BOOK REVIEW

p-adic Deterministic and Random Dynamics, by Andrei Yu. Khrennikov and Marcus Nilsson, Kluwer Academic, Dordrecht/Boston/London, 2004, 270 pp., hardcover, \$99.00, ISBN 1402026595;

Dynamical systems most naturally arise in the setting of trajectories (time orbits) constrained to lie in a phase space, typically a manifold, of a system evolving according to some physical rules. Sampling such a flow at fixed time intervals often reduces the problem to the study of a smooth map on a manifold, and the local properties of such a map are largely governed by the behaviour of the derivative. Thus one quickly arrives at the simplest model system to study: iterates of a linear map on $\mathbb{R}^n$. Specializations and other model systems abound; those most relevant here are rational maps on the Riemann sphere (complex dynamics), attractors for expansive maps, automorphisms of compact groups (algebraic dynamics) and iteration of polynomials in an arithmetic setting. Each of these contributes motivation, ideas, and suggestions for analogies, to the topics in this book.

1. p-adic dynamics in vivo

Before turning to the willful pursuit of p-adic dynamics, it is important to recognize that p-adic phenomena cannot be avoided – they arise in the familiar setting of smooth maps on manifolds. The map

$$f : (z, w) \mapsto (z^2, z/2 + w/4)$$

of the solid torus $S = \{z \in \mathbb{C} \mid |z| = 1\} \times \{w \in \mathbb{C} \mid |w| \leq 1\}$ has a natural attractor $\Lambda = \bigcap_{n=0}^{\infty} f^n(S)$ on which the interesting part of the dynamics lives, in the sense that points away from this set are pulled towards it under iteration, and points on it move around in a complex fashion. The set Λ is not a manifold but is not far from being one: it locally resembles a product of local fields, in this case $\mathbb{R} \times \mathbb{Q}_2$. Indeed, there is a local isometry from Λ to $\mathbb{R} \times \mathbb{Q}_2$ carrying the action of f to the map $(s, t) \mapsto (2s, 2t)$. Thus the natural class of simplest models for such maps begins with automorphisms of solenoids: a solenoid is a compact, connected, finite-dimensional group. The most familiar examples are tori; Λ is a simple non-toral example of a solenoid.

Studying the measurable dynamics of an ergodic algebraic automorphism T of a compact group X, preserving Haar measure λ, again forces one into p-adic dynamics. The most fundamental ergodic or stochastic property of such a map is that it is measurably isomorphic to a Bernoulli shift; this was shown for X a torus by Katznelson, then generalized by Lind [18] and others. The generalization involves understanding the role played by p-adic hyperbolicity in automorphisms of solenoids; indeed the fundamental Diophantine estimate used by Katznelson finds its most natural proof in work of Lind and Schmidt [20] using the product formula for p-adic valuations.

2000 *Mathematics Subject Classification.* 37F99 (11S85 28C10 28D15 37H99 37N25 92B20).

A natural property that involves both the measurable and the topological structure of an algebraic dynamical system is the following: a group automorphism T is exponentially recurrent if, for any non-empty open set U,

$$\lambda\left(\{x \in U \mid \min\{j > 0 \mid T^j x \in U\} = n\}\right) \longrightarrow 0$$

exponentially fast as $n \to \infty$ (λ again is Haar measure). This rapid decay is necessary but not sufficient for T to be isomorphic to a Bernoulli shift via an almost continuous conjugacy. Lind [19] showed that exponential recurrence follows from ergodicity; the proof involves reduction to automorphisms of solenoids, and for these again an understanding of p-adic hyperbolicity is required.

Computing the topological or measure-theoretic entropy of a compact group automorphism reduces again to the solenoid case, and there an adelic covering space arises naturally in the calculation: Lind and Ward [22] gave an adelic proof of a result of Yuzvinskiĭ [31] by showing that if T is the map dual to the automorphism induced by an invertible linear map $A : \mathbb{Q}^d \to \mathbb{Q}^d$, then the entropy is given by

$$h(T) = \sum_{p \leq \infty} \log^+ |\mu_{i,p}|_p \tag{1}$$

where $\mu_{i,p}$ denotes the eigenvalues of A with multiplicity in an algebraic closure of $\mathbb{Q}_p$ (as usual $\mathbb{Q}_\infty = \mathbb{R}$), and $\log^+ x = \max\{0, \log x\}$.

A simple illustration of the arithmetic issue behind all this is given by an example from [19]: consider the compact group automorphism dual to the action of the matrix

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 6/5 \end{bmatrix}$$

on $\mathbb{Q}^2$. A neighbourhood of the identity in the character group of $\mathbb{Q}^2$ is isometric to an open subset of the adele ring $\mathbb{Q}_{\mathbb{A}}^2$; on the Archimedean quasi-factor $\mathbb{R}^2$ the map has eigenvalues of unit modulus, and the only hyperbolic behaviour comes from the 5-adic quasi-factor. Thus the recurrence properties and the positive entropy of this map only become visible in a non-Archimedean place.

There are many other situations in which p-adic dynamical considerations arise without being sought, including the study of periodic orbits under a typical group automorphism [29] and an explanation for the rounding errors in iteration of (Euclidean) lattice maps by Bosio and Vivaldi [5].

Ultrametric valuations on global fields of positive characteristic and their completions also arise naturally in several dynamical problems. Examples include a convenient geometrical description of the simplest linear cellular automata [30], the structure of algebraic $\mathbb{Z}^d$-actions of entropy rank one in work of Einsiedler and Lind [10], and the mixing structure of algebraic dynamical systems on zero-dimensional compact groups in work of Masser [23] and Schmidt [26].

Thus p-adic and non-Archimedean dynamics is not esoteric: it is forced on us in the study of attractors of smooth maps, of algebraic dynamical systems, and in many other ways.

2. p-adic dynamics in vitro

Motivated by the many parts of mathematics in which it is natural to treat the p-adic completion of the rationals on an equal footing with the real one (including number theory, algebraic and arithmetic geometry, model theory, and so on), many mathematicians have studied p-adic analogs of familiar dynamical systems.

An extremely incomplete list follows: Herman and Yoccoz [13] found p-adic analogs of the role of bad Diophantine approximation in local linearization of dynamics. Lubin introduced the use of formal groups from local arithmetic geometry as a tool for studying iterates of p-adic analytic maps. Hsia [14], Li [17], Benedetto [3], Rivera-Letelier [25] and others have systematically studied iteration of rational maps over the p-adics, discovering both analogs of known phenomena in complex dynamics and novel uniquely ultrametric phenomena. Two special cases are multiplication by a p-adic number and iteration of a monomial; one or other of these have been studied from an ergodic theory viewpoint by Parry and Coelho [9], the authors, and others. An accessible overview of much of this work may be found in an article by Bryk and Silva [6]. In a different direction, Morton and Silverman [24], Arrowsmith and Vivaldi [1], and others, including the authors of the book under review, have also studied iteration of rational maps over global fields using p-adic techniques, a subject that might be called arithmetic dynamics.

3. Prospects

There are several directions in which p-adic, ultrametric, or adelic ideas appear potentially productive, from which I have selected the following. More extensive lists may be found in the p-adic dynamics bibliography maintained by Silverman [27] and the algebraic dynamics bibliography of Vivaldi [28].

In mathematical physics, Brekke, Freund, Peter, Olson, and Witten [7], [11], [12] and others have studied string theories in which the world sheet coordinates are p-adic. Khrennikov [15] and others have related certain difficulties in mathematical models for biological, physical, probabilistic and neurological phenomena to the Archimedean property of the reals, motivating attempts to use the p-adic numbers and their natural hierarchical structure to model physical and biological systems.

Besser and Deninger [4] found an analog of the Mahler measure (and in particular of (1) and its generalization to algebraic actions of higher rank) that is p-adic valued, and raised the interesting question of whether there is a dynamically meaningful notion of p-adic valued entropy to go with it. Two constructions are given of the p-adic Mahler measure in [4]. The first is a 'local' integral related to Shnirel′man integration directly analogous to the classical Mahler measure. The second arises by integration of a syntomic regulator with the p-adic class of a torus in the singular homology group of a variety associated to the underlying polynomial. It is possible that the p-adic valued probabilities studied by Khrennikov, Yamada, and van Rooij [16] may help to make progress with finding a dynamical meaning for the former; the dynamical notions behind the latter are a mystery.

The explicit decomposition into local entropy contributions (an instance of the 'local-to-global' principle) in (1) has not yet been fully understood for the corresponding entropy calculation for actions of higher rank groups by Lind, Schmidt and Ward [21].

In a different direction, Call and Silverman [8] and others have studied morphic heights associated to morphisms of varieties; these have a decomposition into local heights much like the local contributions to entropy in (1). Morphic heights are a powerful tool in p-adic dynamics, and the full story of their relationship with entropy is not yet clear.

Finally, there is a great deal of research activity in the study of dynamics on Berkovich spaces; the notes of Baker and Rumely [2] give an overview. These spaces may prove to be the right setting for many unifying principles and new results in ultrametric dynamics.

4. What is in this book?

This book begins with an elementary account of p-adic numbers, p-adic analysis and p-adic dynamics. Basic classifications of fixed points, the small denominators phenomena, minimality and ergodicity are described. The last six chapters are more closely concerned with aspects of p-adic dynamics and p-adic modelling studied by the University of Växjö group comprising Khrennikov, Lindahl, Nilsson, Nyqvist and Svensson. The topics here are neural networks, random dynamics, probability distributions on a model for mental states, wavelets, and probability (in each case, a p-adic valued model).

In one or two of these topics the treatment does not entirely give the impression of being definitive – some fairly fundamental aspects of the development involve choices that may change as the subject matures. Much of this is an inevitable consequence of the subject. For example, it seems impossible to find a p-adic-valued probability theory that is simultaneously monotonic (if event A is a subset of event B then the probability of A is 'less' than the probability of B; this includes making up a notion of order or partial order in a field that cannot itself carry a natural structure of an ordered field), compatible with the von Mises relative frequency picture, and countably additive.

I am not qualified to comment on the biological and neurological models, but the breadth of scientific enquiry presented here is certainly thought-provoking. The earlier chapters on p-adic dynamics in general are a useful introduction. The subject of p-adic dynamics as a whole would benefit from a more extensive account, which might add to the topics contained in this book several others, including: entropy and heights, some of the more sophisticated connections between p-adic dynamics and arithmetic, the increasingly important study of homogeneous spaces of p-adic algebraic groups, a unification of non-Archimedean dynamics in zero and positive characteristic, the more recent work on the dynamics of rational maps over p-adic projective space, and the exciting prospects afforded by the study of dynamics on the Berkovich projective line.

This is an interesting text that will serve graduate students and others well as an introduction to an attractive field, and as a source of striking and provocative problems.

References

[1] D. Arrowsmith and F. Vivaldi, *Geometry of p-adic Siegel discs*, Phys. D **71** (1994), no. 1-2, 222–236. MR **MR1264116 (95d:**11162)

[2] M. Baker and R. Rumely, *Analysis and dynamics on the Berkovich projective line*, `arXiv:math.NT/0407433`, 2004.

[3] R.L. Benedetto, *Components and periodic points in non-Archimedean dynamics*, Proc. London Math. Soc. (3) **84** (2002), no. 1, 231–256. MR **MR1863402 (2002k:**11215)

[4] A. Besser and C. Deninger, *p-adic Mahler measures*, J. Reine Angew. Math. **517** (1999), 19–50. MR **MR1728549 (2001d:**11070)

[5] D. Bosio and F. Vivaldi, *Round-off errors and p-adic numbers*, Nonlinearity **13** (2000), no. 1, 309–322. MR **MR1734635 (2000k:**37130)

[6] J. Bryk and C. E. Silva, *Measurable dynamics of simple p-adic polynomials*, Amer. Math. Monthly **112** (2005), no. 3, 212–232. MR MR2125384

[7] L. Brekke, P. Freund, M. Olson, and E. Witten, *Non-Archimedean string dynamics*, Nuclear Phys. B **302** (1988), no. 3, 365–402. MR **MR947888 (89k:**81131)

[8] G. S. Call and J. H. Silverman, *Canonical heights on varieties with morphisms*, Compositio Math. **89** (1993), no. 2, 163–205. MR **MR1255693 (95d:**11077)

[9] Z. Coelho and W. Parry, *Ergodicity of p-adic multiplications and the distribution of Fibonacci numbers*, Topology, ergodic theory, real algebraic geometry, Amer. Math. Soc. Transl. Ser. 2, vol. 202, Amer. Math. Soc., Providence, RI, 2001, pp. 51–70. MR **MR1819181 (2002e:**11103)

[10] M. Einsiedler and D. Lind, *Algebraic $\mathbb{Z}^d$-actions of entropy rank one*, Trans. Amer. Math. Soc. **356** (2004), no. 5, 1799–1831 (electronic). MR **MR2031042 (2005a:**37009)

[11] P. Freund and M. Olson, *Non-Archimedean strings*, Phys. Lett. B **199** (1987), no. 2, 186–190. MR **MR919703 (89d:**81094)

[12] P. Freund and E. Witten, *Adelic string amplitudes*, Phys. Lett. B **199** (1987), no. 2, 191–194. MR **MR919704 (89d:**81095)

[13] M. Herman and J.-C. Yoccoz, *Generalizations of some theorems of small divisors to non-Archimedean fields*, Geometric dynamics (Rio de Janeiro, 1981), Lecture Notes in Math., vol. 1007, Springer, Berlin, 1983, pp. 408–447. MR **MR730280 (85i:**12012)

[14] L.-C. Hsia, *A weak Néron model with applications to p-adic dynamical systems*, Compositio Math. **100** (1996), no. 3, 277–304. MR **MR1387667 (97j:**14024)

[15] A. Khrennikov, *Non-Archimedean analysis: quantum paradoxes, dynamical systems and biological models*, Mathematics and its Applications, vol. 427, Kluwer Academic Publishers, Dordrecht, 1997. MR **MR1746953 (2001h:**81004)

[16] A. Khrennikov, S. Yamada, and A. van Rooij, *The measure-theoretical approach to p-adic probability theory*, Ann. Math. Blaise Pascal **6** (1999), no. 1, 21–32. MR **MR1693138 (2000f:**46102)

[17] H.-C. Li, *p-adic dynamical systems and formal groups*, Compositio Math. **104** (1996), no. 1, 41–54. MR **MR1420709 (98a:**11163)

[18] D. A. Lind, *The structure of skew products with ergodic group automorphisms*, Israel J. Math. **28** (1977), no. 3, 205–248. MR MR0460593 (57 #586)

[19] ———, *Ergodic group automorphisms are exponentially recurrent*, Israel J. Math. **41** (1982), no. 4, 313–320. MR **MR657863 (83i:**28022)

[20] D. Lind and K. Schmidt, *Bernoullicity of solenoidal automorphisms and global fields*, Israel J. Math. **87** (1994), no. 1-3, 33–35. MR **MR1286813 (95e:**28013)

[21] D. Lind, K. Schmidt, and T. Ward, *Mahler measure and entropy for commuting automorphisms of compact groups*, Invent. Math. **101** (1990), no. 3, 593–629. MR **92j:**22013

[22] D. A. Lind and T. Ward, *Automorphisms of solenoids and p-adic entropy*, Ergodic Theory Dynam. Systems **8** (1988), no. 3, 411–419. MR **MR961739 (90a:**28031)

[23] D. W. Masser, *Mixing and linear equations over groups in positive characteristic*, Israel J. Math. **142** (2004), 189–204. MR **MR2085715 (2005e:**37011)

[24] P. Morton and J.H. Silverman, *Periodic points, multiplicities, and dynamical units*, J. Reine Angew. Math. **461** (1995), 81–122. MR **MR1324210 (96b:**11090)

[25] J. Rivera-Letelier, *Dynamique des fonctions rationnelles sur des corps locaux*, Astérisque (2003), no. 287, xv, 147–230. MR **MR2040006 (2005f:**37100)

[26] K. Schmidt, *Dynamical systems of algebraic origin*, Birkhäuser Verlag, Basel, 1995. MR **97c:**28041

[27] J. Silverman, `www.math.brown.edu/~jhs/MA0272/ArithDynRefsOnly.pdf`.

[28] F. Vivaldi, `www.maths.qmw.ac.uk/~fv/database/algdyn.bib`.

[29] T. Ward, *Almost all S-integer dynamical systems have many periodic points*, Ergodic Theory Dynam. Systems **18** (1998), no. 2, 471–486. MR **MR1619569 (99k:**58152)

[30] ———, *Additive cellular automata and volume growth*, Entropy **2** (2000), no. 3, 142–167 (electronic). MR **MR1882488 (2004h:**37014)

[31] S. A. Yuzvinskiĭ. *Calculation of the entropy of a group-endomorphism*, Sibirsk. Mat. Ž., **8** (1967), 230–239. MR MR0214726 (35#5575)

Thomas Ward

School of Mathematics, University of East Anglia, Norwich U.K.

E-mail address: `t.ward@uea.ac.uk`