BOOK REVIEWS

DYNAMICAL SYSTEMS OF ALGEBRAIC ORIGIN
(Progress in Mathematics)

By K S : 328 pp., DM.138.–, 3 7643 5174 8 (Birkhäuser, 1995).

Ergodic theory studies group actions given by representations $T: G \to MPT(X)$, where $MPT(X)$ is the group of invertible measure-preserving transformations of a probability space X. The basic problem is to determine when two such actions are 'equivalent' (there are several different notions of equivalence), by producing invariants of the action preserved by the equivalence.

Traditionally, G is often either $\mathbb{Z}$ or $\mathbb{R}$. In both of these cases, the motivation is to study probabilistic properties of flows (or maps induced by sections of flows) arising from dynamical systems that evolve in time, represented by a parameter running through $\mathbb{R}$.

The study of ergodic properties (invariants useful for detecting non-equivalence) for actions of other groups has, however, always been of importance. Here the group does not represent a 'time' evolution, but most ergodic-theoretical notions still make sense.

Zimmer proved a remarkable rigidity result for actions of semi-simple Lie groups: let G_1 and G_2 be connected simple Lie groups of rank greater than or equal to two and with trivial centres, and let T_1, T_2 be actions of G_1, G_2 that are orbit equivalent (there is a measure-preserving map from X_1 to X_2 taking each G_1-orbit onto a G_2-orbit). Then the groups G_1 and G_2 are isomorphic, and under this isomorphism the actions must also be isomorphic (see [**5**]).

Actions of amenable groups are, in contrast, all orbit equivalent, and the ergodic theory of amenable group actions is in many ways similar to the traditional setting of $\mathbb{Z}$-actions: entropy theory, the isomorphism theorem for Bernoulli shifts under certain assumptions, and so on (see [**3**]). Actions of groups like $\mathbb{Z}^d$ appear in the work of Furstenberg, Katznelson and Weiss on connections between recurrence in ergodic theory and topological dynamics and combinatorics, and of course ergodic theory owes a historical debt to statistical mechanics where $\mathbb{Z}^d$-actions arise naturally (see [**4**]). However, tractable examples of $\mathbb{Z}^d$-actions are not easy to come by, particularly actions of positive entropy. Commuting toral automorphisms and cellular automata (homeomorphisms of shift spaces commuting with the shift map) give rise to very interesting $\mathbb{Z}^d$-actions, but these are automatically of zero entropy for $d > 1$. Subshifts of finite type can give positive entropy actions, but the entropy seems impossible to compute in general, and there are difficulties of a logical nature with developing a general theory. An additional problem is the fact that no positive-entropy smooth $\mathbb{Z}^d$-action can exist on a finite dimensional manifold for $d > 1$.

Schmidt's book is the culmination of a broad research programme into a special class of $\mathbb{Z}^d$-actions: those in which X is a compact abelian group, and each transformation in the image of the action T is a group automorphism. We call such actions algebraic. For $d = 1$ the measurable structure is well understood: single automorphisms of compact abelian groups are isomorphic to Bernoulli shifts if they are mixing, and they are therefore classified by their entropy, which is easy to compute. For $d > 1$, the situation is more complicated, as illustrated by the following example due to Ledrappier [**2**]. Let A be a compact abelian group, and let X_A denote the subset of $A^{\mathbb{Z}^2}$ of points a with $a(n+1,m)+a(n,m)+a(n,m+1) = 0_A$ for all $(n,m) \in \mathbb{Z}^2$. Then X_A is a closed subgroup of $A^{\mathbb{Z}^2}$, and the natural shift action of $\mathbb{Z}^2$ restricts to an algebraic $\mathbb{Z}^2$-action S_A on X_A. Ledrappier showed that when $A = \mathbb{Z}/2\mathbb{Z}$, this action is mixing but not mixing of all orders, showing that the measurable structure of mixing $\mathbb{Z}^2$-actions is quite different to the case of algebraic $\mathbb{Z}$-actions.

Kitchens and Schmidt produced a unified algebraic framework for studying algebraic actions in [**1**]. If T is an algebraic $\mathbb{Z}^d$-action on the compact abelian group X, then the dual automorphisms of the d commuting automorphisms generating T give the discrete group $M = \hat{X}$ (the character group of X) the structure of a module over the ring $R_d = \mathbb{Z}[u_1^{\pm 1}, \dots, u_d^{\pm 1}]$. That is, multiplication by the monomial $u_1^{n_1} \dots u_d^{n_d}$ on M is dual to the automorphism $T(n_1, \dots, n_d)$ of X. This correspondence goes both ways, so any R_d-module M gives rise by duality to an algebraic $\mathbb{Z}^d$-action T^M. For example, the example of Ledrappier given above with $A = \mathbb{Z}/2\mathbb{Z}$ corresponds to the module $R_2/\langle 2, 1+u_1+u_2 \rangle$. The language of commutative algebra and the decomposition methods via prime ideals associated to a module make it possible to 'translate' dynamical properties of the original $\mathbb{Z}^d$-action into arithmetic and geometric properties of prime ideals associated to the corresponding module. The current state of this dynamics-algebra dictionary—in whose compilation Schmidt has played the role of a Samuel Johnson—is elegantly presented, with complete proofs and in a unified fashion.

To whet appetites, I shall mention three entries.

First, the entropy of T^M has been computed. If M is cyclic, of the form $R_d/\mathfrak{p}$ for a prime ideal $\mathfrak{p}$, then $h(T^M)$ is zero unless $\mathfrak{p} \neq \{0\}$ is principal, in which case

$$h(T^M) = \int_0^1 \dots \int_0^1 \log |f(e^{2\pi i s_1}, \dots, e^{2\pi i s_d})| \, ds_1 \dots ds_d \quad \text{if } \mathfrak{p} = \langle f \rangle,$$

and the entropy is infinite if the ideal is 0. For example, if in Ledrappier's example the alphabet group A is replaced by the additive circle $\mathbb{T}$, we obtain a system whose corresponding module is $R_2/\langle 1+u_1+u_2 \rangle$; the entropy of this action is

$$h(S_{\mathbb{T}}) = \frac{3\sqrt{3}}{4\pi} \sum_{n=1}^{\infty} \left(\frac{n}{3}\right) \frac{1}{n^2},$$

where $(\frac{n}{3})$ is the Legendre symbol.

Secondly, algebraic $\mathbb{Z}^d$-actions turn out to be isomorphic to higher-dimensional Bernoulli shifts if they have completely positive entropy. A key ingredient in this proof is the Artin–Whaples formula for valuations of a number field.

Finally, it is known that T^M is mixing of all orders if and only if for every associated prime ideal $\mathfrak{p}$ of M, either $\mathfrak{p} = \langle p \rangle$ for some rational prime p, or $\mathfrak{p} \cap \mathbb{Z} = \{0\}$ and the algebraic $\mathbb{Z}^d$-action corresponding to the cyclic module $R_d/\mathfrak{p}$ is mixing. This result requires recent deep quantitative theorems on S-units due to Schlickewei, and shows, for example, that the $\mathbb{Z}^2$-action given by the (natural invertible extension of the) maps $x \mapsto 2x$ and $x \mapsto 3x$ on $\mathbb{T}$ is a mixing of all orders, zero entropy, Markov shift.

Many topics beyond and around the development of this dictionary are covered, including in Chapter I a thorough treatment of actions of countable groups by automorphisms of compact metrizable groups, in Chapter VII a discussion of delicate new invariants for the study of certain zero-dimensional zero-entropy actions, and in Chapter IX some recent and remarkable results on rigidity and cohomological rigidity for algebraic $\mathbb{Z}^d$-actions.

Since the first few pages are rather intimidating in their generality, it is a good idea to begin as one might with a detective story: turn to Section 31 (the last one), where the following surprising rigidity result is proved. If $\phi: X_A \to X_A$ is a measure-preserving Borel map from Ledrappier's example (as above, with $A = \mathbb{Z}/2\mathbb{Z}$) to itself, that commutes with the $\mathbb{Z}^2$-action S_A, then ϕ is almost everywhere equal to a continuous onto group homomorphism. This fascinating new rigidity phenomenon should help to motivate the reader through the preceding three hundred pages.

Motivational and historical material is given throughout, and there are a huge number of examples to illustrate the results. Any ergodic theorist or graduate student with some background in ergodic theory will find this book a rich source of ideas and problems, and it is the only coherent and unified source for the recent developments in algebraic $\mathbb{Z}^d$-actions. In addition, there is much here to interest people in number theory and commutative algebra. An extensive bibliography, index of notation and index complete this essential reference in algebraic ergodic theory.

References

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FINITE MODEL THEORY
(Perspectives in Mathematical Logic)

By H -D E and J F : 327 pp., DM.148.–, 3 540 60149 X (Springer, 1995).

Model theory is the branch of mathematical logic that deals with the expressive limits of language, by examining the relationship between logic on the one hand and algebraic, mathematical structures on the other. The language of primary interest in model theory is the first-order predicate calculus, and to a large extent, model theory can be characterised as the study of the relation of elementary equivalence. This equivalence relation between structures tells us which structures are not distinguishable in first-order logic, and hence permits us to determine the limits of expressibility.

Model theory has in the past been mainly concerned with infinite structures. This is partly because one of the goals of mathematical logic was the elucidation of questions of infinity arising in the foundations of mathematics, and partly because