THE NUMBER OF DISTINCT EIGENVALUES OF ELEMENTS IN FINITE LINEAR GROUPS

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Abstract Let G be a finite (non-abelian) irreducible linear subgroup over the complex numbers, and let g be an element of G of prime order p. Suppose that g does not belong to a proper normal subgroup of G. We show that the number of distinct eigenvalues of g can only be $p, p-1, p-2, (p+1)/2$ or $(p-1)/2$. Moreover, we provide a full classification of such groups G provided g has at most $p-2$ distinct eigenvalues.

1. Introduction

Let $\mathbb{C}$ denote the field of complex numbers and $GL(n,\mathbb{C})$ the group of all $(n\times n)$-matrices over $\mathbb{C}$.

Theorem 1.1. *Let G be a finite irreducible subgroup of $GL(n,\mathbb{C})$ and $n>1$. Let $g\in G$ be an element of prime order p. Suppose that g does not belong to a proper normal subgroup of G. Then the number of distinct eigenvalues of g can only be $p, p-1, p-2, (p+1)/2$ or $(p-1)/2$.*

Let $Z(G)$ denote the center of a group G. In fact we obtain full classification of finite irreducible subgroups $G\subset GL(n,\mathbb{C})$ generated by the conjugacy class of an element g such that $g^p\in Z(G)$ and g has at most $p-2$ distinct eigenvalues. To state the result we need some notation.

For a matrix g denote by $\operatorname{Spec} g$ the set of all distinct eigenvalues of g, disregarding the multiplicities, and by $\deg g$ their number. (As g is of finite order then $\deg g$ is exactly the degree of the minimum polynomial of g.) Let ε be a primitive p-root of 1. Set $\nu_p=\{\varepsilon^i\}$ where i runs over non-quadratic residues modulo p, and $\mu_p=\{\varepsilon^i\}$ where i runs over quadratic residues modulo p. We also set $\bar\nu_p=\nu_p\cup 1$, $\bar\mu_p=\mu_p\cup 1$, and $\rho_p^j=\{1,\varepsilon,\dots,\varepsilon^{p-1}\}\setminus\{\varepsilon^j,\varepsilon^{-j}\}$ where $0\neq j<p$. Our notations for classical groups and for groups of Lie type are standard. The alternating group of order $n!/2$ is denoted by A_n, and J_2 is a sporadic simple group of order 604800.

Theorem 1.2. *Let G be a finite irreducible subgroup of $GL(n,\mathbb{C})$ and, for a prime $p>3$, let $g\in G$ with $g^p\in Z(G)$. Suppose that $\langle g^G\rangle=G$ and $\deg g<p-1$. Then one of the following holds:*

Mathematics Subject Classification 20H20, 20C15.

(A) $\deg g = n = (p-1)/2$ and $G/Z(G) \cong PSL(2,p)$; in addition, $\operatorname{Spec} g = \alpha \cdot \nu_p$ *or* $\alpha \cdot \mu_p$ *for some* $\alpha \in \mathbb{C}$.

(B) $\deg g = (p+1)/2$ and one of the following holds:

(1) *$G/Z(G) = PSp(2k,p)$, $gZ(G)$ is a transvection in $G/Z(G)$; furthermore, either $k > 1$ and $n = (p^k \pm 1)/2$ or else $k = 1$ and $n = (p+1)/2$;*

(2) *$G/Z(G) = F_p^{2k} \cdot Sp(2k,p)$, a semidirect product of $Sp(2k,p)$ and the additive group of the natural $Sp(2k,p)$-module F_p^{2k}, $k \geq 1$, the projection of g into $Sp(2k,p)$ is a transvection and $n = p^k$;*

(3) *$G/Z(G) = A_6 \cong PSL(2,9)$, $p = 5$ and $n = 3$;*

(4) *$G/Z(G) = A_7$, $p = 7$ and $n = 4$.*

(5) *$G/Z(G) = J_2$, $p = 5$ and $n = 6$;*

(6) *$G/Z \cong A_5 \times A_5$, $p = 5$ and $n = 4$.*

In addition, $\operatorname{Spec} g = \alpha \cdot \bar{\nu}_p$ *or* $\alpha \cdot \bar{\mu}_p$ *where α is as above.*

(C) $\deg g = n = p-2$, $G/Z(G) = SL(2,q)$ for q even. In addition, $\operatorname{Spec} \theta(g) = \alpha \cdot \rho_p^j$ *for some j with $1 \leq j \leq (p-1)/2$.*

Remarks (1) The set ρ_5^j coincides with $\bar{\nu}_5$ or $\bar{\mu}_5$. (2) The list above does not indicate all isomorphisms between simple groups, for instance, $SL(3,2)$ for $p = 7$ appears as $PSL(2,7)$ in case (A).

Theorem 1.2 extends some earlier results. If G is p-soluble, or more generally, contains a non-central normal p'-subgroup then a similar result essentially belongs to Hall and Higman [10]. The estimate $\deg g \geq (p-1)/2$ under assumptions of Theorem 1.1 has been obtained by Robinson [16]. For the particular case where $n < p$ the result of Theorem 1.2 can be extracted from [28].

Our proof uses the classification of finite simple groups. The best result not using the classification is that of Robinson [15].

Notation p is a fixed prime integer (usually $p > 3$). If H is a group then H' is the commutator subgroup of H and $Z(H)$ is its center. For $a, b \in H$ we set $[a,b] = aba^{-1}b^{-1}$. We denote by ρ_H the regular character, the regular representation of H or regular H-module; similarly, 1_H denotes the trivial character, the one-dimensional trivial representation of H or the trivial H-module. The symbol $\phi \in \operatorname{Irr} H$ means that ϕ is an irreducible character, irreducible representation or irreducible $\mathbb{C}H$-module. If $K \subset H$ are finite groups, M is an H-module and L is an FH-module, we denote by $M|_K$ the restriction of M to K, and we write $L \in M|_K$ (resp., $L \notin M|_K$), in order to say that L is (resp., is not) a submodule of $M|_K$ (and the same for representations). The meaning of the symbols $\operatorname{Spec} h$, $\deg h$, μ_p, ν_p, $\bar{\mu}_p$, $\bar{\nu}_p$, ρ_p^j is defined prior Theorem 1.2. The greatest common divisor of integers m, n is denoted by (m,n).

Our notation for sporadic simple groups agrees with [2]. For a group H we say that H is a finite algebraic group if there is an algebraic group G and a Frobenius endomorphism F of G such that $H = \{g \in G : F(g) = g\}$. We say that H is reductive (respectively, connected, simply connected) if G is reductive (respectively, connected, simply connected). For uniformity purposes, we use symbols $GL^{\varepsilon}(n, q)$ with $\varepsilon = \pm 1$ to refer to group $GL(n, q)$ if $\varepsilon = 1$ and $U(n, q)$ if $\varepsilon = -1$. Similar meaning is assigned to the symbol $SL^{\varepsilon}(n, q)$.

2. Preliminaries and some known results

Theorem 2.1. [26] *Let G be a quasi-simple finite group with cyclic Sylow p-subgroup. Let M be a non-trivial irreducible $\mathbb{C}G$-module. Let $g \in G$ be an element of order p such that $\deg g|_M < p - 1$. Then one of the following holds:*

(1) *$G \cong SL(2, q)$, $q > 4$ is even, $p = q+1$, $\dim M = \deg g = p-2$ and $\operatorname{Spec} g|_M = \rho_p^j$ for some j;*

(2) *$G/Z(G) \cong PSL(2, p)$, $\dim M = \deg g = (p - 1)/2$ and $\operatorname{Spec} g|_M = \mu_p$ or ν_p;*

(3) *$G/Z(G) \cong PSL(2, p)$, $\dim M = \deg g = (p + 1)/2$ and $\operatorname{Spec} g|_M = \bar{\mu}_p$ or $\bar{\nu}_p$;*

(4) *$G/Z(G) = A_6 \cong PSL(2, 9)$, $p = 5$ and $\dim M = \deg g = 3$ and $\operatorname{Spec} g|_M = \bar{\mu}_5$ or $\bar{\nu}_5$;*

(5) *$G/Z(G) = A_7$, $p = 7$ and $\dim M = \deg g = 4$ and $\operatorname{Spec} g|_M = \bar{\mu}_7$ or $\bar{\nu}_7$.*

Theorem 2.2. [27] *Let G be a non-trivial covering of the alternating group A_n where $n > 7$. Let $1_G \neq \theta \in \operatorname{Irr} G$. Then $\deg \theta(g) \geq p - 1$.*

Remark. If $n \leq 7$ and $p > 3$ then Sylow p-subgroups of G are cyclic. This case is contained in Theorem 2.1.

Lemma 2.3. *Suppose that $G/Z(G)$ is a simple group with cyclic Sylow p-subgroup. Let M be an irreducible $\mathbb{C}G$-module of dimension greater than 1 and let $g \in (G \setminus Z(G))$. Suppose $g^p \in Z(G)$ and $\deg g|_M < p - 1$. Then $G/Z(G)$ is as in cases (1) - (5) of Theorem 2.1 and there is a root of unity α such that $\alpha \cdot \operatorname{Spec} g|_M$ is as in Theorem 2.1.*

Proof. Observe that $G = G'Z(G)$ and $G/Z(G) \cong G'/Z(G')$. Therefore, there is a p-element $h \in G'$ such that $gh \in Z(G)$. As Sylow p-subgroup of $G/Z(G)$ are cyclic, $|Z(G')|$ is coprime to p (see for instance [26, Lemma 3.3]). Therefore, $h^p = 1$. Obviously, $\deg g|_M = \deg h|_M$ so h satisfies the hypothesis of Theorem 2.1 which yields the lemma.

Theorem 2.4. *Let $g \in G$ where $G/Z(G)$ is a simple group of Lie type in characteristic p and $g^p = 1$. Let $\theta \in \operatorname{Irr} G$, and $1 < \deg \theta(g) < p-1$. Then $G/Z(G) = PSp(2n, p)$, $n \geq 1$, $\dim \theta = (p^n \pm 1)/2$ and g is a transvection. In addition, $\operatorname{Spec} \theta(g) = \alpha\bar{\mu}_p$ or $\alpha\bar{\nu}_p$ for some $\alpha \in \mathbb{C}$ unless $n = 1$ and $\dim \theta = (p-1)/2$ in which case $\operatorname{Spec} \theta(g) = \alpha\mu_p$ or $\alpha\nu_p$. Moreover, there exists θ satisfying the above conditions.*

Proof. It is well known that $(|Z(G)|, p) = 1$ for $p > 3$. So the result is available from [23] (see also [24]) except for the dimension of θ. These have been determined in [24] for cases (i) - (iv), for case (v) this has been done in [19].

Lemma 2.5. [11] *Let X be an irreducible cyclic subgroup of $GL(n, q)$. Then $|X|$ divides $q^n - 1$. If $|X| = p$ then $n = m_p(q)$.*

Lemma 2.6. [11] *Let V be a vector space endowed by a non-degenerate bilinear or quadratic form, and let $I(V)$ denote the group of isometries of V. Let X be an irreducible cyclic subgroup of $I(V)$. Then one of the following holds:*

(i) $I(V) = U(n, q)$, n is odd, $|X|$ divides $q^n + 1$;

(ii) $I(V) = Sp(2n, q)$ and $|X|$ divides $q^n + 1$;

(iii) $I(V) = O^-(2n, q)$ and $|X|$ divides $q^n + 1$.
In addition, if $|X| = p$ then $n = m_p(q^2)$.

Lemma 2.7. (Hall-Higman) *Let $T = RH$ be a finite linear group where $H = \langle h \rangle$ is a cyclic p-subgroup and R is a normal p'-subgroup. Suppose that $[h, R] \neq 1$. Then* $\deg h \geq p - 1$.

Proof. Let ψ be a representation of T over an algebraically closed field of characteristic p obtained from T by reduction modulo p. Then $\psi|_R$ is faithful and $\deg g \geq \deg \psi(g)$. By a theorem of Hall and Higman (see [5, VII.10.2]) $\deg \psi(g) \geq p - 1$. So the lemma follows.

Lemma 2.8. [8, §13.2] *Let H be a finite connected algebraic group, R a parabolic subgroup of H and U the unipotent radical of R. Then $C_H(U) = Z(H) \cdot Z(U)$.*

Lemma 2.9. (Borel-Tits, see [8, §13.1]) *Let H be a finite reductive algebraic group in characteristic r and $g \in G$. If g normalizes an r-subgroup of H then g belongs to a parabolic subgroup of H.*

Lemma 2.10. *Let G be a finite reductive group in characteristic $r \neq p$ and let $g \in G \setminus Z(G)$ be a p-element. Let ϕ be an irreducible representation of G and* $\dim \phi > 1$. *If $g \in G$ normalizes an r-subgroup in G then* $\deg \phi(g) \geq p - 1$.

Proof. By Lemma 2.9, g belongs to a parabolic subgroup P of H. Let U be the unipotent radical of P. By Lemma 2.8, $\mathbb{C}_H(U) = Z(G) \cdot Z(U)$. Hence $[g, U] \neq 1$ so the lemma follows from Lemma 2.7.

Lemma 2.11. *Let $G = SL^\varepsilon(n, q)$ and $h \in H = GL^\varepsilon(n, q)$ be a semisimple element. Then $H = C_H(g)G$.*

Proof. Suppose first that g is irreducible. Then $|C_H(g)| = q^n - \varepsilon$ (where n is odd if $\varepsilon = -1$). It is well-known that the mapping $C_H(g) \to GL^\varepsilon(1, q)$ defined for $g \in C_H(g)$ by $g \to \det g$ is surjective. Therefore, $|C_H(g)G/G| = q - \varepsilon$ and the result follows. The general case can be easily reduced to this one.

Lemma 2.12. *Let T a Sylow p-subgroup of $GL(n,q)$ where $(p,q)=1$.*

(*i*) (see [8, §10]) *If T contains an irreducible subgroup of order p then T is cyclic. In addition, $|T|$ is the p-part of q^n-1 and n is the minimum positive integer such that p divides q^n-1.*

(*ii*) *Express $q-1$ in the form $p^a l$. If T contains an irreducible cyclic p-subgroup $C=\langle c\rangle$ such that* $\mathrm{Id}\neq c^p\in Z(GL(n,q))$ *then $n=p$ and $|c|=p^{a+1}$. In addition, $|T|=p^{ap+1}$ and some conjugate of T consists of monomial matrices.*

(*iii*) (see [21]) *If T is irreducible and primitive then either T is cyclic or $p=n=2$.*

Proof. Claim (ii) can be easily deduced from [21].

In Lemma 2.13 below ρ_p denote the set of all p-roots of unity and $\lambda_p=\rho_p\setminus 1$. The meaning of the symbols $\mu_p,\nu_p,\bar\mu_p,\bar\nu_p$ is defined in the introduction. The symbol $\mu_p\nu_p$ means the set $\{ab: a\in\mu_p, b\in\nu_p\}$ (ignoring repetitions).

Lemma 2.13. (1) *If $p\equiv 1(\mathrm{mod}\,4)$ and $p\neq 5$ then $\nu_p^2=\mu_p^2=\rho_p=\bar\mu_p\mu_p=\bar\nu_p\nu_p$, $\mu_p\nu_p=\bar\mu_p\nu_p=\bar\nu_p\mu_p=\lambda_p$.*

(2) *If $p\equiv 3(\mathrm{mod}\,4)$ and $p\neq 3$ then $\nu_p^2=\mu_p^2=\lambda_p=\bar\mu_p\mu_p=\bar\nu_p\nu_p$ and $\nu_p\mu_p=\rho_p=\bar\nu_p^2=\bar\mu_p^2$.*

(3) *If $p=5$ then $\mu_5^2=\bar\nu_5$, $\nu_5^2=\bar\mu_5$, $\mu_5\nu_5=\lambda_5=\bar\mu_5\nu_5=\mu_5\bar\nu_5$, $\bar\mu_5^2=\bar\nu_5^2=\rho_5$, $\mu_5^3=\nu_5^3=\lambda_5$.*

(4) *If $p=3$ then $\mu_3^2=\nu_3$, $\nu_3^2=\mu_3$, $\mu_3\nu_3=\{1\}$, $\bar\mu_3\nu_3=\bar\nu_3$, $\bar\nu_3\nu_3=\bar\mu_3$, $\bar\mu_3\bar\nu_3=\rho_3=\bar\mu_3^2=\bar\nu_3^2$.*

Proof. It suffices to determine μ_p^2,ν_p^2 and $\mu_p\nu_p$ as other cases easily follow from these. Let A (resp., B) denote the subset of all squares (resp., non-squares) in $F_p^*=F_p\setminus 0$. Fix $\beta\in B$. Then

$$\mu_p^2=\{\varepsilon^i:\ i=x^2+y^2,\ x,y\in F_p^*\},\ \nu_p^2=\{\varepsilon^i:\ i=\beta(x^2+y^2),\ x,y\in F_p^*\},$$

$$\mu_p\nu_p=\{\varepsilon^i:\ i=x^2+\beta y^2,\ x,y\in F_p^*\}.$$

In order to determine μ_p^2,ν_p^2 and $\mu_p\nu_p$ consider a vector space over F_p of dimension 2 with basis e_1,e_2. Let W' denote the subset of vectors proposional to neither e_1 nor e_2. Define on W three quadratic forms by setting $(e_1,e_2)=0$ and

(1) $(e_1,e_1)=(e_2,e_2)=1$;

(2) $(e_1,e_1)=(e_2,e_2)=\beta$;

(3) $(e_1,e_1)=1$, $(e_2,e_2)=\beta$.

Set $T_i=\{(w,w)\}_{w\in W'}$ where $i=1,2,3$ and (w,w) is computed in cases (1), (2), (3), respectively. Obviously, $\mu_p^2=\{\varepsilon^i:\ i\in T_1\}$, $\nu_p^2=\{\varepsilon^i:\ i\in T_2\}$, $\mu_p\nu_p=\{\varepsilon^i:\ i\in T_3\}$. As $aw\in W'$ if $w\in W'$, $a\in F_p^*$, we observe that $AT_i=T_i$. Hence $A\subseteq T_i$ (resp., $B\subseteq T_i$) if and only if $1\in T_i$ (resp., $\beta\subseteq T_i$).

It is well-known that $A\cup B\subseteq\{(w,w)\}_{w\in W}$. Therefore, it suffices to decide when $1,\beta\in T_i$. Let W_0 denote the subset of $W\setminus W'$ of vectors w such that $(w,w)=1$ or

β. Clearly, W_0 consists of 4 vectors $\pm e_1, \pm e_2$, and for all them $(w,w) = 1$ in case (1), $(w,w) = \beta$ in case (2), while in case (3) two vectors $w = \pm e_1$ satisfy $(w,w) = 1$ and two vectors $w = \pm e_2$ satisfy $(w,w) = \beta$. The total numbers of vectors $w \in W$ with $(w,w) = 1$ (or $(w,w) = \beta$) is equal to $|O(W)|/4$, and $|O(W)| = 2(p \pm 1)$ (where sign $-$ has to be chosen if W is anisotropic). Hence we arrive the condition $p \pm 1 > 8$ in cases (1), (2), and $p \pm 1 > 4$ in case (3) to guarantee that $A \cup B \subseteq T_i$. The cases $p < 8$ can be easily inspected.

By using the character tables given in [2] one can easily establish Lemmas 2.14 and 2.15 below:

Lemma 2.14. ([2]) *Let $G/Z(G)$ be a sporadic simple group and let $\theta \in \mathrm{Irr}\, G$. Suppose that $g^p = 1$ and $1 < \deg\theta(g) < p-1$. Then $G/Z(G) \cong J_2$, $\dim\theta = 6$, $p = 5$, $\mathrm{Spec}\,\theta(g) = \bar{\mu}_5$ or $\bar{\nu}_5$ and every eigenvalue of g is of multiplicity 2.*

Lemma 2.15. *Let G be the universal covering of one of the following groups: $SL(4,2)$, $PSL(3,4)$, $PSU(6,2)$, $SU(4,3)$, $Sp(6,2)$, $\Omega^+(8,2)$, $\Omega(7,3)$, $G_2(4)$. Let $g \in G$ be of prime order p. Let $\theta \in \mathrm{Irr}\, G$ and $\dim\theta > 1$. Then $\deg\theta(g) > p-2$.*

Proof. All these groups are available in [2] and the result can be deduced by inspection. (In view of Theorem 2.1 one has only to inspect primes p for which Sylow p-subgroups of G are not cyclic.)

3. Classical groups

In this section we prove the following theorem.

Theorem 3.1. *Let G be a group such that $G/Z(G)$ is classical, that is, there is a subgroup $Z \subseteq Z(G)$ such that G/Z is isomorphic to a group H such that either $SL^\varepsilon(k,q) \subseteq H \subseteq GL^\varepsilon(k,q)$ or $Sp(k,q) \subseteq H \subseteq CSp(k,q)$ or $\Omega(k,q) \subseteq H \subseteq O(k,q)$ with $k > 5$ odd or $\Omega^\pm(k,q) \subseteq H \subseteq O^\pm(k,q)$ with $k > 6$ even. Let θ be an irreducible representation of G of dimension greater than 1. Let $g \in G$ be a semisimple element of prime order p. Then either $\deg\theta(g) > p-2$ or case (C) of Theorem 1.2 holds.*

Lemma 3.2. *Let $H = GL^\varepsilon(n,q)$ and $H' \subseteq G \subseteq H$. Let $t \in H$ be a semisimple element and let $m(t)$ be the least natural number k such that $t^k \in Z(G)$. Suppose that $1 < \deg\phi(t) < m(t)$ for some representation $\phi \in \mathrm{Irr}_F\, G$. Then there is $\tau \in \mathrm{Irr}\, H$ such that $1 < \deg\tau(t) < m(t)$.*

Proof. Let τ be an irreducible representation of H such that ϕ is a constituent of $\tau|_G$. By Clifford's theorem, $\tau|_G = \oplus_i\phi_i$ where ϕ_i are irreducible and conjugate to ϕ by elements of H. By Lemma 2.11, every coset hG contains an element from $C_H(t)$. Therefore, $\mathrm{Spec}\,\phi_i(t) = \mathrm{Spec}\,\phi(t)$ as desired.

Lemma 3.3. *Let $q > 3$ and $SL^\varepsilon(3,q) \subseteq G \subseteq GL^\varepsilon(3,q)$. Let $g \in G$ be a semisimple element of order $p > 3$. Let $\theta \in \mathrm{Irr}\, G$ where $\dim\theta > 1$. Then $\deg\theta(g) > p-2$.*

Proof. We can assume that $G = GL^{\varepsilon}(3,q)$ by Lemma 3.2. As the characters of G are known, one could compute the eigenvalue multiplicities of g in every irreducible representation of G. (In a more conceptual way one can argue as follows (sketch). For $\varepsilon = 1$ the lemma follows from Theorem 2.1 and Lemma 2.7. Let $\varepsilon = -1$. If $(p, q+1) \neq 1$ then the lemma follows from Theorem 2.1. Otherwise, $g = hz$ where $z \in Z(G)$ and h is contained in a subgroup H of G isomorphic to $SU(2,q)$. It follows from Theorem 2.1 that q is even and all non-trivial irreducible constituents of $\theta|_H$ are of dimension $q-1$ and are equivalent to each other. By [20, Theorem 1.4], θ is a Weil representation of G. In this case the irreducible constituents of $\theta|_H$ are known and at least two of them are non-equivalent.

Lemma 3.4. *Let F be an arbitrary field. Let p be a prime and let $a, b \in GL(r,F)$ be semisimple matrices such that a^p and $[a,b] \neq \mathrm{Id}$ are scalar. Then $\deg b = p \cdot \deg b^p$, and $\operatorname{Spec} b$ consists of all p-roots of the elements of $\operatorname{Spec} b^p$.*

Proof. Observe that $[a, b^p] = [a,b]^p = [a^p, b] = \mathrm{Id}$. In particular, $[a,b] = \omega \cdot \mathrm{Id}$, where ω is a primitive p-root of 1 in K Let W be the natural module for $GL(r,F)$, $\lambda \in \operatorname{Spec} b^p$ and W_λ the λ-eigenspace of b^p in W. Obviously, $aW_\lambda = W_\lambda$ and $bW_\lambda = W_\lambda$. As $[a,b] \neq \mathrm{Id}$ is scalar, $[a,b]|_{W_\lambda} \neq \mathrm{Id}$. Let β be an eigenvalue of b in W_λ and denote by $W_{\lambda\beta}$ the β-eigenspace of b in W_λ. Observe that $\beta^p = \lambda$. Pick $0 \neq w \in W_{\lambda\beta}$. We have $ba^i w = [b, a^i] a^i b w = \omega^i \beta a^i w$. This means that $a^i w$ belongs to the $\omega^i\beta$-eigenspace of b in W_λ ($i = 2, \dots, p-1$). Hence $\beta, \omega\beta, \dots, \omega^{p-1}\beta$ are eigenvalues of b in W_λ.

Lemma 3.5. *Let $p > 2$ be a prime and $SL^{\varepsilon}(p,q) \subset X \subseteq GL^{\varepsilon}(p,q)$. Let $x \in X$ be an irreducible p-element such that $x^p = \alpha \cdot \mathrm{Id}$.*

(1) *There exists matrices $d, d_1 \in X$ such that $d^p = d_1^p = \mathrm{Id}$, $[d, d_1] = \mathrm{Id}$, $[d_1, x] = d$ and $[d, x] = \omega \cdot \mathrm{Id}$ where $\omega \neq 1$ is p-root of 1.*

(2) *Let F be an algebraically closed field of characteristic f with $(f, pq) = 1$. Let ϕ be an irreducible representation of X with $\dim \phi > 1$. Then $\deg \phi(x) = p$ and if $\phi(x^p) = \varepsilon \cdot \mathrm{Id}$ then $\operatorname{Spec} \phi(x)$ consists of all p-roots of ε.*

Proof. Since x is irreducible, $\alpha \neq 1$. As α is a p-element, p divides $q - \varepsilon$. Let V denote the natural $GL^{\varepsilon}(p,q)$-module. (1) Let $0 \neq v_1 \in V$ and $v_i = xv_{i-1}$ for $i = 1, \dots, p-1$. Then $xv_p = \alpha v_p$ and $\det x = \alpha$. As x is irreducible, $v_1, \dots, v_p$ is a basis in V. If $\varepsilon = -1$, there is $v_1 \in V$ such that the basis $v_1, \dots, v_p$ is orthonormal. (This is well-known and can be easily deduced from the fact that $|GL^{\varepsilon}(p,q)| = p \cdot |GL^{\varepsilon}(1,q)|$.) As X contains $SL^{\varepsilon}(p,q)$ and $\alpha \neq 1$, we conclude that X contains all matrices of determinant α^l for every integer l. Let $1 \neq \omega \in F_{q^{(3-\varepsilon)/2}}$ with $\omega^p = 1$. Set $d = \mathrm{diag}(\omega, \omega^2, \dots, \omega^{p-1}, 1)$ and $d_1 = \mathrm{diag}(\dots, \omega^{-i(i-1)/2}, \dots)$ where we indicate the term on the i-th position. Then $d, d_1 \in X$. One can straightforward check that $[x, d_1] = d$ and $[d, x] = \omega \cdot \mathrm{Id}$.

(2) Let d, d_1 be as above. Let $\lambda \neq 1$ be an eigenvalue of $\phi(d)$ and W the eigenspace of λ. Then $\dim W < \dim \phi$ and $C_X(d)W = W$ so $d_1 W = W$. If $[\phi(d), \phi(x)] =$

$\phi([d,x]) \neq \mathrm{Id}$ then (2) follows from Lemma 3.4. If $[\phi(d),\phi(x)] = \mathrm{Id}$ then $\phi(x)W = W$ and there is an eigenvalue $\mu \neq 1$ of $\phi(d_1)|_W$ (as $\phi([d_1,x]) = \lambda \cdot \mathrm{Id}$) and let W_1 be the μ-eigenspace of d_1 in W. Then $W_1 \neq W$ and again the result follows from Lemma 3.4.

Lemma 3.6. *Theorem* 3.1 *is true if* $SL^\varepsilon(n,q) \subseteq G \subseteq GL^\varepsilon(n,q)$ *and* $(p, q-\varepsilon) \neq 1$.

Proof. Let V be the natural module for G. Let W be a minimal non-degenerate g-submodule of V of dimension at least 3 such that $g|_W$ is non-trivial. To justify the existence, let L be a minimal g-submodule of V. Then $\dim L$ is either p or 1, see Lemma 2.12. In the former case choose $W = L$. In the latter case g is diagonalizable (under an orthogonal basis if G is unitary). Therefore, as $\dim V > 2$, we can choose W to be non-degenerate of dimension 3. By Lemma 2.11 it suffices to prove the lemma for $G = GL^\varepsilon(n,q)$. Let $X_W = \{x \in G : x|_{W^\perp} = \mathrm{Id}\}$ and $X = \langle X_W, g\rangle$. Let K be the kernel of the restriction homomorphism $\eta : X \to X|_W$. Obviously, $K \subseteq Z(X)$ and $X = X_W \cdot K$. Therefore, $g = g_1 g_2$ where $g_1 \in X_W$ and $g_2 \in K$. Let τ be an irreducible constituent of $\theta|_X$ of dimension greater that 1. Suppose the lemma is false. Then $\deg \tau(g) = \deg \tau(g_1) < p-1$. If $\dim W = 3$, this contradicts Lemma 3.3. (The case $q = 2$ is irrelevant as $p > 3$.) If $\dim W = p$, this contradicts Lemma 3.5.

Let V be a symplectic, orthogonal or unitary space of dimension m. Then we denote by $I(V)$ its group of isometries.

Lemma 3.7. *Let* $h \in H = I(V)$ *be an element of prime order* $p > 2$. *Suppose that* h *stabilizes no isotropic (singular) subspace of* V.

(1) *There is an orthogonal decomposition* $V = V_1 \oplus \cdots \oplus V_l$ *as a direct sum of* h*-submodules such that* $h|_{V_i} \neq \mathrm{Id}$ *and* V_i *is irreducible for all* $i \leq l-1$, *and* $\dim V_1 = \cdots = \dim V_{l-1}$. *In addition, either* $h|_{V_l}$ *is irreducible and* $\dim V_l = \dim V_1$ *or* $h|_{V_l} = \mathrm{Id}$ *and* V_l *is anisotropic (non-singular).*

(2) *If a Sylow* p*-subgroup of* H *is not cyclic then* $g|_{V_2} \neq \mathrm{Id}$, *except, possibly, when* V *is unitary of dimension* 2 *and* p *divides* $q+1$ *or* V *orthogonal of dimension at most* 4.

Proof. (1) Let V_1 be an irreducible submodule of V. By Maschke's theorem, we can choose V_1 with $h|_{V_1} \neq \mathrm{Id}$. By assumption, V_1 is non-degenerate, hence so is $V_1^\perp$. If $h|_{V_1} \neq \mathrm{Id}$, the claim follows by induction. Otherwise, V_1 is anisotropic (non-singular). The assertion on dimension follows from Lemma 2.12.

(2) By Lemma 2.12(i), $l \neq 1$. Suppose $g|_{V_2} = \mathrm{Id}$. By (1), V_2 is anisotropic, so V is not symplectic, $\dim V_2 \leq 2$ if V is orthogonal and $\dim V_2 \leq 1$ if V is unitary. Let N be the normalizer of $\langle h\rangle$ in a Sylow p-subgroup S of H that contains h. Then $NV_2 = V_2$. If $N|_{V_2} \neq \mathrm{Id}$ then $N|_{V_2}$ has an element of order p whence $\dim V_1 \leq \dim V_2$. So if V is orthogonal then $\dim V \leq 4$. If V is unitary then $\dim V \leq 2$, as required. Therefore, we are left with the case where $N|_{V_2} = \mathrm{Id}$. Then we can apply this argument to $N_1 = N_S(N)$ to conclude that $N_1 V_2 = V_2$ and $N_1|_{V_2} = \mathrm{Id}$. It follows that $SV_2 = V_2$

and $S|_{V_2} = \mathrm{Id}$. Then $SV_1 = V_1$ and by Lemma 2.12(i) S is cyclic, which is not the case.

Lemma 3.8. *Let* $\dim V = m$ *and either* $SL(m,q) \subseteq G \subseteq GL(m,q)$ *or* $I(V)' \subseteq G \subseteq I(V)$. *Let* $g \in G$ *be an irreducible element. Then* g *is conjugate in* G *to* m *distinct elements* g^i *unless* V *is orthogonal and* $G \neq I(V)$ *in which case* g *is conjugate in* G *to at least* $m/2$ *distinct elements of shape* g^i. *In particular, if* $m > 2$ *then* g *is conjugate to some* g^i *with* $i \neq 1, -1$.

Proof. Let X be the group generated by g, and let K be the span of X in $\operatorname{End} V$. Then K is a field by Schur's lemma. As K is irreducible, $\dim K = \dim V = m$. In addition, $C_G(g) = K \cap G$. Therefore, the number in question is equal to the order of $\Gamma := N_G(X)/C_G(X)$. As Γ is a subgroup of the Galois group of K, it is cyclic. Let $a \in N_G(X)$ be such that $aC_G(X)$ is a generator of Γ. It is well known that $|\Gamma| = m$ if $G = I(V)$ or $GL(m,q)$. (The case of orthogonal groups in characteristic 2 is discussed in detail in [12, Lemma 4.3.15].) So we are done in the symplectic group case. For groups $GL^\varepsilon(m,q)$ observe that $GL^\varepsilon(m,q) = C_{GL^\varepsilon(n,q)}(g) \cdot SL^\varepsilon(n,q)$ (Lemma 2.11) so a can be found in $SL^\varepsilon(n,q)$. For an orthogonal space $I(V)/\Omega(V)$ is of exponent 2, hence $a^2 \in \Omega(V)$ and the result follows.

Lemma 3.9. *Let* V *be an orthogonal space and let* $V = V_1 \oplus V_2$ *where* V_1, V_2 *are non-degenerate subspaces of* V *of equal dimension* $2m$ *orthogonal to each other. Suppose that* $g \in G = \Omega(V)$ *preserves* V_1, V_2 *and acts irreducibly in each of them. Then* g *is conjugate to* g^i *for at least* $2m$ *distinct* j *with* $1 \leq j < |g|$.

Proof. Set $g_i = g|V_i$ for $i = 1, 2$. By Lemma 3.8 that there is an element $x_1 \in N_{I(V_1)}(\langle g_1 \rangle)$ such that $x_1^j g_1 x_1^{-j}$ contains $2m$ distinct elements. Let ν be the spinor norm of x_1. Both V_1, V_2 are of Witt index $m-1$ as they contain irreducible elements (Lemma 2.6). Therefore, V_1, V_2 are isomorphic as orthogonal spaces, so one can fix bases in V_1, V_2 with the same Gram matrix. As g_i is irreducible, Sylow p-subgroups S_i of $\Omega(V_i)$ are cyclic (Lemma 2.12). In view of Sylow's theorem, we can assume that S_1 and S_2 consist of the same set of matrices (under above bases), as well as their subgroups T_i of order p. Therefore, this also holds for $N_i := N_{I(V_i)}(T_i)$. It follows that N_2 contains an element x_2 of spinor norm ν which induces an automorphism of order $2m$ on T_2. Then the spinor norm of $a = \operatorname{diag}(x_1, x_2)$ is 1, that is, $a \in \Omega(V)$. As we can assume $T_i = \langle g_i \rangle$, the lemma follows.

Proof of Theorem 3.1. Suppose the contrary. Let M be the module afforded by θ and V the natural module for H which we also view as a G-module. In view of Theorem 2.1 and Lemma 2.3 we can assume that Sylow p-subgroups of H are not cyclic. Hence we can also assume that $k > 2$. Then g is reducible as otherwise Sylow p-subgroups of G are cyclic (Lemma 2.12). If H is orthogonal, we can assume that $H = \Omega(k,q)$ as $g \in G'$ in this case. By Lemma 2.7, if g normalizes an r-subgroup of G then g centralizes it. It follows from Lemma 2.8 that h, the projection of g in H, cannot belong to a parabolic subgroup of $GL(k,q)$, $U(k,q)$, $Sp(k,q)$ or $\Omega(k,q)$, respectively. As the unipotent radical of any parabolic subgroup in each of these groups belongs to H', we conclude that g stabilizes no isotropic (or singular) subspace of V. As g is reducible, $H' \neq SL(k,q)$.

Let V be the natural module for H. We can express $V = V_1 \oplus V_2$ where V_1, V_2 are non-degenerate g-stable subspaces of V and $g|_{V_1} \neq \mathrm{Id}$ is irreducible. Suppose first that $\dim V_1 = 1$. Then V is a unitary space and p divides $q+1$. This case has been settled in Lemma 3.6.

Suppose that $\dim V_1 > 2$. Observe that $I(V_1)$ is not solvable. (Indeed, as g_1 is irreducible, we have to inspect only the case where $I(V_1) \cong U(3,2)$ which group has no element of order $p > 3$.) Let $X_1 = \{x \in G : x|_{V_1^\perp} = \mathrm{Id}\}$. As $g|V_1$ is irreducible, p does not divide $q \pm 1$. Hence $g \in G'$ so we can assume that $G = G'$. Set $X = \langle X_1, g\rangle$. It is easy to observe that $X = X_1' Z(X_1)$ so there is $y \in X_1$ of order p such that $g = yz$ with $z \in Z(X)$, $z^p = 1$. Let M be the module afforded by θ and N an irreducible submodule of $M|_X$ of dimension greater that 1. Then $\deg(g|_N) \leq p-2$. Obviously, $\deg(g|_N) = \deg(h|_N)$ which contradicts Theorem 2.1 unless $X_1'/Z(X_1') \cong PSL(2,q)$ with q even or $q = 9$. This is only possible when $X_1' = \Omega^-(4,q)$ where q is even and $p = q^2+1$. (The case with $q = 9$ is realized only for $3 \cdot PSL(2,9)$ which cannot occur here.) So V is an orthogonal space. (This settles the case of unitary groups as $\dim V_1$ is odd for them.) Moreover, $\deg h|_N < p-1$ implies that $\mathrm{Spec}(h|_N) = \rho_p^j$. As Sylow p-subgroups of G are not cyclic, $g|_{V_2} \neq 1$ by Lemma 3.7. So there is a non-degenerate g-submodule $V_3 \subseteq V_2$ such that $g|_{V_3}$ is irreducible and non-trivial. Observe that $\dim V_3 = 4$ as $p = q^2+1$. Set $W = V_1 \oplus V_3$ so $\dim W = 8$. Again set $X_W = \{x \in G : x|W^\perp = \mathrm{Id}\}$ and $X = \langle X_W, g\rangle$. Then $g = hz$ where $h \in X_W$ and $z \in Z(X)$. Let now N denote an irreducible constituent of the restriction $M|_X$ and we can assume $\dim N > 1$. Then $\deg(g|_N) = \deg(g_1|_N)$ and $\mathrm{Spec}(g_1|_N) = \rho_p^j$. Therefore, $\mathrm{Spec}(g|_N) = \alpha \cdot \rho_p^j$ for some $\alpha \in \mathbb{C}$ with $\alpha^p = 1$. This contradicts Lemma 3.9 applied to $\Omega(W)$.

Let $\dim V_1 = 2$. Then p divides $q+1$ so $q > 3$. Suppose first that V is symplectic, $k = 4$ and $q > 4$. Let X be the stabilizer of V_1 in G. Then $X = X_1X_2$ (a direct product) where $X_i \cong Sp(2,q)$ for $i = 1,2$. Clearly, $g \in X$ and $g = g_1g_2$ where $1 \neq g_i \in X_i$ and $g_i^p = 1$ for $i = 1,2$. If ϕ is an irreducible constituent of $\theta|_X$ then $\phi = \phi_1 \otimes \phi_2$ where ϕ_i is an irreducible representation of X_i. Therefore, $\phi(g) = \phi(g_1) \otimes \phi(g_2)$. Then $\deg\theta(g) < p-1$ implies $\deg\phi(g) < p-1$ and $\deg\phi_i(g_i) < p-1$ for $i = 1,2$. By Theorem 2.1, this is impossible unless ϕ_i is trivial or q is even and $p = q+1$, and in this case $\mathrm{Spec}\,\phi_i(g_i) = \rho_p^j$ for some j. By Lemma 2.13, $\deg(\phi(g_1) \otimes \phi(g_2)) \geq p-1$ if both ϕ_i are non-trivial and $p > 5$. As $q > 4$, we observe that $p > 5$. So either ϕ_1 or ϕ_2 is trivial. This is true for every irreducible constituent ϕ of θ. As $1 \in \rho_p^j$, it follows that $\mathrm{Spec}\,\theta(g_1) = \rho_p^j$. However, g_1 belongs to a parabolic subgroup of G which is a contradiction as shown above. Let $q = 4$ so $p = 5$. Using the character table of $G = Sp(4,4)$ one can easily show that $\deg\theta(g) \geq 4$. Let $\dim V > 4$. Then g stabilizes a non-degenerate subspace W of V of dimension 4. Let Y be the stabilizer of W in G. Then $Y = Y_1Y_2$ where $Y_1 \cong Sp(4,q)$ and $Y_2 \cong Sp(k-4,q)$. In addition, $g \in Y$ and $g = g_1g_2$ where $g_i \in Y_i$ are of order p. So the argument used for X_i leads to the conclusion that $\deg\theta(g) > p-2$.

Let V be orthogonal. Then $\dim V > 6$ so g stabilizes a non-degenerate subspace W of dimension 6 and $g|_W$ is non-trivial. Again, define Y to be the stabilizer of W in G. Then $g \in Y'$ and $Y' = Y_1Y_2$ where $Y_1 \cong \Omega^{\pm}(6,q)$ and $Y_2 \cong \Omega^{\pm}(k-6,q)$. Express $g = g_1g_2$ for $g_i \in Y_i$. As above, consider an irreducible constituent ϕ of $\theta|_Y$ such that $\phi(g_1)$ is not identity. Then $\phi = \phi_1 \otimes \phi_2$ where ϕ_i is an irreducible representation of Y_i for $i = 1,2$. Therefore, $\phi(g) = \phi(g_1) \otimes \phi(g_2)$. Observe that $Y_1/Z(Y_1) \cong PSL(4,q)$ or $PSU(4,q)$. This cases have been already settled. Hence $\deg\phi_1(g_1) > p-2$ so $\deg\phi(g) > p-2$ and the result follows.

4. Exceptional groups of Lie type

Throughout this section G denotes a simply connected finite quasi-simple group of Lie type. This means that there is a simple symply connected algebraic group $\bar{G}$ and an algebraic group endomorphism $F : \bar{G} \to \bar{G}$ such that $G = \{x \in \bar{G} : F(x) = x\}$. As above, p is distinct from the defining characteristic of G, and we always assume $p > 2$. In this section we assume that G is not classical. We study the minimum polynomial degrees of elements of order $p > 3$ in G and in related groups. The groups ${}^2B_2(q)$ and ${}^2G_2(q)$ are not discussed here as their Sylow p-subgroups are cyclic for $p > 3$. We denote by S a Sylow p-subgroup of G. A subgroup A of G and the meaning of the symbol r_{m_p} are introduced in Lemma 4.3 below. C_k denotes the cyclic subgroup of order k. In this section we prove

Theorem 4.1. *Let G be a quasi-simple group of exceptional Lie type and let $g \in G$ be a semisimple element of order p. Let θ be a non-trivial irreducible representation of G. Then $\deg\theta(g) > p-2$.*

Lemma 4.2. [8, §10.1] *Let $\Phi_m(x)$ denote the cyclotomic polynomial for m-th roots of 1, and $\Pi\Phi_m^{r_m}(x)$ a polynomial associated with G, see* [8, Table on page 111]. *Then S is cyclic if and only if there is exactly one m such that p divides $\Phi_m(q)$ and $r_m = 1$ for this m.*

Observe that $\Phi_1(q) = q-1$, $\Phi_2(q) = q+1$, $\Phi_3(q) = q^2+q+1$, $\Phi_4(q) = q^2+1$, $\Phi_5(q) = q^4+q^3+q^2+q+1$, $\Phi_6(q) = q^2-q+1$, $\Phi_8(q) = q^4+1$, $\Phi_{10}(q) = q^4-q^3+q^2-q+1$, $\Phi_{12}(q) = q^4-q^2+1$, $\Phi_{18}(q) = q^6-q^3+1$, $\Phi_{24}(q) = q^8-q^4+1$.

Lemma 4.3. [8, §10.1] *Let G be a quasi-simple group of Lie type and $\Pi\Phi_m^{r_m}(x)$ the polynomial associated with G, see* [8, p. 111]. *Let $p > 2$ be a prime and S a Sylow p-subgroup of G. Then S contains a homocyclic normal subgroup A of rank r_{m_p} and exponent $e_p = |\Phi_{m_p}(q)|_p$ where $|n|_p$ denotes the p-part of a natural number n.*

Table 1 gives an information about the group A in Lemma 4.3 provided $r_{m_p} > 1$ (equivalently, S is not cyclic, see [8, §10.1]). It is extracted from [8, Table 4.1] and [13]. The first column of Table 1 contains the groups of exceptional Lie type that contains a non-cyclic Sylow p-subgroup for a prime $p > 2$. The algebraic group notation makes no difference between quasi-simple and simple groups but this yields no confusion in our use of Table 1. (The reader can consult [13] for more details.) The second column gives $\Phi_m^{r_m}(q)$ with $r_m > 1$. For the primes p such that $m = m_p$ the third column gives the index of A in S, provided it is greater than 1. The last two columns contain an (incomplete) list of subgroups H of G containing A. In Table 1 the sign $\times$ stands for the direct product. By $X \circ Y$ we denote a central product of X and Y. In general, $X \circ Y$ differs from $X \times Y$, however, the equality may happen for certain values of q. For the groups ${}^2B_2(q)$ and ${}^2F_4(q)$ we assume that $q = 2^{2a+1}$ with $a > 0$, as in [8].

TABLE 1

G	$\Phi_m^{r_m}(q)$ $r_m > 1$, $m = m_p$	$S : A$	H
$G_2(q)$	$\Phi_1^2(q)$	3	$SL_3(q)$
$G_2(q)$	$\Phi_2^2(q)$	3	$SU_3(q)$
${}^3D_4(q)$	$\Phi_1^2(q)$	9	$SL_3(q)$
${}^3D_4(q)$	$\Phi_2^2(q)$	9	$SU_3(q)$
${}^3D_4(q)$	$\Phi_3^2(q)$		$SL_3(q) \times C_{q^2+q+1}$
${}^3D_4(q)$	$\Phi_6^2(q)$		$SU_3(q) \times C_{q^2-q+1}$
$F_4(q)$	$\Phi_1^4(q)$	9	$SL_3(q) \circ SL_3(q)$, $B_4(q)$
$F_4(q)$	$\Phi_2^4(q)$	9	$SU_3(q) \circ SU_3(q)$, $B_4(q)$
$F_4(q)$	$\Phi_3^2(q)$		$SL_3(q) \circ SL_3(q)$
$F_4(q)$	$\Phi_4^2(q)$		$B_4(q)$
$F_4(q)$	$\Phi_6^2(q)$		$SU_3(q) \circ SU_3(q)$
${}^2F_4(q)$	$\Phi_1^2(q)$		$C_2(q)$
${}^2F_4(q)$	$\Phi_2^2(q)$	3	$SU_3(q)$
${}^2F_4(q)$	$\Phi_4^2(q)$		${}^2B_2(q) \times {}^2B_2(q)$
$E_6(q)$	$\Phi_1^6(q)$	81,5	$SL_6(q) \circ SL_2(q)$
$E_6(q)$	$\Phi_2^4(q)$	9	$SL_6(q) \circ SL_2(q)$, $D_5(q)$
$E_6(q)$	$\Phi_3^3(q)$		$SL_3(q) \circ SL_3(q) \circ SL_3(q)$, $F_4(q)$

$E_6(q)$	$\Phi_4^2(q)$		$D_5(q)$
$E_6(q)$	$\Phi_6^2(q)$		$F_4(q)$, $SU_3(q)\circ SU_3(q)$
${}^2E_6(q)$	$\Phi_1^4(q)$	9	$SU_6(q)\circ SL_2(q)$, ${}^2D_5(q)$
${}^2E_6(q)$	$\Phi_2^6(q)$	81,5	$SU_6(q)\circ SL_2(q)$
${}^2E_6(q)$	$\Phi_3^2(q)$		$SL_3(q)\circ SL_3(q)$
${}^2E_6(q)$	$\Phi_4^2(q)$		${}^2D_5(q)$, $F_4(q)$
${}^2E_6(q)$	$\Phi_6^3(q)$		$SU_3(q)\circ SU_3(q)\circ SU_3(q)$
$E_7(q)$	$\Phi_1^7(q)$	81,5,7	$SL_8(q)$
$E_7(q)$	$\Phi_2^7(q)$	81,5,7	$SU_8(q)$
$E_7(q)$	$\Phi_3^3(q)$		$E_6(q)$, $SL_6(q)\circ SL_3(q)$
$E_7(q)$	$\Phi_4^2(q)$		$SL_8(q)$
$E_7(q)$	$\Phi_6^3(q)$		${}^2E_6(q)$, $SU_6(q)\circ SU_3(q)$
$E_8(q)$	$\Phi_1^8(q)$	$3^5, 25, 7$	$SL_5(q)\circ SL_5(q)$, $SL_9(q)$
$E_8(q)$	$\Phi_2^8(q)$	$3^5, 25, 7$	$SU_5(q)\circ SU_5(q)$, $SU_9(q)$
$E_8(q)$	$\Phi_3^4(q)$		$SL_3(q)\circ SL_3(q)\circ SL_3(q)\circ SL_3(q)$
$E_8(q)$	$\Phi_4^4(q)$	5	$SU_5(q^2)$, $D_8(q)$,
$E_8(q)$	$\Phi_5^2(q)$		$SL_5(q)\circ SL_5(q)$
$E_8(q)$	$\Phi_6^4(q)$		$SU_3(q)\circ SU_3(q)\circ SU_3(q)\circ SU_3(q)$
$E_8(q)$	$\Phi_8^2(q)$		$D_8(q)$
$E_8(q)$	$\Phi_{10}^2(q)$		$SU_5(q)\circ SU_5(q)$
$E_8(q)$	$\Phi_{12}^2(q)$		${}^3D_4(q^2)$

Lemma 4.4. *Let G be a group given by the 1-st column of Table 1. Suppose that $p>3$ divides the number in the 2nd column. Then S is contained in one of the subgroups listed in Table 1, except when $p=5$, $G=E_8(q)$ and $m_5=1,2$.*

Proof. Observe that $\Phi_m^{r_m}(q)$ divides $|H|$ except when H is a product of groups with common center. More precisely, if H in Table 1 is a central product $H_1\circ\ldots\circ H_k$ then $\Phi_m^{r_m}(q)$ divides $|H_1|\cdots|H_k|$. Therefore, p^{r_m} divides H unless p divides the order of the common subgroup of $H_1\cdots H_k$. As $p>3$, this happens only when $p=5$, $G=E_8(q)$ and $m_5=1,2$, see Table 1. Therefore, with this exception, $|A|$ divides $|H|$ and the result follows provided $S=A$. Suppose that $S\neq A$. In all other cases in Table 1 where $A\neq S$ one observes that $|S|$ divides $|H|$ and the lemma follows.

Lemma 4.5. *Let r denote Lie rank of G and $p>3$. Then every p-element $g\in G$ of order at most e_p is conjugate to an element in A.*

Proof. Suppose first that $r_{m_p}=r$. Observe that A is a subgroup of a maximal torus T in G, see [8, §10] or [1, Section 3]. Let $\bar{G}$ be an algebraic group associated with G, and let $\bar{T}$ be a maximal torus of $\bar{G}$ containing T. It is well-known that g is conjugate in $\bar{G}$ to an element $\bar{g} \in \bar{T}$. Set $e = e_p$. Let A_0 be the subgroup of elements $a \in A$ with $a^e = 1$. Then $|A_0| = e^r$ (as $r = r_{m_p}$). Therefore, A_0 coincides with the subgroup of elements $t \in \bar{T}$ of exponent e, so $\bar{g} \in A_0 \subseteq A \subseteq T \subset G$. By [17, Part E, Ch.I, 3.4], the elements $g, \bar{g}$ are conjugate in G if $C_G(g)$ is connected. As G is simply connected (which is assumed everywhere in this section unless the opposite is stated), $C_G(g)$ is connected, see [17, Ch. II, 3.9].

Let $r_{m_p} < r$. For classical groups the result stated is well-known. Let G be exceptional. If Sylow p-subgroups of G are cyclic, they are conjugate to A. Otherwise, inspection of Table 1 leaves us with the case where $G = E_8(q)$ with $p = 5$ and $m_5 = 4$ (here $m_5 \neq 1, 2$ as we assume that $m_5 < r = 8$). In this case S is contained in a group isomorphic to $SU(5, q^2)$. For this group the lemma is true, so it is true for G.

Lemma 4.6. *Suppose that $p > 3$ and S is not cyclic.*

(1) *S is contained in a proper semisimple subgroup of G except, possibly, when $G = {}^3D_4(q)$, $m_p = 3, 6$ or $G = E_8(q)$, $p = 5$ and $m_5 = 1, 2$.*

(2) *S is contained in a proper quasi-simple subgroup of G except, possibly, when $G = {}^3D_4(q)$ and $m_p = 3, 6$ or $G = {}^2F_4(q)$, $m_p = 4$ or $G = E_6(q)$, $m_p = 1$ or $G = {}^2E_6(q)$, $m_p = 2$ or $G = E_8(q)$, $m_p = 5, 6, 10$ or $p = 5$ for $m_5 = 1, 2$.*

(3) *S is contained in a central product of classical groups except, possibly, when $G = {}^3D_4(q)$, $m_p = 3, 6$ or $G = {}^2F_4(q)$, $m_p = 4$ or $G = E_8(q)$, $m_p = 12$ or $m_p = 1, 2$ for $p = 5$. In addition, at least one of the factors is not of type $A_1(q)$.*

(4) *Let $g \in S$ be of order p. Suppose that g is not contained in a parabolic subgroup of G. Then there is a quasi-simple classical subgroup X not of type $A_1(q)$ and $x \in X$ of order p such that $x^{-1}g \in C_G(X)$ except when $G = {}^2F_4(q)$ and $m_p = 4$. In the exceptional case X can be chosen of type ${}^2B_2(q)$.*

Proof. Item (1) - (3) follows from Lemma 4.4 by straightforward inspection of Table 1. Consider (4). Observe that g centralizes no unipotent element (hence no non-trivial semisimple subgroup (see Lemma 2.9)). If we are not in an exceptional case of item (3) then g is contained in a subgroup $H = XY$ where X is a quasi-simple classical subgroup not of type $A_1(q)$ and Y is a product of classical groups. If G is of type ${}^3D_4(q)$ then $H = XY$ where X is of type $SL(3, q)$ or $SU(3, q)$ and Y is abelian. Using this, one observes that if G is of type $E_8(q)$ and $m_p = 12$ then $g \in H \cong {}^3D_4(q^2)$ hence $g \in XY$ where X is of type $SL(3, q^2)$ or $SU(3, q^2)$ and Y is abelian. If $G = {}^2F_4(q)$ and $m_p = 4$ then $H = XY$ where X and Y are of type ${}^2B_2(q)$. Observe that X does not centralizes g and $(|Z(X)|, p) = 1$ (see Table 1; observe that if $p = 5$ then $m_p \leq 4$). Therefore, $g = g_1g_2$ where $g_1 \in X$ and $g_2 \in Y$ are of order p. Then $g_1^{-1}g$ centralizes X as required. If $p = 5$, $G = E_8(q)$ and $m_5 = 1, 2$ then A is contained in a subgroup H isomorphic to $SL(9, q)$ or $SU(9, q)$. By Lemma 4.5 g is conjugate to an element from A.

Lemma 4.7. *Theorem* 4.1 *is true if G is a central extension of $F_4(2)$ or ${}^2E_6(2)$.*

Proof. If $G/Z(G) \cong F_4(2)$ then one can inspect the character table of G in [2]. Let G be a central extention of ${}^2E_6(2)$ and $N = G/Z(G)$. One can assume that G is universal. Then $|Z| = 12$. Let Z_0 be the subgroup of order 4 in Z. Then $X = G/Z_0$ is isomorphic to the universal Chevalley group ${}^2E_6(2)$. Let $\bar{g}$ be the projection of g into N. As $q = 2$ here and $p > 3$, the cases with $m_p = 1, 2, 6$ can be ignored. By Table 1, $\bar{g}$ is contained in a group H isomorphic to $F_4(2)$ if $m_p = 3$, and to ${}^2D_5(2)$ if $m_p = 4$. In the former case the lemma follows from the above. Group ${}^2D_5(2)$ has trivial Schur multiplier. Therefore, if $m_p = 4$ then the lemma follows from the result for ${}^2D_5(2) \cong \Omega^-(10, 2)$.

Proof of Theorem 4.1. Let S be a Sylow p-subgroup of G. If S is cyclic, the result follows from Theorem 2.1. Suppose that S is not cyclic. In view of Lemma 4.7 we can ignore exceptional central extensions of $G/Z(G)$. By Lemma 4.6, there is a quasi-simple subgroup X of G which is either of type ${}^2B_2(q)$ or of classical in the same characteristic and of rank greater than 1 such that g can be expressed as $g = g_1g_2$ where $g_1 \in X$ and $[g_2, X] = 1$. Set $Y = \langle X, g\rangle$ (so $g_2 \in Z(Y)$). Let ϕ be a non-trivial irreducible constituent of $\theta|_Y$. Suppose the contrary, that $\deg\theta(g) < p-2$. Then $\deg\phi(g) < p-2$ and $\deg\phi(g_1) < p-2$. This contradicts Theorem 3.1 if X is classical. If $X = {}^2B_2(q)$ then Sylow p-subgroups of X are cyclic so the result follows from Theorem 2.1.

5. Almost simple groups

A group H is called *almost simple* if H' is quasi-simple. (Commonly, one requires $Z(H) = 1$ but we prefer to drop this condition.) In this section we consider groups $G = \langle g^G\rangle$ where $g \in G$ is of prime order $p > 3$ and g induces an outer automorphism on G' where G' is quasi-simple.

Theorem 5.1. *Let $G = \langle g, G'\rangle = \langle g^G\rangle$ where G' is a quasi-simple normal subgroup of G and $g^p \in Z(G)$. Let $\theta \in \operatorname{Irr} G$ and $\dim\theta > 1$. Suppose that $\deg\theta(g) < p-1$. Then $G = G' \cdot Z(G)$ where G' is a quasi-simple group from Theorem* 1.2.

The case where $G = G'$ has been settled in the previous sections, and the case with $G = G' \cdot Z(G)$ follows from Lemma 2.3. Therefore, we assume that $g \notin G' \cdot Z(G)$. None of alternating or sporadic simple groups has an outer automorphism of odd order. Therefore, $G'/Z(G')$ is a simple group of Lie type. Groups of Lie type with exceptional center are listed in [8, page 72]. One observes that none of them has an outer automorphism of order greater than 3. Therefore, we only need to deal with groups G such that $Z(G')$ is not exceptional, that is, G' is isomorphic to a finite algebraic groups. Moreover, we can assume that G is simply connected.

Lemma 5.2. [8, 7.3] *Let G be a quasi-simple finite group of Lie type defined over a field of characteristic r and let α be an outer automorphism of G of prime order $p > 3$. Then one of the following holds:*

(1) *α is inner-diagonal;*

(2) *α is conjugate in* $\mathrm{Aut}\, G$ *to a field automorphism of G (in the sense of Steinberg* [18]*).*

Lemma 5.3. *Let G be a quasi-simple finite group of Lie type defined over a field of characteristic r and let α be an outer automorphism of G of prime order p. Suppose that any α-stable r-subgroup of G is contained in $C_G(\alpha)$. Then α is the product of an inner and a diagonal automorphisms of G, and α normalizes no non-trivial r-subgroup of G. If $p > 3$ then G is a central quotient of $SL_n^\varepsilon(q)$ and α is induced by an inner automorphism of $GL_n^\varepsilon(q)$. In addition, p divides n and $q-\varepsilon$.*

Proof. The first claim is a particular case of [6, Lemma 2.1]. The second one follows immediately from well-known facts on diagonal automorphisms. (As α is not inner, p divides the order of $Z(\overline{H})$ where $\overline{H}$ is simply connected algebraic group associated with H. It follows that H is of type $A_{n-1}^\varepsilon(q)$ and p divides n. As $G \neq G' \cdot Z(G)$, then p divides $q-\varepsilon$.)

Proof of Theorem 5.1. Let $H = G'$. By the observation in the beginning of this section we can assume that H can be embedded into the related algebraic group $\tilde{H}$. By Lemma 5.3 $G = G_1 \cdot Z(G)$ where G_1 is contained in $\tilde{H}$ (every diagonal automorphism of H is known to be induced by an inner automorphism of $\tilde{H}$.) We can further assume that $\tilde{H}$ is simply connected. Let r denote the defining characteristic of H. Let α be the automorphism induced on H by g-conjugation so α is not inner. By Lemma 2.7, each α-stable abelian r-subgroup A of H belongs to $C_H(\alpha)$. By Lemma 5.3, α is a product of inner and a diagonal automorphisms of H and H is of type $A_{n-1}^\varepsilon(q)$ and $n > 2$ as p divides $(n, q-\varepsilon)$. Therefore, we can assume that $SL^\varepsilon(n,q) \subseteq G_1 \subseteq GL^\varepsilon(n,q)$ so the result follows from Theorem 3.1. This completes the proof.

6. General case

In this section we prove the following:

Proposition 6.1. *Theorem* 1.2 *is true if G is not almost simple.*

Lemma 6.2. *Group G in Theorem* 1.2 *is primitive.*

Proof. Suppose the contrary. Then G transitively permutes subspaces of a direct decomposition $V = V_1 \oplus \cdots \oplus V_k$. As $G = \langle g \rangle^G$, there is a g-orbit of size p on $\{1, \ldots, k\}$. Let $v \in V_i$ and $gV_i \neq V_i$. Then $g^i v$ are linear independent for $i = 1, \ldots, p$, so the minimum polynomial of g is of degree p.

Let $S(G)$ denote the maximal solvable normal subgroup of G. The following two lemmas state standard facts of the Clifford theory. In Lemmas 6.3, 6.4, 6.7 and Theorem 6.6 below F denotes an algebraically closed field of any characteristic and F_q the field of q elements.

Lemma 6.3. *Let $G \subset GL(n, F)$ be an irreducible primitive tensor-indecomposable subgroup. Then one of the following holds:*

(1) *G has an irreducible normal subgroup N that is a central product of non-abelian quasi-simple groups transitively permuted by G when it acts on N via conjugation;*

(2) *there is a prime r such that $E = O_r(G)$ is irreducible. In addition, $EZ(G)/Z(G)$ is elementary abelian of order n^2.*

Lemma 6.4. *Let $G \subset GL(n, F)$ be an irreducible primitive tensor-indecomposable subgroup. Suppose that $S(G) \neq Z(G)$, equivalently, case* (2) *of Lemma* 6.3 *holds. Set $E = O_r(G)$ and $W =: EZ(G)/Z(G)$.*

(1) *W is elementary abelian.*

(2) *View W as a vector space over F_r. Then the commutator mapping $E \times E \to Z(E)$ induces on W a non-degenerate alternating bilinear form and the conjugation action of G on $E(G)$ induces a homomorphism $\pi : G \to Sp(W) \cong Sp(2k, r)$ which kernel is E and the image is an irreducible subgroup of $Sp(2k, r)$.*

Lemma 6.5. [22] *Assume the hypothesis of Lemma* 6.4 *and suppose that $G = \langle g^G \rangle$ where $g^r \in Z(G)$. In addition, suppose that* $\deg g < p$. *Then* $\deg g = (p+1)/2$, *$\pi(g)$ is a transvection in $Sp(2k, r)$ and every abelian g-invariant subgroup of E belongs to $C_G(g)$. Conversely, if $\pi(g)$ is a transvection and every abelian g-invariant subgroup of E belongs to $C_G(g)$ then* $\deg g = (p+1)/2$. *In addition,* $\operatorname{Spec} g = \alpha \cdot \bar{\mu}_r$ *or* $\alpha \cdot \bar{\nu}_r$ *for some $\alpha \in F$.*

Theorem 6.6. *Let $G \subset GL(n, F)$ be a primitive irreducible tensor-indecomposable subgroup such that $S(G) \neq Z(G)$. Let $g \in G$ be such that $G = \langle g^G \rangle$ and $g^p \in Z(G)$. Suppose that* $\deg g < p - 1$. *Then n is a p-power, $G/(O_p(G)Z(G)) \cong Sp(2k, p)$ and the projection of g into $Sp(2k, p)$ is a transvection. In addition,* $\operatorname{Spec} g = \alpha\bar{\mu}_p$ *or* $\alpha\bar{\nu}_p$ *for some $\alpha \in F$.*

Proof. It follows from Lemma 6.3 that $n = r^k$ for some prime r and an integer $k > 0$, and G has a normal irreducible r-subgroup E. Set $N = Z(G)E$. By Lemma 2.7, if $r \neq p$ then $\deg g \geq p - 1$. So $r = p$. Then G/N is isomorphic to an irreducible subgroup of $Sp(2k, p)$ by Lemma 6.4. Let $\pi : G \to Sp(2k, p)$ be the quotient group homomorphism and $b = \pi(g)$. By Lemma 6.5, $\deg g = p$ unless $\pi(g)$ is a transvection in $Sp(2k, p)$. By a theorem of Mclaughlin [14], $\pi(G) = Sp(2k, p)$. The claim on $\operatorname{Spec} g$ follows from Lemma 6.5.

Lemma 6.7. *Let $G \subset GL(n, F)$ be a irreducible primitive tensor-indecomposable subgroup. Suppose that $S(G) = Z(G)$. Suppose that $G = \langle g^G \rangle$ where $g^p \in Z(G)$ and $g \notin G' \cdot Z(G)$. Then* $\deg g = p$.

Proof. By Lemma 6.3, $N = N_1 \circ \cdots \circ N_k$ where $N_1, \ldots, N_k$ are quasi-simple. We can assume that $g^i N_1 g^{-i} = N_{i+1}$ for $i = 1, \ldots, p-1$. As N_i is simple, its order is divisible by at least three primes. Hence there is a prime r such that r is distinct from the characteristic of F, $r \neq p$ and N_1 contains a subgroup C_1 of order r. Set $A = \pi_{i=0}^{p-1} g^i C_1 g^{-i}$ so A is a g-invariant abelian subgroup of N and A is not contained in $C_G(g)$. By Lemma 2.7, $\deg g = p$.

Lemma 6.8. *Theorem* 1.2 *is true if G is primitive and tensor-indecomposable.*

Proof. If $S(G) = Z(G)$ then either G is almost simple or the minimal non-central normal subgroup of G is a central product of at least p quasi-simple non-abelian groups. The latter case is considered in Lemma 6.7. The former case is done in Theorem 5.1.

This lemma shows that we are left with tensor decomposable groups. Let G be tensor-decomposable. Then $G \subseteq GL(n_1, \mathbb{C}) \otimes GL(n_2, \mathbb{C})$ where $n_1, n_2 > 1$. This yields projective representations $\phi_i : G \to GL(n_i, \mathbb{C})$ and then $\phi(G) \subseteq G_1 \otimes G_2$ where $G_i = Z\phi_i(G)$ for $i = 1, 2$ and Z_i the group of scalar matrices in $GL(n_i, \mathbb{C})$. Therefore, for each $g \in G$ there are $g_i \in G_i$ $(i = 1, 2)$ such that $g = g_1 \otimes g_2$. If $g^p \in Z(G)$ then $g_i^p \in Z_i$. If $\deg g < p - 1$ then $\deg g_i < p - 1$ for $i = 1, 2$. Moreover, $\deg g_i < p - 1$. Therefore, we can use induction.

Lemma 6.9. *Let $G = \langle g^G \rangle \subset GL(n, \mathbb{C})$ be tensor-decomposable. Suppose that $\deg g < p - 1$. Then $G/Z \cong A_5 \times A_5$, $p = 5$ and $n = 4$.*

Proof. According to the above comments $G \subset G_1 \otimes G_2$ where $G_i \subset GL(n_i, \mathbb{C})$ with $n_1 n_2 = n$ and $n_i > 1$. Clearly, G_i are irreducible. Suppose first that G_1 and G_2 are tensor-indecomposable. Express $g = g_1 g_2 \in G$ where $g_i \in G_i$ for $i = 1, 2$ and $g^p \in Z(G)$. Therefore, $\operatorname{Spec} g = \alpha(\operatorname{Spec} g_1)(\operatorname{Spec} g_2)$ where $\alpha \in \mathbb{C}$. So $\deg g_i < p - 1$. Observe that g_2 is not scalar as $G = \langle g^G \rangle$. By Lemma 6.8, for $g_i \in G_i$ one of the cases in Theorem 1.2 holds. In particular, we can assume that $\operatorname{Spec} g_i \in \{\mu_p, \nu_p, \bar{\mu}_p, \bar{\nu}_p, \rho_p^j\}$. Obviously, $\operatorname{Spec} g_i$ cannot be ρ_p^j. Furthermore, by Lemma 2.13 $\deg g > p - 2$ unless $p = 5$ and $\operatorname{Spec} g_1 = \operatorname{Spec} g_2 = \mu_5$ or ν_5. In particular, $\deg g_1 = \deg g_2 = 2$. Therefore, for both $g_i \in G_i$ case (A) of Theorem 1.2 holds so $G_i/Z(G_i) \cong PSL(2, 5) \cong A_5$. As the universal central extension of A_5 is isomorphic to $SL(2, 5)$, we can assume that $G_1 \cong G_2 \cong SL(2, 5)$. Moreover, $n_i = 2$ as $\deg g_i = 2$ and only 2-dimensional representations of $SL(2, 5)$ satisfy this property.

Proof of Theorem 1.2. Suppose the contrary. By Lemma 6.2 G is primitive. If G is tensor-decomposable, the result follows from Lemma 6.9 and leads to case (B6) of the theorem. Suppose that G is tensor-indecomposable. If $S(G) \neq Z(G)$ then the result follows from Theorem 6.6 and leads to case (B2) of the theorem. Suppose that $S(G) = Z(G)$. By Lemma 6.3 G is almost simple. Then $G = G' \cdot Z(G)$ by Theorem 5.1. In this case with no loss of generality we can assume that $G = G'$, that is, G is quasi-simple. This case splits to four subcases depending on whether $G/Z(G)$ is sporadic, alternating, classical or of exceptional Lie type. Sporadic groups yield only one case where $G/Z(G) \cong J_2$ by Lemma 2.14. The case of alternating groups has been settled in [27], see Theorem 2.2. This class of groups yields only examples (B3) and (B4) in the statement (as well as $2 \cdot A_5 \cong SL(2,5)$ and $A_5 \cong SL(2,4)$ which are viewed here as classical groups.) For $G/Z(G)$ being of Lie type g is either semisimple or unipotent. The latter case has been known much earlier, see Theorem 2.4. This yields cases (A), (B1) of the theorem. Let g be semisimple. If Sylow p-subgroup of G are cyclic, the result is available from [26] as is recorded in Theorem 2.1. From this we obtain case (C) of the theorem. It remains to treat the case where Sylow p-subgroups of G are not cyclic. Non-standard coverings for groups of Lie type are considered in Lemma 2.15 and Lemma 4.7. The case where p divides $|Z(G)|$ is considered in Lemma 3.6. The case where $(|Z(G)|, p) = 1$ and G is classical is contained in Theorem 3.1. Groups of exceptional Lie type are examined in Theorem 4.1.

References

[1] M. Broué and G. Malle, Théorèmes de Sylow génériques pour les groupes réductifs sur les corps finis, *Math. Annalen* 292(1992), 241 - 262.

[2] J. Conway, R. Curtis, S. Norton, R. Parker and R. Wilson, *Atlas of finite groups*, Clarendon Press, Oxford, 1985.

[3] D.I. Deriziotis, *Conjugacy classes and centralizers of semisimple elements in finite groups of Lie type*, Vorlesungen aus dem Fachbereich Mathematik, University of Essen, 1984, 148pp.

[4] D.I. Deriziotis and G. Michler, Character table and blocks for finite simple triality groups $^3D_4(q)$, *Trans. Amer. Math. Soc.* 303(1987), 39 - 70.

[5] W. Feit. *The representation theory of finite groups*, North-Holland, Amsterdam, 1982.

[6] P. Fleischmann, W. Lempken and A. Zalesski, Linear goups over $GF(2^k)$ generated by a conjugacy class of a fixed point free element of order 3, *J. Algebra* 244(2001), 631 - 663.

[7] P. Gerardin, Weil representations associated to finite fields, *J. Algebra* 46(1977), 54 - 101.

[8] D. Gorenstein and R. Lyons. *The local structure of finite groups of characteristic 2 type*, Memoirs of the Amer. Math. Soc. 42(1983), no. 276, 731 pp.

[9] R. Guralnick, K. Maagard, J. Saxl and Pham Huu Tiep, Cross characteristic representations of odd characteristic symplectic groups and unitary groups, *J. Algebra* 257(2002), 291–347.

[10] Ph. Hall and G. Higman, On the p-length of p-soluble groups and reduction theorem for Burnside's problem, *Proc. London Math. Soc.* 6(1956), 1-42.

[11] B. Huppert, Singer-Zyklen in klassischen Gruppen, *Math. Z.* 117(1970), 141-150.

[12] P. Kleidman and M. Liebeck, *Subgroup of classical groups.* Cambridge Univ. Press, Cambridge, 1990. (London Math. Soc. Lecture notes no.129.)

[13] M. Liebeck, J. Saxl and G. Seitz, Subgroups of maximal rank in finite exceptional groups of Lie type, *Proc. London Math. Soc.* (3) 65(1992), 297-325.

[14] J. McLaughlin, Some groups generated by transvections, *Archiv Math.* 18(1967), 364 - 368.

[15] G.R. Robinson, Remarks on reduction $(\bmod p)$ of finite complex linear groups, *J. Algebra* 83(1983), 471 - 476.

[16] G.R. Robinson, On eigenvalues of elements of finite linear groups, *J. Algebra* 165(1994), 113-121.

[17] T. Springer and R. Steinberg, Conjugacy classes, In: *Seminar on algebraic groups and related finite groups*, Ed. A. Borel, Springer-Verlag, Berlin, 1970.
[18] R. Steinberg, *Lectures on Chevalley groups*, Yale Univ., 1967.
[19] Pham Huu Tiep and A.E. Zalesski, Minimal characters of the finite classical groups, *Comm. Alg.* 24(1996), 2093 - 2167.
[20] Pham Huu Tiep and A.E. Zalesski, Some characterizations of the Weil representations of the symplectic and unitary groups, J. Algebra 192(1997), 130-165.
[21] A. Weir, Sylow p-subgroups of the classical groups over finite fields with characteristic prime to p, *Proc. Amer. Math. Soc.* 6(1955), 529 - 533.
[22] A.E. Zalesski, The normalizer of the extraspecial linear group (in Russian). *Vesti AN BSSR, ser. fiz.-mat. navuk* 1985, no.6, 11 - 16.
[23] A. Zalesski, Spectra of elements of order p in representations of Chevalley groups of characteristic p (in Russian). *Vesti AN BSSR, ser. fiz.-mat. navuk* 1986, no.6, 20 - 25.
[24] A.E. Zalesski, Eigenvalues of matrices of complex representations of finite groups of Lie type. In: "Algebra: Some current trends", Proc. conf. in Algebra (Varna, 1986). Lecture notes in Math. vol. 1352, Springer-Verlag, 1988, 206-218.
[25] A.E. Zalesski, The eigenvalue 1 of matrices of complex representations of finite Chevalley groups, *Proc. Steklov Inst. of Math.* 1991, issue 4, 109-119.
[26] A.E. Zalesski, Minimal polynomials and eigenvalues of p-elements in representations of groups with a cyclic Sylow p-subgroup, *J. London Math. Soc.* (2) 59(1999), 845 - 866.
[27] A.E. Zalesski, The eigenvalues of matrices of projective complex representations of alternating groups (In Russian), *Vesti Akad. Navuk Belarusi*, Ser. fiz., mat., inform., 1996, no.3, 41 - 43.
[28] J. Zhang, Complex linear groups of degree at most $p-1$, *Contemp. Math.* 82(1989), 243-254.

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