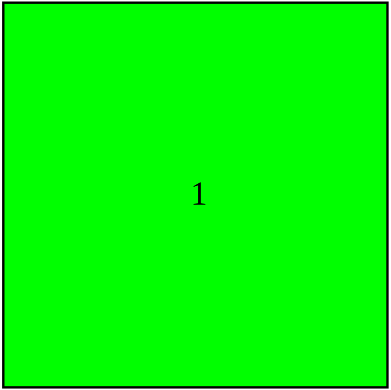

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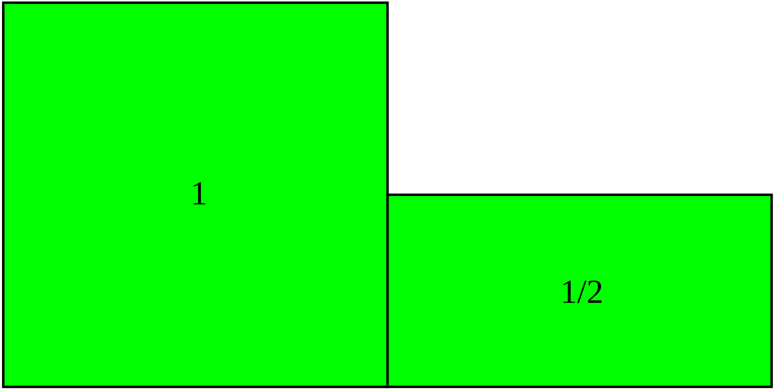

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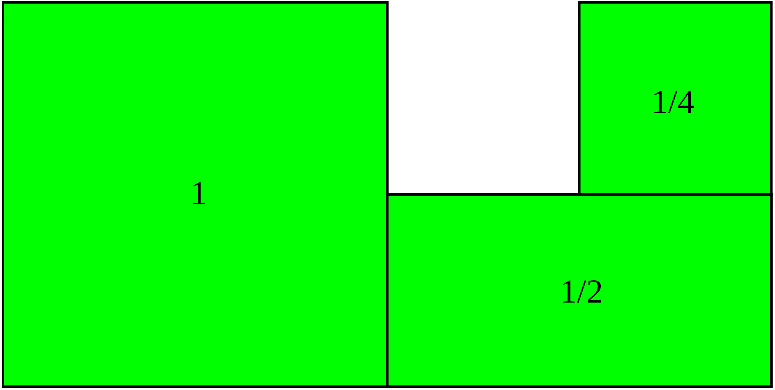

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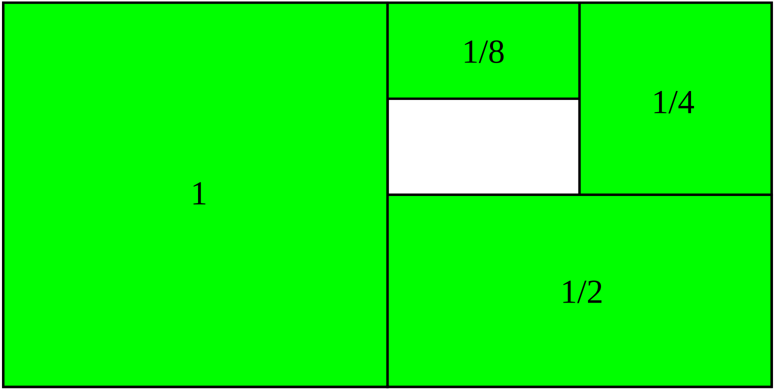

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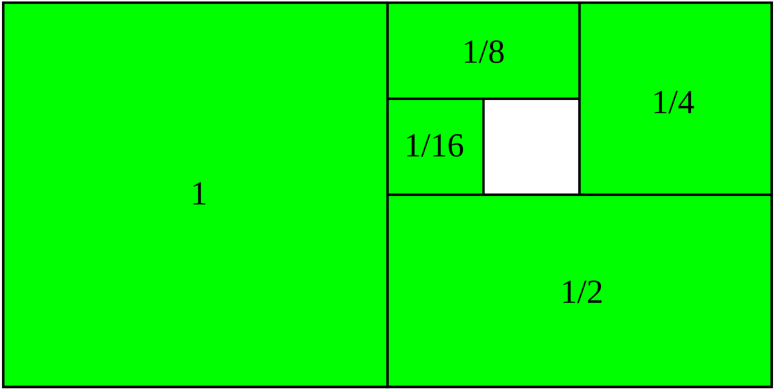

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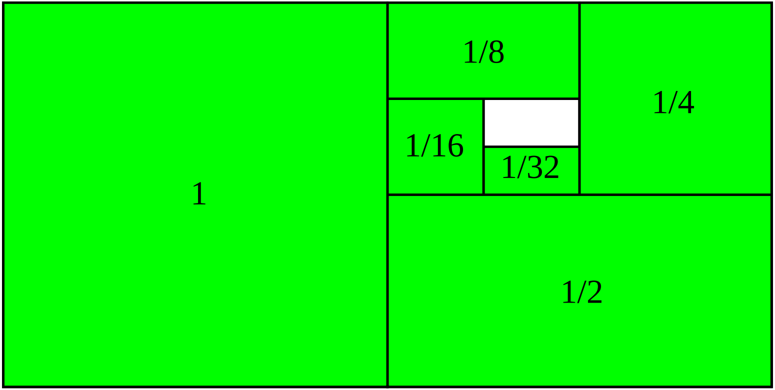

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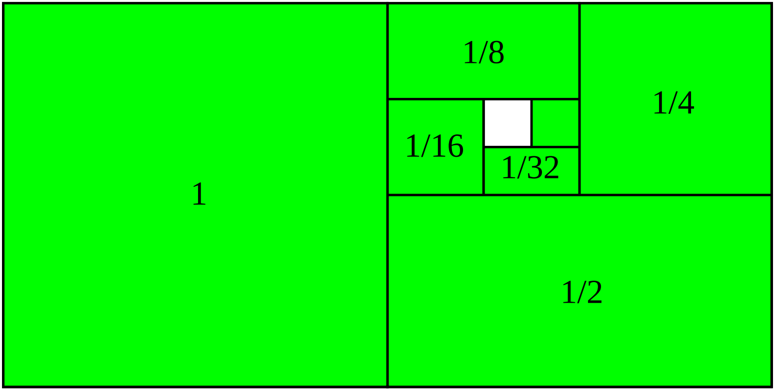

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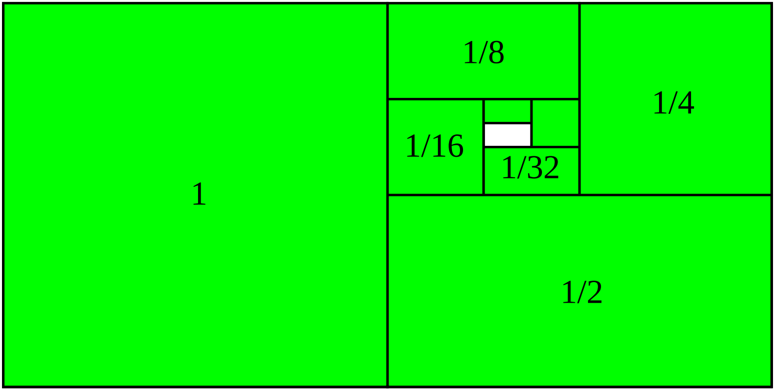

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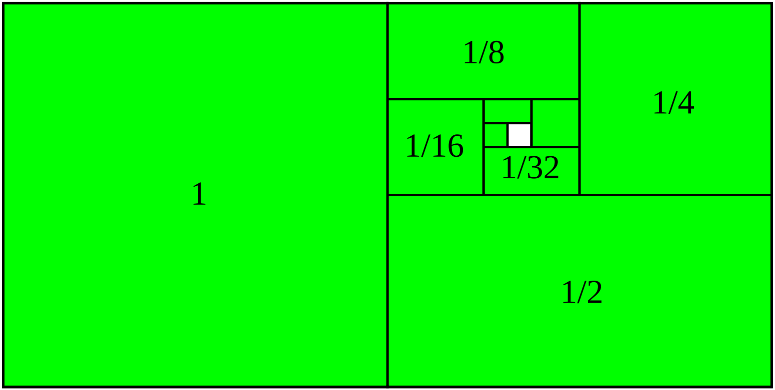

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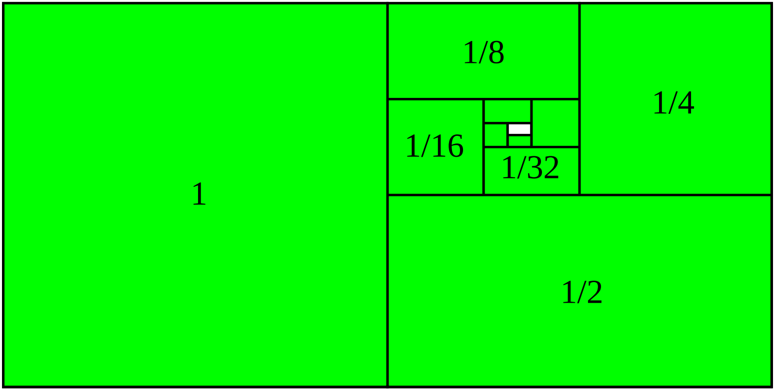

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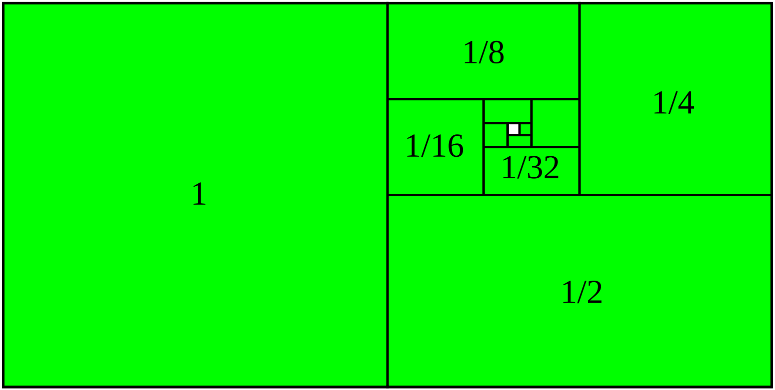

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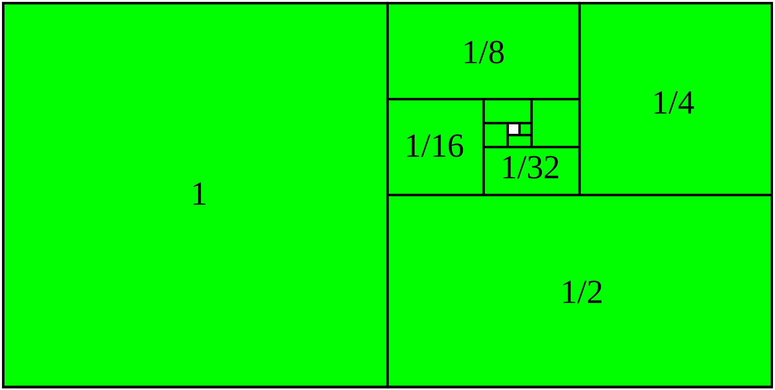

The green area is

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \frac{1}{256} + \frac{1}{512} + \frac{1}{1024}.$$

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and so on.
In fact, we have $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^k} \geqslant \frac{k}{2}$, so the harmonic series *diverges to infinity*.

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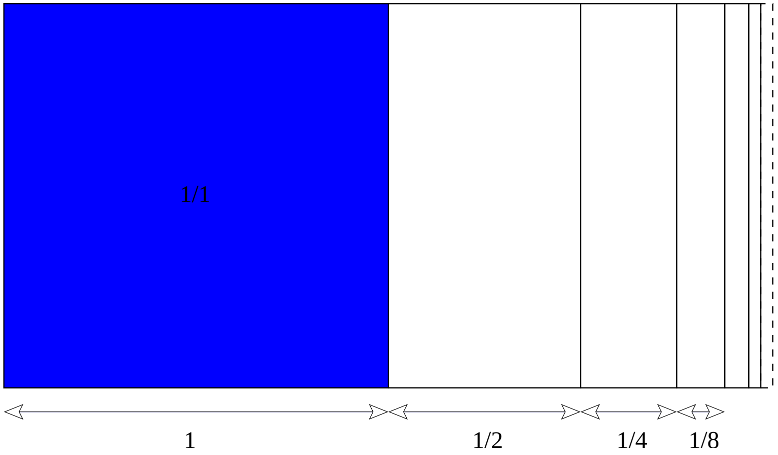

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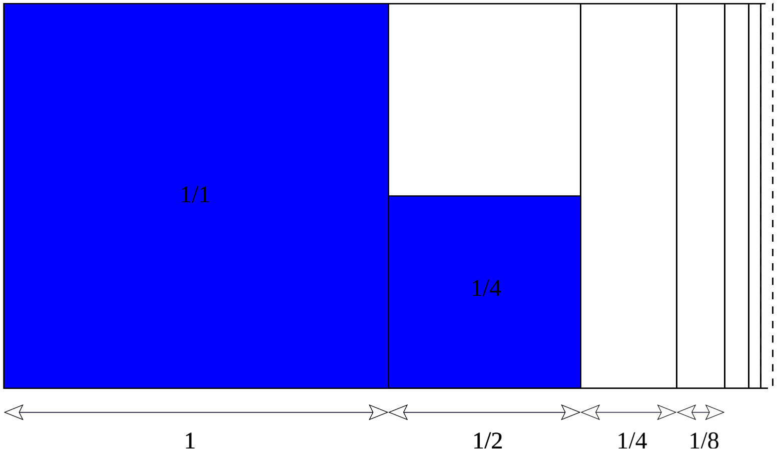

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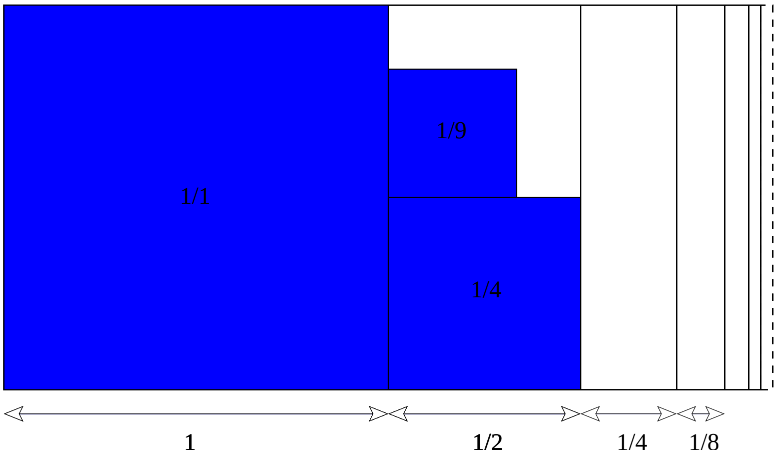

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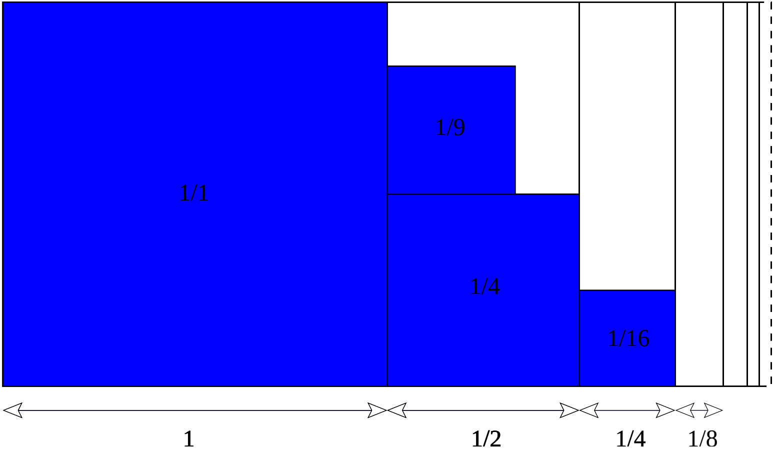

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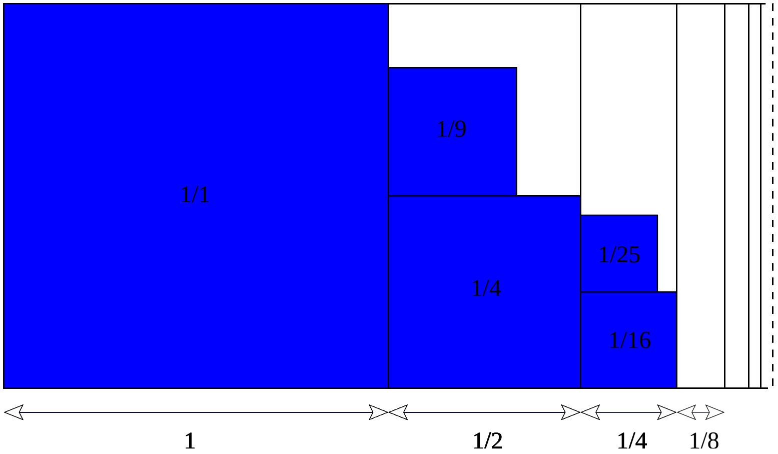

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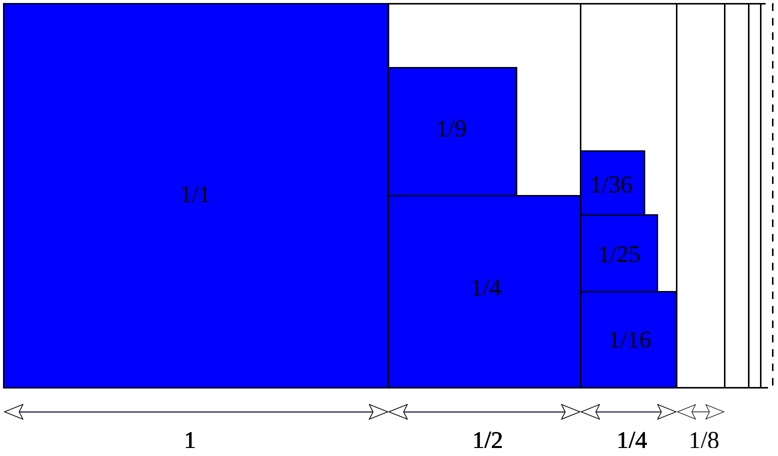

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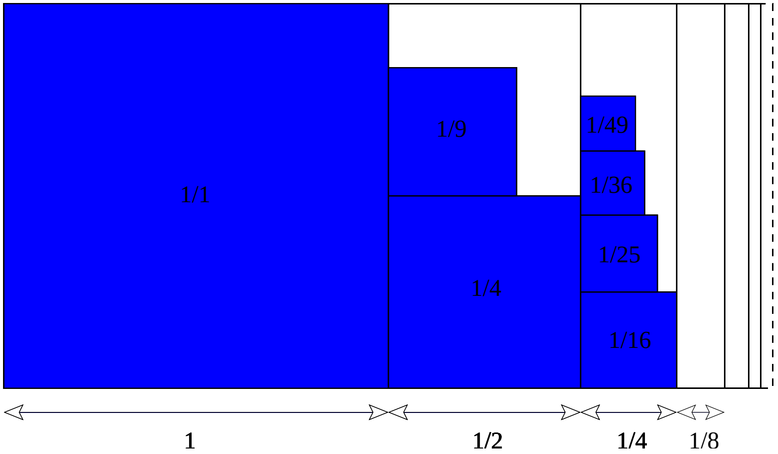

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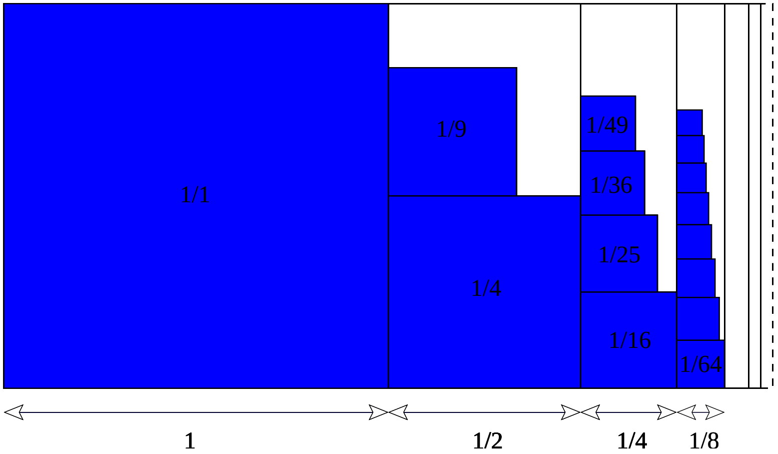

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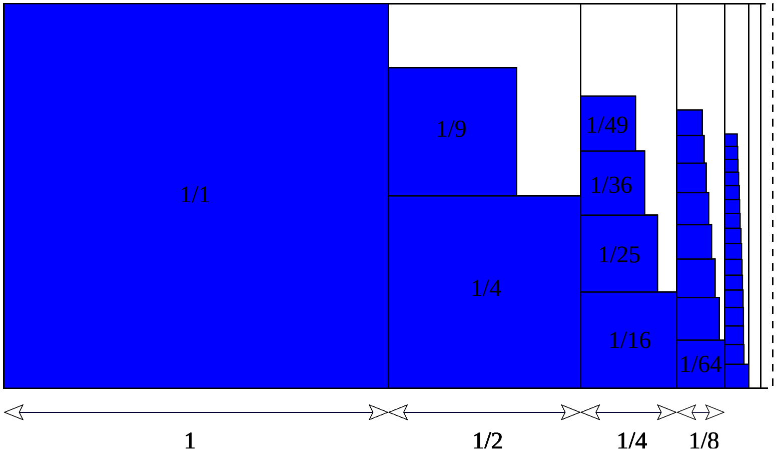

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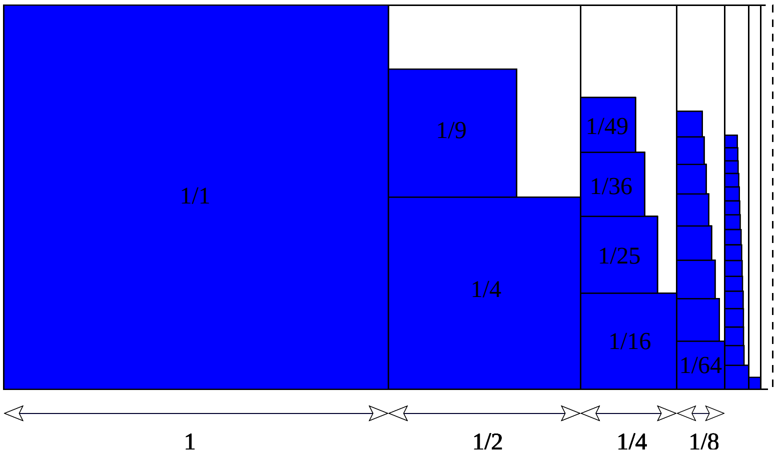

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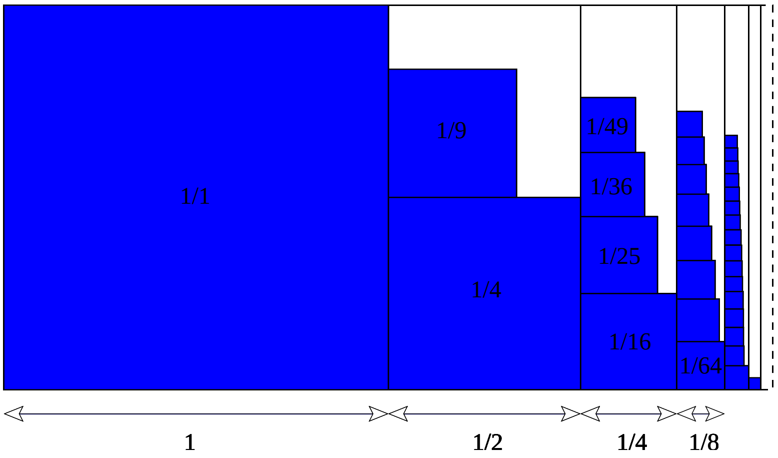

This grouping of terms shows that $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$ converges to some number smaller than 2.

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$$\left(1 - \frac{x^2}{1}\right)\left(1 - \frac{x^2}{4}\right)\cdots\left(1 - \frac{x^2}{N^2}\right) = 1 - x^2\left(\frac{1}{1} + \frac{1}{4} + \cdots + \frac{1}{N^2}\right) + x^4(\cdots)$$

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Euler also knew the *Maclaurin expansion*

$$\frac{\sin \pi x}{\pi x} = 1 - \frac{\pi^2 x^2}{3!} + \frac{\pi^4 x^4}{5!} - \frac{\pi^6 x^6}{7!} + \cdots$$

The Basel problem (posed by Pietro Mengoli in 1644)

Can the sum $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$ be described in terms of known constants?

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and that the zeros of $\frac{\sin \pi x}{\pi x}$ occur at $\pm 1, \pm 2, \pm 3, \ldots$. (What happens at $x = 0$?)

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is correct, but takes a **lot** of work to justify. The key step is to prove that the infinite product converges, and Euler did this nine years later, in 1744.

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so the partial sums must be bounded, so the series converges.

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It is still not known if the same is true for the sum of reciprocal 5th powers.

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If the terms a_n become small fast enough (like $1/n^2$ for example) then the series converges to some number.

If the terms a_n become small, but not very fast (like $1/n$ for example) then the series can diverge to infinity even though the terms are going to zero.

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We left the sum

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on the other hand, the nth prime is much smaller than n^2, so perhaps it is less likely to converge than the sum of reciprocal squares...

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We have learnt **nothing** from the two comments!

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This is not an easy result – if anyone wants to see an account there will be one added to the slides for this talk on my webpage.

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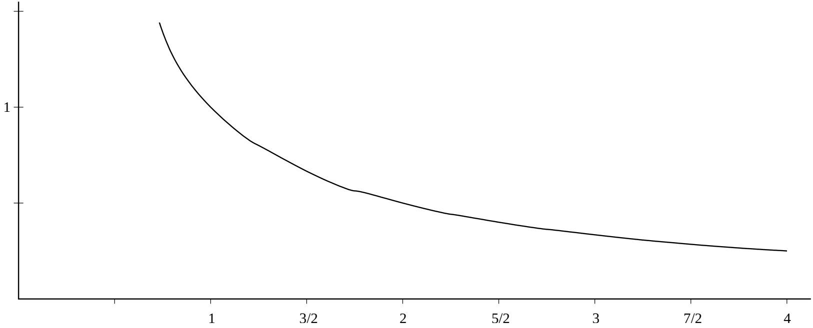

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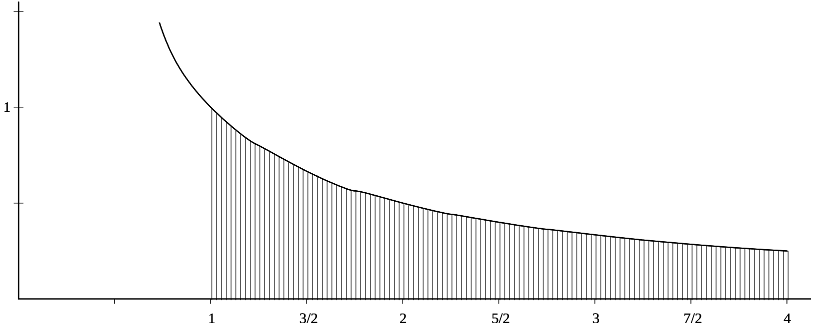

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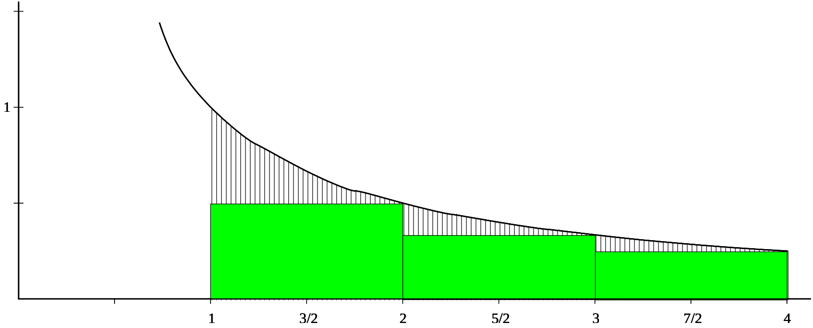

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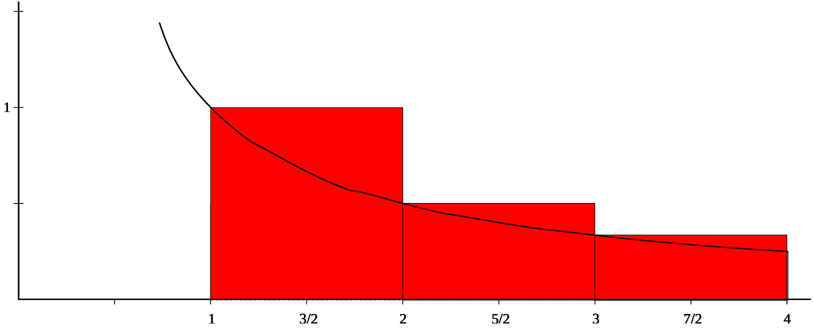

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That is, another proof that the harmonic series diverges to infinity, and some idea of how fast it does so.

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to denote what happens as we let N go to infinity.

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We argue by contradiction: assume that the sum converges.

Then the *tail* of the sum can be made as small as we please: there is some number k with

$$\sum_{n=k}^{\infty} \frac{1}{p_n} < \frac{1}{2}.$$

Now let $N(x)$ be the number of whole numbers smaller than x and only divisible by the primes $p_1, \ldots, p_k$.

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This is impossible, so the assumption that the series converges must have been wrong.