∗ † ‡

Some Aspects of Finite Linear Groups: A Survey

Pham Huu Tiep and A. E. Zalesskii

Dedicated to Prof. A. I. Kostrikin on the occasion of his 70th birthday

Abstract

Finite linear groups have been studied since the beginning of the century. The goal of this survey is to reflect recent results in this area, obtained mostly in the 90's and using the classification of finite simple groups.

Contents.

1 Introduction

After a period of very intensive development in the 60's and 70's, the classification problem for finite linear groups attracted less attention in the ensuing

[*]Part of this work was done while the authors were participating in the special semester on Representations of Algebraic Groups and Related Finite Groups at the Isaac Newton Institute (Cambridge, 1997). It is a pleasure to thank the organizers of the semester M. Broué, R. W. Carter, and J. Saxl for the opportunity to participate, and the staff of the Newton Institute for its hospitality.

[†]The authors are grateful to M. Collins and R. Solomon for their comments on the survey.

[‡]The second author has been supported in part by INTAS

years. One of the reasons for this is certainly the announcement of the complete classification of finite simple groups (CFSG). Using the CFSG and the representation theory of finite groups, one can obtain much stronger results using easier machinery. Although not everyone recognizes/accepts the CFSG theorem, since an essential part of the proof of this fundamental theorem has not been published yet or is still being revised, there is a huge number of publications and interesting results that use the classification.

This survey aims to report on relatively recent results, obtained *using* the classification. (One may also consider the results discussed below as those on representations of groups whose simple sections are known.) We do not try to be complete, but rather to reflect certain aspects of finite linear group theory.

Classification problems for finite linear groups have been studied since the beginning of the century. Of permanent attention were the classifications of

(A) groups of small degrees; and of

(B) groups generated by a conjugacy class of particular elements (reflections, pseudoreflections etc.).

Classical results of Dickson, Mitchell, Blichfeldt, Brauer, Coxeter, Shepard and Todd were partially reflected in the surveys [F1] and [Z1]. Enormous contributions to the topic have been made by Feit and his successors. Feit's survey [F1] is the main source of information about the results of that period. See also [Le] and [Z1].

Continuing in the vein of (A), in Sections 2, 6, 7, and 8 we report on the results towards a classification of linear groups of relatively small degree, both in zero and in prime characteristic. These results rely upon the CFSG, and upon new developments in representation theory of quasi-simple groups. Among these are refinements of the Landázuri-Seitz lower bounds, cf. Sections 3, 4, and a classification of low-dimensional representations of finite groups of Lie type in cross characteristic, cf. Section 5. Sections 9 and 10 represent new results somewhat along the line of (B): classifications of linear groups containing elements with specific minimal polynomial, respectively containing elements with simple spectrum. Finally, Sections 11, 12, and 13 give insight on some closely related activities. In particular, we consider the following questions for quasi-simple groups: when the tensor product of two representations can be irreducible, and which complex representations remain irreducible modulo every prime.

2 Linear Groups of Small Degree

Linear groups of degree < 4 were classified at the beginning of the century, both for the zero and the prime characteristic cases by using a kind of geometrical approach. New ideas, based on the representation theory elaborated by Brauer, Feit, and their successors, made it possible to classify linear groups of degree up to 11 for the zero characteristic case. For the prime characteristic case this was done up to degree 5 by using certain early results of the CFSG. All these results did not use the full classification theorem of finite simple groups by the obvious reason.

The CFSG makes the problem of describing linear groups of small degree much more accessible. Consider first the case of zero characteristic representations. Given an integer n, the list of quasi-simple groups of degree n (which is the core of the problem) may be obtained from degree formulae for irreducible projective representations of simple groups. Such formulae are well known for symmetric and alternating groups since 1901 thanks to Schur [Sch]. For groups of Lie type, these formulae are provided by Lusztig's classification of complex characters ([Lus 2] and [Ca]). A deep result of the Deligne-Lusztig character theory of finite groups G of Lie type, defined over a field of characteristic p, states that each character degree is the product $(G^* : C_{G^*}(s))_{p'} \cdot \psi(1)$ of the p'-part of the length of the conjugacy class of a semisimple element s in the dual group G^* and the degree of a unipotent character ψ of $C_{G^*}(s)$. The first factor in this product is relatively easy to compute, while formulae for the second factor are more complicated. Nevertheless, these are explicit formulae. Character degrees of sporadic simple groups can be read off from [Atlas].

For the prime characteristic case the degree formulae for irreducible projective representations of simple groups are in general not available yet. However, relatively small degrees can be obtained from the study of low-dimensional representations which has been pursued intensively during recent years (see Section 5 below). For sporadic groups the degrees can be taken from the Atlas of Brauer characters [JLPW] published in 1995. However, this atlas does not cover all sporadic groups, although it is constantly updated using new powerful computers and software like GAP, MAPLE, MAGMA, CHEVIE. (Some results are obtained by hand.)

For finite groups of Lie type, the problem splits into the cases of the *defining* and of *cross* characteristic, according to whether the characteristic of the field of the representation in question coincides with or differs from the defining characteristic of the group of Lie type.

The defining characteristic case is very interesting and important. The key fact is that Steinberg's tensor product theorem reduces the problem in question to a similar problem for representations of algebraic groups. Degree formulae for finite groups of Lie type of rank 1, 2, and also of type A_3 are known thanks to Braden, Hagelskjær, Jantzen, and Springer, cf. [An]. In general, Lusztig's conjecture [Lus 1], in combination with Jantzen's translation principle [Ja 2], provides degree formulae for irreducible representations of simple groups of Lie type in the defining characteristic (if p is larger than the Coxeter number of G). Recently, this conjecture has been proved for p big enough (how big would be enough, is not known, however). This major achievement combines the beautiful results of Kazhdan-Lusztig [KaL], [Lus 3] (showing the equivalence of Lusztig's conjecture in the quantum case and in the Kac-Moody case), of

Casian [Cas] and Kashiwara-Tanisaki [KT] (proving Lusztig's conjecture in the Kac-Moody case), and of Andersen-Jantzen-Soergel [AJS] (establishing that the validity of Lusztig's conjecture in the quantum case implies the validity of Lusztig's conjecture for the prime characteristic case for $p \gg 0$). There are also some (partial) results about the dimensions of irreducible representations of $GL_n(p)$ (in the defining characteristic) due to Mathieu-Papadopoulo [MP], Brundan-Kleshchev-Suprunenko [BKS] and Tanisaki [Ta].

The cross characteristic case is in general not easy as well, and the results obtained so far in this case are far from being complete. But there are some methods of describing representations of relatively small degree, which will be discussed below.

Modulo the CFSG and new results in representation theory, quasi-simple linear groups of degree up to 27 have been classified; this was done by Kleidman [Kl] for degree < 13, and by Kondrat'ev [Ko 1], [Ko 2] for degree < 28.

3 Representations of smallest degree in zero characteristic

For finite classical groups, the (nontrivial) irreducible projective complex representations of smallest degree are classified in [TZ 1]. In the cases of $SL_n(q)$, $SU_n(q)$ and $Sp_{2n}(q)$ with q odd, the three lowest degrees are determined in [TZ 1]. The same problem for exceptional groups has been recently solved by Lübeck [Lu] using computer and CHEVIE. Rasala treated a similar problem for the ordinary representations of the symmetric groups in [Ra].

Let $G(q)$ be a finite simple group of Lie type defined over a field of order q, where q is a power of the prime p, and let $\ell(G(q))$ (resp. $d(G(q))$) denote *the smallest integer $t > 1$ such that $G(q)$ has a projective irreducible representation of degree t over a field of characteristic other than p* (resp. *over the field* $\mathbf{C}$ *of complex numbers*). Let $N(G(q))$ be the number of *minimal characters*, that is, (faithful) projective complex characters of $G(q)$ of degree $d(G(q))$. The main result of [TZ 1] (resp. of [Lu]) is Table I (resp. Table II) which list $d(G(q))$ and $N(G(q))$ for finite classical groups (resp. for finite exceptional groups).

TABLE I. $d(G(q))$ and $N(G(q))$ for finite classical groups

$G(q)$	$d(G(q))$	$N(G(q))$	Conditions
$PSL_2(q)$	$(q-1)/2$	2	$2 \nmid q,\ q \neq 9$
$PSL_2(q)$	$q-1$	$q/2$	$2 \mid q,\ q \neq 4$
$PSL_2(4)$	2	2	
$PSL_2(9)$	3	4	
$PSL_n(q)$	$(q^n-q)/(q-1)$	1	$\begin{cases} n \geq 3, \\ (n,q) \neq & (3,2),(3,4), \\ & (4,2),(4,3) \end{cases}$
$PSL_3(2)$	3	2	
$PSL_3(4)$	6	6	
$PSL_4(2)$	7	1	
$PSL_4(3)$	26	2	
$PSp_{2n}(q)$	$(q^n-1)/2$	2	$2 \nmid q,\ n \geq 2$
$PSp_{2n}(q)$	$(q^n-1)(q^n-q)/(2(q+1))$	1	$\begin{cases} 2 \mid q,\ n \geq 2, \\ (n,q) \neq (2,2) \end{cases}$
$PSp_4(2)'$	3	4	
$PSU_n(q)$	$(q^n-q)/(q+1)$	1	$2 \nmid n,\ n \geq 3$
$PSU_n(q)$	$(q^n-1)/(q+1)$	q	$\begin{cases} 2 \mid n,\ n \geq 3, \\ (n,q) \neq (4,2),(4,3) \end{cases}$
$PSU_4(2)$	4	2	
$PSU_4(3)$	6	4	
$P\Omega^+_{2n}(q)$	$(q^n-1)(q^{n-1}+q)/(q^2-1)$	1	$q > 3,\ n \geq 4$
$P\Omega^+_{2n}(3)$	$(q^n-1)(q^{n-1}-1)/(q^2-1)$	2	$q = 3,\ n > 4$
$P\Omega^+_8(3)$	$(q^n-1)(q^{n-1}-1)/(q^2-1)$	6	$q = 3,\ n = 4$
$\Omega^+_{2n}(2)$	$(q^n-1)(q^{n-1}-1)/(q^2-1)$	1	$q = 2,\ n > 4$
$\Omega^+_8(2)$	8	3	
$P\Omega^-_{2n}(q)$	$(q^n+1)(q^{n-1}-q)/(q^2-1)$	1	$n \geq 4$
$\Omega_{2n+1}(q)$	$(q^{2n}-1)/(q^2-1)$	1	$q > 3,\ 2 \nmid q,\ n \geq 3$
$\Omega_{2n+1}(3)$	$(q^n-1)(q^n-q)/(q^2-1)$	1	$q = 3,\ n > 3$
$\Omega_7(3)$	27	2	

In the cross characteristic case it is much more difficult to classify representations of $G(q)$ of, say, the smallest degree. To illustrate the difficulty of this case, let us consider the nontrivial irreducible representations of $Sp_{2n}(q)$, q an odd prime and $n > 1$, over an algebraically closed field of characteristic $r \nmid q$. It is well known that the degree of such a representation Φ is at least $(q^n - 1)/2$, and, moreover, one has at least two such representations of degree $(q^n - 1)/2$, which are obtained by reducing the complex Weil representations of the same degree modulo r. Is there any other representation of this degree? The answer is presumably no, but only very recently this negative answer has been confirmed

TABLE II. $d(G(q))$ and $N(G(q))$ for finite exceptional groups

$G(q)$	$d(G(q))$	$N(G(q))$	Conditions
$E_6(q)$	$q(q^6+q^3+1)(q^4+1)$	1	
$E_7(q)$	$q(q^{14}-1)(q^4-q^2+1)/(q^2-1)$	1	
$E_8(q)$	$q(q^{12}+1)(q^{10}+1)(q^6+1)$	1	
$F_4(q)$	q^8+q^4+1	1	$q \equiv 1(\text{mod}2)$
$F_4(q)$	$q(q^4+1)(q^3-1)^2/2$	1	$q \equiv 0(\text{mod}2)$, $q > 2$
$F_4(2)$	52	1	
${}^2E_6(q)$	$q(q^6-q^3+1)(q^4+1)$	1	
$G_2(q)$	q^3+1	1	$q \equiv 1(\text{mod}3)$, $q > 4$
$G_2(q)$	q^3-1	1	$q \equiv 2(\text{mod}3)$, $q > 2$
$G_2(q)$	q^4+q^2+1	1	$q \equiv 0(\text{mod}3)$, $q > 3$
$G_2(3)$	14	1	
$G_2(4)$	12	1	
${}^3D_4(q)$	$q(q^4-q^2+1)$	1	
${}^2F_4(q)$	$(q^3+1)(q^2-1)\sqrt{q/2}$	2	$q = 2^{2a+1} > 2$
${}^2F_4(2)'$	26	2	
${}^2B_2(q)$	$(q-1)\sqrt{q/2}$	2	$q = 2^{2a+1} > 2$
${}^2G_2(q)$	q^2-q+1	1	$q = 3^{2a+1} > 3$

by Guralnick, Magaard and Saxl in [GMS]. One of the main tools to deal with the modular case is provided by the Landázuri-Seitz lower bounds, see Section 4.

4 The Landázuri-Seitz lower bounds

In the remarkable paper [LS] Landázuri and Seitz found lower bounds for the degrees of non-trivial irreducible projective representations of finite simple groups of Lie type $G(q)$ in cross characteristics. (One should mention that the smallest degree of representations of $G(q)$ in the defining characteristic has been known for a while; usually it is exactly the dimension of the "natural" module for the group in question. Also, similar bounds for the modular projective representations of the alternating groups were obtained by Wagner, cf. [Wag 2], [Wag 3], [Wag 4]). These bounds turn out to be extremely useful in numerous applications. A typical application is as follows. Suppose we want to find all finite groups G with a specific property $(\mathcal{P})$. First one could try to reduce to the case where G is a quasi-simple group. Using the CFSG and assuming that we can deal with the alternating and the sporadic groups separately, we are down to the case where G is a group of Lie type defined over a field $\mathbf{F}_q$. Next, suppose that we are lucky enough to deduce from $(\mathcal{P})$ that G has a nontrival projective representation of certain (known, or known to be bounded) degree d in some characteristic r. It is clear that $\ell(G(q)) \leq d$ in the case where $r \not| q$. Hence the Landázuri-Seitz lower bounds imply that both the rank of G and the size of the ground field $\mathbf{F}_q$ are bounded, and therefore there remains only a finite number of possible candidates for G.

It is clear that $\ell(G(q)) \leq d(G(q))$. Comparing the Landázuri-Seitz bound (as given in [LS]) with $d(G(q))$ given in Tables I and II, one sees that this bound is best possible in many cases. Still, it could be improved in a number of cases. The bounds have been refined by Seitz and Zalesskii [SZ], and improved recently by Guralnick, Hoffman, Magaard, Pentilla, Praeger, Saxl, and Tiep in [GPPS], [GT 1], [Hof 1], [Hof 2], [MT] again. These improvements yield new lower bounds which we record in Table III.

The Landázuri-Seitz lower bounds provide a very efficient tool for classifying linear groups of small degrees, see [Ko 1], as well as the linear groups of relatively small degrees, see Section 8.

5 Low-dimensional representations of finite groups of Lie type in cross characteristics

In a number of applications, it is important to know all cross characteristic representations of a given finite group of Lie type $G(q)$ of a degree d, which is close in some sense to $\ell(G(q))$. For instance, in [GPPS] one needs to find all representations Φ with $\deg(\Phi) \leq 2\ell(G(q))$. It turns out in this kind of application that the group $G(q)$ in question usually has a representation Φ in cross characteristic whose degree is bounded by some upper bound which is larger than $\ell(G(q))$ but much smaller than $\ell(G(q))^2$. In this situation one would like to identify Φ with one of the "known" representations, for instance Weil representations.

At present, the representation theory of finite groups of Lie type in cross characteristics has been developed to a level which makes it possible to solve the following important problem:

Problem 5.1 *Let $G(q)$ be a finite simple group of Lie type and $\varepsilon > 0$. Classify all irreducible projective representations of $G(q)$ in cross characteristic of degree smaller than $\ell(G(q))^{2-\varepsilon}$.*

One should mention the "experimental fact" that $G(q)$ has only a few irreducible representations of degree less than $\ell(G(q))^{2-\varepsilon}$, but it has many irreducible representations of degree $\approx \ell(G(q))^2$. Therefore, it makes sense to classify representations of degree up to $\ell(G(q))^{2-\varepsilon}$.

Problem 5.1 has been solved completely in the case of $PSL_n(q)$.

Theorem 5.2 [GT 1] *Let $n \geq 3$, q a prime power, r a prime not dividing q and $(n,q) \neq (3,2)$, $(3,4)$, $(4,2)$, $(4,3)$, $(6,2)$, $(6,3)$. Suppose $L = PSL_n(q)$ has a projective absolutely irreducible representation Φ of degree d over a field of characteristic p, where*

$$1 < d < \lceil(L) = \begin{cases} (q-1)(q^2-1)/(3,q-1), & \text{if } n=3 \\ (q-1)(q^3-1)/(2,q-1), & \text{if } n=4 \\ (q^{n-1}-1)\left(\frac{q^{n-2}-q}{q-1} - \kappa_{n-2}\right), & \text{if } n \geq 5 \end{cases}.$$

Then Φ is an irreducible constituent of the r-modular reduction of a complex Weil representation of $SL_n(q)$. More precisely, either Φ is equivalent to a unique representation of degree $(q^n-q)/(q-1) - \kappa_n$, or it is equivalent to one of $(q-1)_{r'} - 1$ (inequivalent) representations of degree $(q^n-1)/(q-1)$.

In this theorem, κ_n is defined to be 1 if r divides $(q^n-1)/(q-1)$, and 0 otherwise. If N is a positive integer and r a prime, then N_r denotes the r-part of N, and $N_{r'}$ denotes the r'-part of N. One should mention that $SL_n(q)$ has $q-1$ (irreducible) complex Weil characters, which can be obtained as follows. Let $SL_n(q)$ act on the set of q^n-1 nonzero vectors of the underlying space $\mathbf{F}_q^n$, and let ρ be the corresponding permutation character. Then ρ is the sum of the trivial character and the aforementioned $q-1$ Weil characters.

The corresponding result for the groups $PSL_n(q)$ which are not included in Theorem 5.2 is tabulated in Table IV, taken from [GT 1]. In the same paper a better lower bound for $\lceil(L)$ has also been given.

Hiss and Malle [HM] are obtaining similar results for Problem 5.1 in the cases of $PSU_n(q)$ and of even characteristic classical groups. For other groups of Lie type one has obtained only the following partial results, which are due to Guralnick, Magaard, Pentilla, Praeger, Saxl, and Hoffman.

Theorem 5.3 [GPPS], [GMS] *Let $G = PSp_{2n}(q)$, q an odd prime power, $n \geq 2$ and $(n,q) \neq (2,3)$. Suppose that G has a projective absolutely irreducible representation Φ of degree $d < q^n - 1$ over a field of characteristic r, $r \not| q$. Then one of the following holds.*

(i) *$d = 1$ and Φ is the trivial representation.*

(ii) *$d = (q^n-1)/2$ and Φ is one of the two representations obtained by reducing the complex Weil representations of degree $(q^n-1)/2$ modulo r.*

(iii) *$r \neq 2$, $d = (q^n+1)/2$, and Φ is one of the two representations obtained by reducing the complex Weil representations of degree $(q^n+1)/2$ modulo r.*

Theorem 5.4 [Hof 1] *Let $G = P\Omega^{\epsilon}_{2n}(q)$, $n \geq 4$, or $G = P\Omega_{2n+1}(q)$, $n \geq 3$. Suppose that $(n,q) \neq (3,2)$, $(4,2)$, $(3,4)$, $(4,3)$, and in addition that $(n,q) \neq (4,2)$, $(5,2)$, $(4,4)$, $(5,3)$ in the case $G = P\Omega^{-}_{2n}(q)$. Let Φ be an absolutely irreducible representation of G over a field of characteristic $r \not| q$. Then the following statement hold.*

(i) *If $G = P\Omega^{+}_{2n}(q)$, $n \geq 4$, $q > 3$, then*

$$\frac{(q^n-1)(q^{n-1}+q)}{q^2-1} - 2 \leq \deg(\Phi) \leq \frac{(q^n-1)(q^{n-1}+q)}{q^2-1} + 1$$

or

$$\deg(\Phi) \geq \frac{(q^n-1)(q^{n-1}+q)}{q^2-1} + \frac{q^n-1}{q-1} - 4\,.$$

Moreover, the smallest of these degrees will not occur if r does not divide $(q^n - 1)/(q-1)$.

(ii) *If $G = P\Omega^{+}_{2n}(q)$, $n \geq 4$, $q \leq 3$, then*

$$\frac{(q^n-1)(q^{n-1}-1)}{q^2-1} \leq \deg(\Phi) \leq \frac{(q^n-1)(q^{n-1}-1)}{q^2-1} + 1$$

or

$$\deg(\Phi) \geq \frac{(q^n-1)(q^{n-1}-1)}{q^2-1} + \frac{q^n-1}{q-1} - 2\,.$$

Moreover, if r does not divide $(q^n-1)/(q-1)$ then the last bound is $\frac{(q^n-1)(q^{n-1}-1)}{q^2-1} + \frac{q^n-1}{q-1} - 1$.

(iii) *If $G = P\Omega^{-}_{2n}(q)$, $n \geq 4$, then*

$$\frac{(q^n+1)(q^{n-1}-q)}{q^2-1} - 1 \leq \deg(\Phi) \leq \frac{(q^n+1)(q^{n-1}-q)}{q^2-1}$$

or

$$\deg(\Phi) \geq \frac{(q^n+1)(q^{n-1}-q)}{q^2-1} + \frac{q^{n-1}-1}{q-1} - 3\,.$$

Moreover, the smallest of these degrees occurs if and only if r divides $(q^{n-1} - 1)/(q-1)$.

(iv) *If $G = P\Omega_{2n+1}(q)$, $n \geq 3$, $q > 3$, q odd, then*

$$\frac{q^{2n}-1}{q^2-1} - 2 \leq \deg(\Phi) \leq \frac{q^{2n}-1}{q^2-1} + q - 2$$

or

$$\deg(\Phi) \geq \frac{q^{2n}-1}{q^2-1} + \frac{q^n-1}{q-1} - 4\,.$$

Moreover, the smallest of these degrees will not occur if r does not divide $(q^n - 1)/(q-1)$.

6 Groups of relatively small degree: Zero characteristic

The program of describing finite linear groups whose degree is less than a certain function depending on a group parameter goes back to Brauer. This program was mostly elaborated by Feit and his successors (Blau, Ferguson, Leonard, Lindsey, Robinson, Sibley, Wales, Winter, and others). In general, the goal was to describe primitive linear groups of order divisible by a prime p and of degree $< 2p$. We do not, however, discuss here these well-known results, which were obtained without using the CFSG and described in many surveys. More recent results make use of the CFSG. In particular, complex irreducible finite linear groups of degree at most $p-1$, where p is a prime divisor of the group order, were described by Zhang [Zh 1]. His result extended an earlier result of Ferguson [Fer] who classified the groups of degree at most $p-3$.

As a consequence of the main result of [LS], Landázuri and Seitz determined finite groups of Lie type G with an irreducible projective complex representation of dimension r where r is a prime divisor of $|G|$, see Corollary in [LS]. In [TZ 1] quasi-simple complex linear groups up to dimension $2r$ were determined. (This result depends on the CFSG as well as on the Deligne-Lusztig theory of complex representations of finite groups of Lie type.)

Theorem 6.1 [TZ 1] *Let G be a quasi-simple irreducible complex linear group of degree d. Suppose that $d \leq 2r$ for some prime divisor r of $|G|$, and let $L = G/Z(G)$. Then one of the following holds.*

1) $L = \mathbf{A}_n$, $\max\{9, r\} \leq n \leq 2r+1$, $d = n-1$.

2) $L = PSL_2(q)$, $q \neq 5, 7, 9$, *and one of the following holds.*

2a) $q = 2^a$, $a \geq 3$, $r = 2^a \pm 1$ *is a Fermat or a Mersenne prime,* $d \in \{r, r \mp 1, r \mp 2\}$.

2b) $q = r \geq 11$, $d \in \{r, (r \pm 1)/2, r \pm 1\}$.

2c) *Either* $q \geq 11$ *is a prime or* $q = 3^n$, n *an odd prime. Furthermore,* $r = (q-1)/2$ *and* $d \in \{r, r+1, 2r\}$.

2d) *Either* $q \geq 13$ *is a prime or* $q = 5^n$, n *an odd prime. Furthermore,* $r = (q-1)/4$ *and* $d = 2r$.

2e) $q \geq 13$, $r = (q+1)/2$, *and* $d \in \{r-1, r, 2r-2, 2r-1, 2r\}$.

2f) *Either* $q \geq 11$ *is a prime or* $q = 3^n$, n *an odd prime. Furthermore,* $r = (q+1)/4$ *and* $d \in \{2r-1, 2r\}$.

3) $L = PSL_n(q)$, $n \geq 3$, *and one of the following holds.*

3a) $q = 2$, $n \geq 5$, *either* $r = 2^{n-1} - 1$ *or* $r = 2^n - 1$, *and* $d = 2^n - 2$ *(1 representation).*

3b) $q \geq 3$, n *an odd prime,* $r = (q^n - 1)/(q-1)$. *Furthermore,* $d = r - 1$ *(1 representation) or* $d = r$ *($q-2$ representations).*

4) $L = PSU_n(q)$, $n \geq 3$, *and one of the following holds.*

4a) $q = 2$, $n - 1 \geq 5$ *is an odd prime*, $r = (2^{n-1} + 1)/3$. *Furthermore,* $d = 2r - 1$ *(2 representations) or* $d = 2r$ *(1 representation).*

4b) n *is an odd prime,* $r = (q^n + 1)/(q + 1)$. *Furthermore,* $d = r - 1$ *(1 representation) or* $d = r$ *(*q *representations).*

5) $L = PSp_{2n}(q)$, $n \geq 2$, *and one of the following holds.*

5a) $q = 3$, n *an odd prime,* $r = (3^n - 1)/2$. *Furthermore,* $d = r$ *(2 representations) or* $d = r + 1$ *(2 representations).*

5b) $q = 3$, n *an odd prime,* $r = (3^n + 1)/4$. *Furthermore,* $d = 2r - 2$ *(2 representations) or* $d = 2r$ *(2 representations).*

5c) $q = 5$, n *an odd prime,* $r = (5^n - 1)/4$. *Furthermore,* $d = 2r$ *(2 representations).*

5d) $n = 2^m$, $r = (q^n + 1)/2$. *Furthermore,* $d = r - 1$ *(2 representations) or* $d = r$ *(2 representations).*

6) *Exceptions for alternating and finite classical groups:*

6a) $L = \mathbf{A}_5 = PSL_2(4) = PSL_2(5)$, $(r, d) = (2, 2)$, $(2, 3)$, $(2, 4)$, $(3, 2)$, $(3, 3)$, $(3, 4)$, $(3, 5)$, $(3, 6)$, $(5, 2)$, $(5, 3)$, $(5, 4)$, $(5, 5)$, $(5, 6)$.

6b) $L = \mathbf{A}_6 = PSL_2(9) = Sp_4(2)'$, $(r, d) = (2, 3)$, $(2, 4)$, $(3, 3)$, $(3, 4)$, $(3, 5)$, $(3, 6)$, $(5, 3)$, $(5, 4)$, $(5, 5)$, $(5, 6)$, $(5, 8)$, $(5, 9)$, $(5, 10)$.

6c) $L = PSL_3(2) = PSL_2(7)$, $(r, d) = (2, 3)$, $(2, 4)$, $(3, 3)$, $(3, 4)$, $(3, 6)$, $(7, 3)$, $(7, 4)$, $(7, 6)$, $(7, 7)$, $(7, 8)$.

6d) $L = SL_3(3)$, $r = 13$, $d = 12$ *(1 representation),* $d = 13$ *(1 representation),* $d = 16$ *(4 representations) or* $d = 26$ *(3 representations).*

6e) $L = \mathbf{A}_7$, $(r, d) = (2, 4)$, $(3, 4)$, $(3, 6)$, $(5, 4)$, $(5, 6)$, $(5, 10)$, $(7, 4)$, $(7, 6)$, $(7, 10)$, $(7, 14)$.

6f) $L = PSL_3(4)$, $(r, d) = (3, 6)$, $(5, 6)$, $(5, 8)$, $(5, 10)$, $(7, 6)$, $(7, 8)$, $(7, 10)$.

6g) $L = \mathbf{A}_8 = SL_4(2)$, $(r, d) = (5, 7)$, $(5, 8)$, $(7, 7)$, $(7, 8)$, $(7, 14)$.

6h) $L = \mathbf{A}_9$, $r = 7$, $d = 8$ *(2 representations).*

6i) $L = \mathbf{A}_{11}$, $r = 11$, $d = 16$ *(2 representations).*

6j) $L = PSL_4(3)$, $r = 13$, $d = 26$ *(2 representations).*

6k) $L = SU_3(3)$, $(r, d) = (3, 6)$, $(7, 6)$, $(7, 7)$, $(7, 14)$.

6l) $L = SU_4(2) = PSp_4(3)$, $(r, d) = (2, 4)$, $(3, 4)$, $(3, 5)$, $(3, 6)$, $(5, 4)$, $(5, 5)$, $(5, 6)$, $(5, 10)$.

6m) $L = PSU_4(3)$, $r = 3, 5, 7$, $d = 6$ *(4 representations).*

6n) $L = SU_5(2)$, $r = 5$, $d = 10$ *(1 representation).*

6o) $L = Sp_6(2)$, $r = 5, 7$, $d = 7$ *(1 representation) and* $d = 8$ *(1 representation).*

6p) $L = Sp_4(4)$, $r = 17$, $d = 18$ *(1 representation) or* $d = 34$ *(2 representations).*

6q) $L = \Omega_8^+(2)$, $r = 5, 7$, $d = 8$ *(1 representation).*

6r) $L = \Omega_8^-(2)$, $r = 17$, $d = 34$ *(1 representation).*

7) L *is an exceptional group of Lie type:*

7a) $L = {}^2B_2(8)$, $r = 7, 13$, $d = 14$ *(2 representations).*

7b) $L = {}^3D_4(2)$, $r = 13$, $d = 26$ *(1 representation).*

7c) $L = G_2(3)$, $r = 7, 13$, $d = 14$ *(1 representation).*

7d) $L = G_2(4)$, $r = 7, 13$, $d = 12$ *(1 representation).*

7e) $L = {}^2F_4(2)'$, $r = 13$, $d = 26$ *(2 representations).*

8) *L is a sporadic finite simple group:*

8a) $L = M_{11}$, *either* $r = 5, 11$ *and* $d = 10$ *(3 representations), or* $r = 11$ *and* $d = 11$ *(1 representation), or* $r = 11$ *and* $d = 16$ *(2 representations).*

8b) $L = M_{12}$, *either* $r = 5, 11$ *and* $d = 10$ *(2 representations), or* $r = 11$ *and* $d = 11$ *(2 representations), or* $r = 11$ *and* $d = 12$ *(1 representation), or* $r = 11$ *and* $d = 16$ *(2 representations).*

8c) $L = M_{22}$, *either* $r = 5, 7, 11$ *and* $d = 10$ *(2 representations), or* $r = 11$ *and* $d = 21$ *(3 representations).*

8d) $L = J_2$, *either* $r = 3, 5, 7$ *and* $d = 6$ *(2 representations), or* $r = 7$ *and* $d = 14$ *(3 representations).*

8e) $L = M_{23}$, *either* $r = 11, 23$ *and* $d = 22$ *(1 representation), or* $r = 23$ *and* $d = 45$ *(2 representations).*

8f) $L = HS$, $r = 11$ *and* $d = 22$ *(1 representation).*

8g) $L = J_3$, $r = 17, 19$ *and* $d = 18$ *(4 representations).*

8h) $L = M_{24}$, $r = 23$, $d = 23$ *(1 representation) or* $d = 45$ *(2 representations).* 8i) $L = McL$, $r = 11$ *and* $d = 22$ *(1 representation).*

8j) $L = Ru$, $r = 29$ *and* $d = 28$ *(2 representations).*

8k) $L = Suz$, $r = 7, 11, 13$ *and* $d = 12$ *(2 representations).*

8l) $L = Co_3$, $r = 23$ *and* $d = 23$ *(1 representation).*

8m) $L = Co_2$, $r = 23$ *and* $d = 23$ *(1 representation).*

8n) $L = Co_1$, $r = 13, 23$ *and* $d = 24$ *(1 representation).*

Conversely, if the triple (L, r, d) *satisfies any of these conditions, then some covering group* G *of* L *has an irreducible complex representation of dimension* d.

Remark 6.2 In most cases of Theorem 6.1 the number of representations in question is given. The remaining cases are easy to handle directly. Using the results of [TZ 1] one could also extend Theorem 6.1 up to dimension, say, $r^{3/2}$ (that is, produce a classification of quasi-simple complex linear groups G of degree at most $r^{3/2}$, where r is a prime divisor of $|G|$).

Even more, we think that the following problem does not seem to be hopeless:

Problem 6.3 *Classify all primitive complex finite linear groups* G *of degree at most* d, *where* d *is the maximal order of group elements of* G.

7 Groups and representations of specific degrees

Irreducible representations of prime degree are of particular interest. The reason is that this condition excludes many configurations of specific type such as tensor-decomposable groups (if one imposes the primitivity condition), or nontriviality of the Schur index. Thus, the result is expected to be more transparent. The complex representations of prime degree of finite groups of Lie type were described by Landázuri and Seitz in their remarkable paper [LS] of 1974. The complete list of primitive complex linear groups of prime degree obtained by Dixon and Zalesskii [DZ] is given below. (Of course, the quasi-simple case of this result is now a particular case of Theorem 6.1.) For brevity, we restrict ourselves to exhibiting the list of perfect groups.

Theorem 7.1 [DZ] *Let G be a finite perfect primitive subgroup of $GL_p(\mathbf{C})$, where $p > 5$ is a prime. Then either G has an irreducible normal extraspecial subgroup, or one of the following holds.*

(i) $G \simeq \mathbf{A}_{p+1}$, $p > 5$.

(ii) $G \simeq PSL_2(q)$, *where p, q are subject to one of the following conditions:*

(a) $q = p$;

(b) $p = (q-1)/2$ *with q either a prime or a 3-power;*

(c) $p = (q+1)/2$;

(d) $p = q - 1 = 2^s - 1$ *is a Mersenne prime;*

(iii) $G \simeq PSp_{2n}(q)$, *where $p = (q^n+1)/2$ with n a 2-power, or $p = (3^n-1)/2$ with n an odd prime and $q = 3$.*

(iv) $G \simeq PSU_n(q)$, *where $p = (q^n+1)/(q+1)$ with n an odd prime.*

(v) $p = 7$ *and* $G \simeq Sp_6(2)$.

(vi) $p = 11$ *and* $G \simeq M_{12}$.

(vii) $p = 23$ *and* $G \simeq Co_2$, Co_3, *or* M_{24}.

Remark 7.2 Classification (up to conjugacy) of the groups G occuring in the cases (iii) and (iv) of Theorem 7.1 can be extracted from [TZ 2]: the groups in question arise as irreducible constituents of the so-called Weil representations of $Sp_{2n}(q)$ for q odd, or $SU_n(q)$.

It is probably worthwhile to write down the list of linear groups of degree $2r$ with r a prime. The following is a particular case of Theorem 6.1.

Theorem 7.3 [TZ 1] *Let G be a quasi-simple irreducible complex linear group of degree d. Suppose that d is twice a prime: $d = 2r$. Then one of the following holds for $L := G/Z(G)$.*

1) $L = \mathbf{A}_{2r+1}$, $r \geq 5$ *(1 representation).*

2) $L = PSL_2(q)$, $q \geq 11$, *and one of the following holds.*

2a) *Either q is a prime or $q = 3^n$, n an odd prime; furthermore, $r = (q-1)/2$ ($(q-1)/2$ representations).*

2b) *Either $q \geq 13$ is a prime or $q = 5^n$, n an odd prime. Furthermore, $r = (q-1)/4$ (2 representations).*

2c) $r = (q+1)/2$ *($(q-3)/2$ representations).*

2d) *Either q is a prime or $q = 3^n$, n an odd prime; furthermore, $r = (q+1)/4$ (2 representations).*

3) $L = PSL_n(2)$, *$n - 1 \geq 5$ a prime, and $r = 2^{n-1} - 1$ (1 representation).*

4) $L = PSU_n(2)$, *$n - 1 \geq 5$ is an odd prime, and $r = (2^{n-1} + 1)/3$ (1 representation).*

5) $L = PSp_{2n}(q)$, *n an odd prime. Furthermore, either $q = 3$ and $r = (3^n + 1)/4$ (2 representations), or $q = 5$ and $r = (5^n - 1)/4$ (2 representations).*

6) *Exceptions for alternating and finite classical groups:*

6a) $L = \mathbf{A}_5 = PSL_2(4) = PSL_2(5)$, $r = 2, 3, 5$.
6b) $L = \mathbf{A}_6 = PSL_2(9) = Sp_4(2)'$, $r = 2, 3, 5$.
6c) $L = PSL_3(2) = PSL_2(7)$, $r = 2, 3$.
6d) $L = SL_3(3)$, $r = 13$ *(3 representations).*
6e) $L = \mathbf{A}_7$, $r = 2, 3, 5, 7$.
6f) $L = PSL_3(4)$, $r = 3, 5$.
6g) $L = \mathbf{A}_8 = SL_4(2)$, $r = 7$ *(1 representation).*
6h) $L = PSL_4(3)$, $r = 13$ *(2 representations).*
6i) $L = SU_3(3)$, $r = 3, 7$.
6j) $L = SU_4(2) = PSp_4(3)$, $r = 2, 3, 5$.
6k) $L = PSU_4(3)$, $r = 3$ *(4 representations).*
6l) $L = SU_5(2)$, $r = 5$ *(1 representation).*
6m) $L = Sp_4(4)$, $r = 17$ *(2 representations).*
6n) $L = \Omega_8^-(2)$, $r = 17$ *(1 representation).*

7) *L is an exceptional group of Lie type:*

7a) $L = {}^2B_2(8)$, $r = 7$ *(2 representations).*
7b) $L = {}^3D_4(2)$, $r = 13$ *(1 representation).*
7c) $L = G_2(3)$, $r = 7$ *(1 representation).*
7d) $L = {}^2F_4(2)'$, $r = 13$ *(2 representations).*

8) *L is a sporadic finite simple group:*

8a) $L = M_{11}$, $r = 5$ *(3 representations).*
8b) $L = M_{12}$, $r = 5$ *(2 representations).*
8c) $L = M_{22}$, $r = 5$ *(2 representations).*
8d) $L = J_2$, *either $r = 3$ (2 representations), or $r = 7$ (3 representations).*
8e) $L = M_{23}$, $r = 11$ *(1 representation).*
8f) $L = HS$, $r = 11$ *(1 representation).*
8g) $L = McL$, $r = 11$ *(1 representation).*

Conversely, if the pair (L, r) satisfies any of these conditions, then some covering group G of L has an irreducible complex representation of dimension $2r$.

8 Groups of relatively small degree: Prime characteristic

The modular case has been examined by Blau and Zhang [BZ]. They classified finite linear groups of degree at most $p-1$ over a field of characteristic p, where p is a prime divisor of the group order. In fact, it is assumed in this result that the group in question doesn't contain a normal subgroup which is a group of Lie type in characteristic p.

Theorem 8.1 [BZ] *Let p be a prime and $\mathbf{F}$ a field of characteristic p. Let $G \leq GL_n(\mathbf{F})$ be a perfect finite irreducible subgroup such that p divides $|G|$. Suppose that $n < p$. Then one of the following holds.*

(1) *G contains a normal subgroup isomorphic to a direct product of groups of Lie type of characteristic p.*

(2) *$n = p-1 = 2^m$ and G contains a normal extraspecial 2-subgroup A such that $G/(A \cdot Z(G))$ is $Sp_{2k}(2^l)$ $(kl = m)$, $SO^-_{2k}(2^l)$ $(kl = m)$, $PSL_2(17)$ $(p = 17)$, $\mathbf{A}_5$, $\mathbf{A}_6$ $(p = 5)$.*

(3) *$n = p-1$, G is quasi-simple and one of the following holds:*
(i) *$G = SL_2(q)$, $q \equiv 1(\mathrm{mod}4)$, and $p = (q+1)/2$;*
(ii) *$G = Sp_{2r}(q)$, q is odd, $r > 1$ is a 2-power, and $p = (q^r+1)/2$;*
(iii) *$G = PSU_r(q)$, r is an odd prime, and $p = (q^r+1)/(q+1)$;*
(iv) *$G/Z(G) \simeq \mathbf{A}_7$ and $p = 5$ or 7;*
(v) *$p = 7$ and $G/Z(G) = PSL_3(4)$, $PSU_4(3)$, or J_2;*
(vi) *$G/Z(G) = G_2(4)$, $p = 13$;*
(vii) *$G = M_{12}$ or M_{22}, and $p = 11$;*
(viii) *$G/Z(G) = Suz$, and $p = 13$;*
(ix) *$G/Z(G) = J_3$, and $p = 19$;*
(x) *$G/Z(G) = Ru$, and $p = 29$;*
(xi) *$G = Co_2$, Co_3, and $p = 23$.*

(4) *$n = p-2$ and one of the following holds:*
(i) *$G = PSL_r(q)$, $p = (q^r-1)/(q-1)$, r is a prime, and $r \not| (q-1)$;*
(ii) *$G = \mathbf{A}_p$, and $p > 5$;*
(iii) *$G/Z(G) = \mathbf{A}_6$ or $\mathbf{A}_7$, and $p = 5$;*
(iv) *$G = M_{11}$, and $p = 11$;*
(v) *$G = M_{23}$, and $p = 23$.*

(5) *$n = p-3$, $G/Z(G) = \mathbf{A}_7$, $p = 7$.*

(6) *$n = p-4$, $G = J_1$, and $p = 11$.*

At this point we would like to describe an interesting result obtained recently by Guralnick, Pentilla, Praeger, and Saxl [GPPS]. Let q be a prime power. It was proved in [Zs] that if $e \geq 3$ and $(q,e) \neq (2,6)$, then $q^e - 1$ has a *primitive prime divisor*, i.e. a prime divisor which does not divide $\prod_{1 \leq i < e}(q^i - 1)$. The main result of [GPPS] is a classification of subgroups G of $GL_d(q)$, with orders disivible by a primitive prime divisor r of $q^e - 1$ for some e satisfying $d/2 < e \leq d$. (This condition implies in particular that $d \leq 2r - 3$.) Such groups were extensively studied by Hering, cf. [He 1], [He 2], and Dempwolff [De], in connection with questions about collineation groups of finite translation planes.

At present very little is known about modular irreducible representations of prime degree, in other words, about linear groups of prime degree in positive characteristic.

9 Minimal polynomials of elements of linear groups

The problem of determining minimal polynomials of group elements in group representations is to some extent connected with that of classifying groups of small degree. For instance, the minimal polynomial of each group element in a representation of degree n does not exceed n. We begin with the zero characteristic case. Let G be a primitive finite linear group. For $g \in G$ let $\deg(g)$ denote the degree of the minimal polynomial of g. An element $g \in G$ is said to be *quadratic*, if g is of order > 2 but $\deg(g) = 2$. The following problem has attracted attention since the beginning of the century:

Problem 9.1 *Determine primitive finite linear groups G that are generated by their quadratic elements.*

Blichfeldt (1917) studied complex finite linear groups containing a quadratic element g. He proved that $\deg(g) = 2$ implies $g^m \in Z(G)$, where $m \leq 5$. He also showed that if $m = 5$ and G is a group of (4×4)-matrices then G has an irreducible normal subgroup isomorphic to a direct product of two copies of $SL_2(5)$. Later, his argument was extended in [Z2, §11] to show that the equality $m = 5$ in general implies that G contains an irreducible normal subgroup isomorphic to a direct product of several copies of $SL_2(5)$. The case $m = 4$ was considered by Korljukov [Kor 1]. He showed that if G is irreducible, tensor-indecomposable, primitive and is generated by quadratic elements of order $m = 4$ then G contains an irreducible normal subgroup N such that (i) N is isomorphic to an extraspecial 2-group, (ii) G/N is one of the following groups: $Sp_{2k}(2)$, $O^{\pm}_{2k}(2)$, $\mathbf{S}_{2k+1}$ or $\mathbf{S}_{2k+2}$. He also made a partial contribution to the case $m = 3$ [Kor 2] by showing that if G is as above with $m = 3$ and if, in addition, G contains a non-trivial solvable normal subgroup N then either G is of degree 3^k with $G/N \simeq Sp_{2k}(3)$ or G is of degree 2^k with

$$G/N \in \{U_k(2), Sp_{2k}(2), O^{\pm}_{2k}(2), \mathbf{A}_{2k+1}, \mathbf{A}_{2k+2}\} .$$

The same case $m = 3$ but for quasi-simple groups is considered (in a more general context) in [Z8], using the CFSG. The list of pairs (G, n) (n the degree of G) consists of the following groups: $(SL_2(5), 2)$; $(SL_2(9), 4)$; $(Sp_{2k}(3), (3^k -$

$(-1)^k)/2)$ and $(PSp_{2k}(3), (3^k+(-1)^k)/2)$ with $k>1$; $(U_k(2), (2^k+2(-1)^k)/3)$ and $(U_k(2), (2^k-(-1)^k)/3)$ with $k \not\equiv 0 (\mathrm{mod} 3)$; $(SU_3(3), 6)$; $(2\cdot \mathbf{A}_k, 2^{[k/2]-1})$; $(2\cdot Sp_6(2), 8)$; $(6\cdot PSU_4(3), 6)$; $(2\cdot G_2(4), 12)$; $(2\cdot \Omega_8^+(2), 8)$; $(6\cdot Suz, 12)$; $(2\cdot J_2, 6)$; $(2\cdot Co_1, 24)$.

The restriction $\deg(g)=2$ has been weakened by Robinson [Ro 1], who showed that if G is a finite primitive linear group and $g\in G$ is a noncentral element of prime order p then $\deg(g)\geq (p+3)/4$. This is the best result in this direction obtained without using the CFSG. Using the CFSG Robinson [Ro 2] improved this result by showing that $\deg g \geq (p-1)/2$. For further progress see [Z7] and [Z8]. The last paper contains in particular the following result.

Theorem 9.2 [Z8] *Let G be a quasi-simple finite group, $g\in G$ with $g^p\in Z(G)$. Let Θ be a faithful complex irreducible representation of G such that $1<\deg(\Theta(g))<p$. Then $p>2$, $g^p=1$, and $\deg(\Theta(g))\in\{p-1, p-2, (p+1)/2, (p-1)/2\}$.*

(A) *If $\deg(\Theta(g))=p-1$ then one of the following holds.*

(1) $G=PSL_n(q)$, $p=(q^n-1)/(q-1)$, *n is an odd prime,* $(n,q-1)=1$, *and* $\dim(\Theta)=p-1$.

(2) $G=SL_2(q)$, $q\equiv 1(\mathrm{mod} 4)$, $p=(q+1)/2$, *and* $\dim(\Theta)=p-1$.

(3) $G=SL_2(q)$, *$q>4$ is even,* $p=q+1$, *and* $\dim(\Theta)=p-1$.

(4) $G=SL_2(p)$, $p>3$, *and* $\dim(\Theta)=p-1$.

(5) $G=SL_2(p^2)$, *and* $\dim\Theta=(p^2-1)/2$.

(6) $G=Sp_{2n}(q)$, *$n>1$ is a 2-power,* $q^n\equiv 1(\mathrm{mod} 4)$, $p=(q^n+1)/2$, *and* $\dim(\Theta)=p-1$.

(7) $G=Sp_4(p)$, *g is not a transvection, and* $\dim(\Theta)=(p^2-1)/2$.

(8) $G=Sp_4(p)$, *g is a transvection, and* $\dim(\Theta)=p(p-1)^2/2$.

(9) $G=SU_n(q)$, $n>2$, *q is even,* $p=q+1$, $(n,p)=1$, $\mathrm{rank}(g-z\cdot \mathrm{Id})=1$ *for some* $z\in \mathbf{F}_{q^2}$, *and* $\dim(\Theta)=(q^n+q(-1)^n)/(q+1)$ *or* $(q^n-(-1)^n)/(q+1)$.

(10) $G=PSU_n(q)$, *n is an odd prime,* $(n,q+1)=1$, $p=(q^n+1)/(q+1)$, *and* $\dim(\Theta)=p-1$.

(11) $G=SU_3(p)$, $p>3$, *g is a transvection, and* $\dim(\Theta)=p(p-1)$.

(12) $G=\mathbf{A}_p$, *the alternating group,* $p>5$, *and* $\dim(\Theta)=p-1$.

(13) $G=2\cdot\mathbf{A}_n$, $p=3$ *or* 5, *g is a p-cycle, and Θ is a so-called basic spin representation of dimension* $2^{[n/2]-1}$.

(14) $G/Z(G)=\mathbf{A}_7$, $|Z(G)|=3$ *or* 6, $p=7$, *and* $\dim(\Theta)=6$.

(15) $G=2\cdot Sp_6(2)$, $p=3,5$, *and* $\dim(\Theta)=8$.

(16) $G=6\cdot PSL_3(4)$, $p=7$, *and* $\dim(\Theta)=6$.

(17) $G=6\cdot PSU_4(3)$, $p=3,7$, *and* $\dim(\Theta)=6$.

(18) $G=2\cdot G_2(4)$, $p=3,5,7$ *or* 13, *and* $\dim(\Theta)=12$.

(19) $G/Z(G)=2\cdot\Omega_8^+(2)$, $p=3,5$, *and* $\dim(\Theta)=8$.

(20) $G=M_{11}$, $p=11$, *and* $\dim(\Theta)=10$.

(21) $G=M_{23}$, $p=23$ *and* $\dim(\Theta)=22$.

(22) $G=2\cdot M_{12}$ *or* $2\cdot M_{22}$, $p=11$, *and* $\dim(\Theta)=10$.

(23) $G = 6 \cdot Suz$, $p = 3, 5, 7$ *or* 13, *and* $\dim(\Theta) = 12$.
(24) $G = 3 \cdot J_3$, $p = 19$, *and* $\dim(\Theta) = 18$.
(25) $G = 2 \cdot Ru$, $p = 29$, *and* $\dim(\Theta) = 28$.
(26) $G = 2 \cdot J_2$, $p = 3, 5, 7$, *and* $\dim(\Theta) = 6$.
(27) $G = 2 \cdot Co_1$, $p = 3, 5, 7, 13$, *and* $\dim(\Theta) = 24$.

(B) *If* $\deg(\Theta(g)) = p - 2$, *then* $G = SL_2(q)$, $q \geq 4$ *even*, $p = q + 1$, *and* $\dim(\Theta) = p - 2$.

(C) *If* $\deg(\Theta(g)) = (p+1)/2$, *then one of the following holds.*
(1) $G = Sp_{2n}(p)$, $n > 1$, *g is a transvection, and* $\dim(\Theta) = (p^n + a)/2$ *with* $a \in \{1, -1\}$ *such that* $(p^n + a)/2$ *is even.*
(2) $G = PSp_{2n}(p)$, $n > 1$, *g is a transvection, and* $\dim(\Theta) = (p^n + a)/2$ *with* $a \in \{1, -1\}$ *such that* $(p^n + a)/2$ *is odd.*
(3) $G = SL_2(p)$, $p > 3$, $p \equiv 3(\text{mod} 4)$, *and* $\dim(\Theta) = (p+1)/2$.
(4) $G = PSL_2(p)$, $p \equiv 1(\text{mod} 4)$, *and* $\dim(\Theta) = (p+1)/2$.
(5) $G = 3 \cdot \mathbf{A}_6$, $p = 5$, *and* $\dim(\Theta) = 3$.
(6) $G = 2 \cdot \mathbf{A}_7$, $p = 7$, *and* $\dim(\Theta) = 4$.
(7) $G = 2 \cdot J_2$, $p = 5$, *and* $\dim(\Theta) = 6$.

(D) *If* $\deg(\Theta(g)) = (p-1)/2$, *then one of the following holds.*
(1) $G = SL_2(p)$, $p \equiv 1(\text{mod} 4)$, *and* $\dim(\Theta) = (p-1)/2$.
(2) $G = PSL_2(p)$, $p > 3$, $p \equiv 3(\text{mod} 4)$, *and* $\dim(\Theta) = (p-1)/2$.

Remark 9.3 Even though in the above theorem only $\dim(\Theta)$ is given, the list of representations Θ can be read off from [Atlas] (for G a "small" group or sporadic group) and from [TZ 1] (for G a special linear, symplectic, or unitary group).

It would be desirable to obtain a modular analogue of Theorem 9.2, that is, to classify all triples (G, Θ, g), G a finite quasi-simple group, g an element of order p modulo $Z(G)$, and Θ a faithful absolutely irreducible representation of G over a field of characteristic $0 < r \neq p$ such that $\deg(\Theta(g)) < p$. This problem in the case G is a group of Lie type in characteristic p has been reduced substantially in [Z3] to that of identifying certain modular representations of $Sp_{2n}(p)$. The latter step is being completed in [GT 2]. If $r = p$ one probably cannot hope for a complete answer to the question, especially for groups of Lie type. However, for a group G of Lie type Suprunenko [Su 2] provides an algorithm for computing $\Theta(g)$. This implies an explicit description of irreducible representations Θ of G such that $\deg(\Theta(g)) = p$ for some $g \in G$, see [Su 3].

Linear groups over finite fields of characteristic $r > 3$ generated by quadratic r-elements were classified by Thompson in 1971 [Th 1]. The case of characteristic 3 was studied by Ho [Ho 1], [Ho 2], but his results do not give a complete list of the groups in question. In characteristic 2, all involutions are quadratic. Therefore, to create a sensible question in the characteristic 2 case one has to change the problem a little bit. Now we require the irreducible linear group $G \leq GL(V)$ in characteristic 2 to have a 2-subgroup A such that $[[V, A], A] = 0$ and $|A| \geq 4$. This kind of quadratic module V turns out to be interesting in the classification of finite simple groups (because of applications in weak-closure arguments and in the amalgam method). These modules have been investigated by Meierfrankenfeld and Stroth, cf. [MS 1], [MS 2]. A new approach towards a "third generation" proof of the CFSG Theorem, suggested by Meierfrankenfeld, Stellmacher, and Stroth, requires the analysis of certain classes of modules, the so-called F_1-modules and F_2-modules, which are closely related to quadratic modules. This is being treated in work of Meierfrankenfeld and Stroth (in preparation).

Let G be a quasi-simple group. Aschbacher [As 2] classified irreducible $\mathbf{F}_2G$-modules V such that $\dim(\mathrm{Id} + g)V \leq 2$ for some involution $g \in G$. Irreducible finite linear groups generated by transvections were described earlier by McLaughlin [M1], [M2]. In a more general context, Wagner [Wag 1], [Wag 5] classified irreducible finite linear groups generated by pseudo-reflections, i.e. elements $g \in GL(V)$ such that $\dim(\mathrm{Id} - g)V = 1$. See also [Ka], [ZS].

There are other partial results for the prime characteristic case. The representations of $\mathbf{A}_n$ over a field of characteristic 2 in which a 3-cycle acts fixed-point-freely were determined by Mullineux [Mu]. A similar result for spin modules of $\mathbf{A}_n$ was obtained by Meierfrankenfeld [Me]. Irreducible representations of simple groups of Lie type in characteristic 2 were classified by Wilson [Wi]. Further progress concerning representations in characteristic 2 has been achieved in [FLZ]. In [Z7] for quasi-simple groups G with cyclic Sylow p-subgroups there are determined a non-trivial representation Φ over a field of characteristic p and a p-element g such that $\deg(\Phi(g)) < |g|$.

10 Groups containing a simple spectrum matrix

A further problem of interest is that of determining group representations containing a matrix X with a simple spectrum (both for prime and zero characteristic). This means that every eigenvalue of X occurs with multiplicity 1. Groups with this property appear in numerous applications. If G is a group, $g \in G$, Ψ is an irreducible representation of G such that $\Psi(g)$ has a simple spectrum, then, obviously, $\deg(\Psi) \leq m(g)$, where $m(g)$ means the order of g modulo $Z(G)$. For representations over a field of characteristic 0 this condition turns out to be very restrictive. For example, consider G to be $SL_n(q)$, $n > 2$. Then the maximal order of an element of G is $(q^n - 1)/(q - 1)$, and the minimum degree of a complex (nontrivial) representation of G is one less, i.e. $(q^n - 1)/(q - 1) - 1$. This observation does not work for some other groups such as alternating and symplectic ones. However, the aforementioned problem does not seem to be very difficult. The situation is more profound in the modular case where representations considered are over a field of prime characteristic p. As a first step one needs to determine the representations of finite simple groups of Lie type of characteristic p with this property. This was done in [SZ 2] for classical groups and in [SZ 3] for all exceptional groups.

Another closely related problem is to determine the modular group representations containing a matrix with a single Jordan block. We mention the following interesting result of Suprunenko in this direction:

Theorem 10.1 [Su 1] *Let* $\mathbf{F}$ *be an algebraically closed field of characteristic* $p > 0$ *and let* $G \subset GL_n(\mathbf{F})$, $n > 1$, *be an irreducible semisimple algebraic group such that a matrix* $g \in G$ *is similar to a single Jordan block. Then* G *is simple and isomorphic to* $SL_n(\mathbf{F})$, $SO_n(\mathbf{F})$, $Sp_n(\mathbf{F})$, $G_2(\mathbf{F})$, $SL_2(\mathbf{F})$, *or* $PSL_2(\mathbf{F})$. *If* G *is of type* G_2 *then* $n = 7$ *if* $p > 2$ *and* $n = 6$ *if* $p = 2$. *If* G *is of type* A_1 *then* $1 < n \leq p$.

In more general context the problem of determining Jordan structure of unipotent elements in group representations is of high importance. This problem will be discussed in Suprunenko's survey [Su 3] in this volume.

11 Tensor products

Let G be a finite group, Φ and Ψ two irreducible representations of G (over the same field). Then usually $\Phi \otimes \Psi$ is not irreducible, and the problem of decomposing $\Phi \otimes \Psi$ into irreducibles is important in applications. In this section we consider the following problem.

Problem 11.1 *Determine all triples* (G, Φ, Ψ) *such that* G *is a finite quasi-simple group,* Φ *and* Ψ *are representations of* G *of degree* > 1 *over an algebraically closed field* $\mathbf{F}$ *and* $\Phi \otimes \Psi$ *is irreducible.*

One of the motivations for studying this problem is that one often needs to exclude the case where an irreducible linear group is tensor-decomposable. Another motivation comes from the classification of maximal subgroups of finite classical groups $G(q)$. According to Aschbacher's Theorem [As 1], if M is a maximal subgroup of $G(q)$, then either M belongs to one of 8 collections $\mathbf{C}_i$, $1 \leq i \leq 8$ of "natural" subgroups of $G(q)$, or $M \in \mathbf{S}$, a collection of quasi-simple groups which act irreducibly on the natural module V of $G(q)$. In this scheme, $\mathbf{C}_4$ consists of stabilizers of tensor decompositions of V. Conversely, if $M \in \cup_{i=1}^{8} \mathbf{C}_i$, then M is maximal up to exceptions determined by Kleidman and Liebeck [KL]. Now suppose $M \in \mathbf{S}$. Could it happen that $M < N < G$ for some $N \in \mathbf{C}_4$, in other words, can the (irreducible) representation of M on V be tensor-decomposable?

Problem 11.1 for solvable groups G has been investigated by Isaacs [Is]. In particular, he conjectures that if the solvable group G has two faithful complex characters α and β such that $\alpha\beta$ is irreducible, then G is abelian. He also proved this conjecture in the case where every minimal normal subgroup of G is cyclic.

In what follows, we assume that G is a finite quasi-simple group, Φ and Ψ are representations of G of degree > 1 over an algebraically closed field $\mathbf{F}$ of characteristic r, such that

$(\star)$: $\Phi \otimes \Psi$ is irreducible.

Case 1 : G is a finite group of Lie type defined over a field of characteristic p and $\mathrm{char}(\mathbf{F}) = p$.

Each irreducible representation of G lifts to that of $\mathbf{G}$, the algebraic group corresponding to G. Steinberg's tensor product theorem then reduces the problem to the case where the highest weights λ and μ of Φ and Ψ, respectively, are restricted. (This means that the coefficients a_i, b_j of the expansion $\lambda = \sum_i a_i\omega_i$, and $\mu = \sum_j b_j\omega_j$ do not exceed $p-1$, where the ω_i's are the fundamental weights of $\mathbf{G}$.) Moreover, if $(\mathbf{G}, p) \in \{(Sp, 2), (F_4, 2), (G_2, 3)\}$ the representation $\Phi \otimes \Psi$ is irreducible whenever λ and μ are restricted and (up to reordering Φ and Ψ) $a_i = 0 = b_j$ for those (i, j) for which the root α_i is short and α_j is long. Seitz [Se, 1.6] showed that for λ and μ restricted there are no more cases where $\Phi \otimes \Psi$ could be irreducible. Thus one gets a complete list of pairs (Φ, Ψ) with property $(\star)$ in case 1.

Case 2: G is a symmetric or alternating group.

It has recently been shown by Bessenrodt and Kleshchev [BK] that $(\star)$ cannot happen for $G = \mathbf{S}_n$ if $\mathbf{F} = \mathbf{C}$. Also, all possible examples in the case $G = \mathbf{A}_n$ and $\mathbf{F} = \mathbf{C}$ are found in [BK]; namely, $(\star)$ occurs in this case if and only if $n = a^2$ for some integer $a \geq 3$, Φ is the (nontrivial component of the) natural permutation representation, and Ψ is any of the two representations corresponding to the Young diagram (a^a).

The cases of spin representations and modular representations of $\mathbf{S}_n$ and $\mathbf{A}_n$ are still open. We note that there are examples of complex spin representations of $\mathbf{A}_9$, $\mathbf{A}_{10}$ with property $(\star)$. Also, some examples of irreducible tensor products of representations of $\mathbf{S}_n$ in characteristic 2 have been found by Gow and Kleshchev [GK].

Case 3 : G is one of the 26 sporadic finite simple groups (or their covers).

For all sporadic groups (in the case $\mathbf{F} = \mathbf{C}$), and for many of them (in the modular case) one can use GAP to find all examples of pairs (Φ, Ψ) with property $(\star)$. For instance, if $\mathbf{F} = \mathbf{C}$ then $(\star)$ occurs precisely for $2 \cdot M_{12}$, $12 \cdot M_{22}$, M_{23}, M_{24}, $2 \cdot J_2$, $3 \cdot J_3$, $2 \cdot Co_1$, Co_2, Co_3, $6 \cdot Fi_{22}$, $3 \cdot Fi'_{24}$, $3 \cdot Suz$, $6 \cdot Suz$, $3 \cdot McL$, $2 \cdot Ru$, $3 \cdot ON$, Th, BM, M. In many examples, either Φ or Ψ is the representation of G of smallest degree. However, some examples don't seem to have any pattern. Below we list a couple of examples for $G = BM$,

$2 \cdot BM$:

$$\Phi(1) = 4371, \quad \Psi(1) = 53936390144$$
$$\Phi(1) = 96256, \quad \Psi(1) = 90807234375$$

and for $G = M$:

$$\Phi(1) = 196883, \quad \Psi(1) = 8980616927734375$$
$$\Phi(1) = 19360062527, \quad \Psi(1) = 8980616927734375$$
$$\Phi(1) = 21296876, \quad \Psi(1) = 3503434660075044981$$

Case 4 : The cross characteristic case.

Theorem 11.2 [MT] *Let G be a finite quasi-simple group of Lie type defined over $\mathbf{F}_q$, and assume that $r = \mathrm{char}(\mathbf{F}) = 0$ or $r \not| q$. Then the situation $(\star)$ cannot happen, except possibly for*
(i) $q \leq 3$;
(ii) $G = Sp_{2n}(5)$;
(iii) $G = Sp_{2n}(q)$, $2|q$;
(iv) $G = G_2(q)$, $3|q$;
(v) $G = F_4(q)$, $2|q$;
(vi) $G = {}^2F_4(q)$, $2|q$.

We do have examples (of complex representations) for $G = Sp_{2n}(3)$, $Sp_{2n}(5)$, $SU_n(2)$. A typical example: Φ and Ψ are Weil representations of distinct degrees for G.

One can remove the cases (iv) – (vi) in Theorem 11.2 if $\mathbf{F} = \mathbf{C}$.

An interesting by-product of [MT] is a new lower bound $q^2(q^2+1)$ for the minimum degree $\ell(G)$ of nontrivial projective representations of $G = G_2(q)$ in cross characteristic, provided that $q = 3^f > 3$. (The Landázuri-Seitz bound for this group is $q(q^2-1)$.)

12 Global irreducibility

Definition 12.1 Let G be a finite group and Φ an irreducible complex representation of degree > 1 of G. We say that Φ is *globally irreducible* if Φ remains absolutely irreducible under reduction modulo every prime p, and *strongly globally irreducible* if Φ is globally irreducible and has Schur index 1. If Φ is globally irreducible and faithful, then we also call G *globally irreducible.*

Globally irreducible groups are interesting for integral lattice theory. The class of rational globally irreducible groups was distinguished by Thompson in [Th 2] in the course of constructing the sporadic finite simple group $Th = F_3$ [Th 3]. Namely, he showed that if G is a finite group and V is a $\mathbf{Q}G$-module of dimension > 1 such that $V(\mathrm{mod} p)$ is irreducible for all primes p, then V contains a unique (up to scalar) G-stable lattice Λ. Moreover, V supports a unique (up to scalar) G-invariant scalar product $(\cdot,\cdot)$, and under $(\cdot,\cdot)$ the lattice Λ is even unimodular. At the time of [Th 2] there were only three known examples of primitive, tensor-indecomposable, rational globally irreducible groups G; namely

(i) G is $W(E_8)$, $2\cdot Co_1$, or F_3.

Correspondingly, Λ is the ubiquitous Korkine-Zolotarev root lattice E_8 (which is the unique even unimodular Euclidean lattice of rank 8), the famous Leech lattice Λ_{24} (which is the unique even unimodular lattice of rank 24 with minimum 4), and the Thompson-Smith lattice of rank 248. Notably, E_8 gives rise to the densest sphere packing in $\mathbf{R}^8$, and Λ_{24} produces the best known sphere packing in $\mathbf{R}^{24}$. Even though it was proved by Minkowski that good sphere packings exist in any dimension, it still remains an open problem (already mentioned by Hilbert in his 1900 famous list of open problems) how to construct good sphere packings explicitly. In the late 80's Elkies [El] and Shioda [Sh] contructed several new lattices which produce sphere packings that are denser than any previously known one in certain dimensions n, $65 \leq n \leq 4000$. Their construction uses the Mordell-Weil groups of certain elliptic curves over function fields. To better understand them from the group-theoretic viewpoint, Gross [Gr] suggested a generalization of the concept of rational global irreducibility. This concept led to several new series of lattices, which are interesting from the point of view of both group theory and algebraic geometry. Interested readers are referred to [Gr] and [T3]. Here we mention only that our notion of global irreducibility differs, however, from the one invented by Gross, which is more arithmetical.

It is easy to give examples of monomial strongly globally irreducible groups. However, primitive globally irreducible groups seem to be a very rare phenomenon. In fact, at the moment there are only a few types of examples in large dimension:

(ii) $G = 2\cdot\mathbf{S}_n$, a double cover of the symmetric group $\mathbf{S}_n$ with n even, in the so-called basic spin representations of degree 2^n, cf. [Wa];

(iii) $G = 2\cdot\mathbf{A}_n$, with n odd, in the basic spin representations of degree 2^{n-1}, cf. [Gow] and [T2];

(iv) $G = Sp_{2n}(p)$, p an odd prime, in Weil representations of dimension $(p^n-1)/2$, cf. [SZ 1]. We mention that this is the only case when a Weil representation of a finite classical group is globally irreducible [T4].

(v) This example is in a sense a characteristic 2 analogue of the previous one. Let $\mathbf{E}$ be an extraspecial 2-group of order 2^{1+2k} considered in its irreducible representation of degree $n = 2^k$, realized over $\mathbf{Q}(i)$, $i = \sqrt{-1}$. Let $\mathbf{E}_1 = \mathbf{E} * \mathbf{Z}_4$ be the group obtained from $\mathbf{E}$ by adding the scalar matrix $i \cdot I$. Let N be the normalizer of $\mathbf{E}_1$ in $GL_n(\mathbf{Q}(i))$. Then N is globally irreducible. Also, certain subgroups of N are globally irreducible as well. To construct them, observe that $N/(Z(N)\mathbf{E}) \simeq Sp_{2k}(2)$, and that the reduction of N modulo 2 gives the so-called spin representation of $Sp_{2k}(2)$. Let V be the natural module for $Sp_{2k}(2)$ over $\mathbf{F}_2$, and for $i < k$ let T_i be the stabilizer in $Sp_{2k}(2)$ of a non-degenerate subspace of V of dimension $2i$. Let N_i be the pullback of T_i in N. Then N_i is globally irreducible.

Problem 12.2 *Classify quasi-simple globally irreducible groups.*

Global irreducibility of the groups $SL_2(q)$, ${}^2B_2(q)$, $SL_3(q)$, $SU_3(q)$, and $Sp_4(q)$ has been investigated in [T1], [T3], [T5]. It turned out that these groups have no globally irreducible represenions, except the Weil representations of degree $(p^n-1)/2$ of $Sp_{2n}(p)$, $n=1,2$. Kleshchev and Premet [KP] have recently shown that $\mathbf{S}_n$, $n \geq 5$, has no globally irreducible representation. (Observe that the proof of this result can be substantially shortened if one makes use of a very recent result of James and Mathas [JM 2] determining which Specht modules of $\mathbf{S}_n$ remian irreducible modulo 2.) Based on these partial results one might conjecture that if G is a quasi-simple globally irreducible group and G is not too small, then G is one of the groups mentioned in the examples (i) – (iv) above.

The following problem is clearly related to Problem 12.2 and has its own interest in the representation theory of finite groups of Lie type.

Problem 12.3 *Let G be a quasi-simple finite group of Lie type. Determine all nontrivial irreducible complex representations Φ of G which remain irreducible modulo the defining characteristic p.*

We note that, according to the Fong-Swan-Isaacs-Rukolaine Theorem [Sw], if G is p-solvable then any absolutely irreducible p-modular representation of G lifts to a complex representation.

Let G be a quasi-simple finite group of Lie type in characteristic p. It is known that the Steinberg representation is irreducible modulo p (and also modulo some other primes $r \neq p$). Are there other examples? The first not obvious example was found in [SZ 1] for the group $G = Sp_{2n}(p)$ with p any odd prime, to be the Weil representations of dimension $(p^n \pm 1)/2$. More recently, the "small" groups $SL_2(q)$, ${}^2B_2(q)$, $SL_3(q)$, $SU_3(q)$, and $Sp_4(q)$ have been handled in [T1], [T3], [T5]. For these groups, if Φ is a complex irreducible representation which remains irreducible modulo p, then either Φ is the Steinberg representation, or one of the following holds:

(a) Φ is a Weil representation of degree $(p^n \pm 1)/2$ of $Sp_{2n}(p)$ (and $n=1,2$);
(b) Φ is a representation of degree $p-1$ of $SL_2(p)$;
(c) Φ is a representation of degree 3 of $SL_3(2) \simeq PSL_2(7)$;
(d) Φ is a Weil representation of degree 7 of $SU_3(3)$;
(e) Φ is a representation of degree 10 of $Sp_4(3)$;
(f) Φ is a representation of degree 52 of $Sp_4(5)$.

(Observe that neither the Steinberg representation, nor the representation Φ described in the cases (d), (e), and (f) is globally irreducible.)

The method used in [T1], [T3], [T5] is based on Jantzen's reduction formula [Ja 1]. Namely, first one can write down any irreducible complex character of G in terms of Deligne-Lusztig characters $R^G_T(\theta)$. Next, Jantzen's reduction formula expresses $R_T^G(\theta)(\mathrm{mod}\, p)$ as a combination of Brauer characters of Weyl modules. Finally, one needs to find the multiplicity of any irreducible Brauer character in the Brauer character of any Weil module. (The answer to the last question is given by Lusztig's conjecture, which has now been proved for very large p.) Unfortunately, it is very difficult to use Jantzen's reduction formula directly for groups of Lie type of large rank, since it then involves too many summands.

It is worthwhile to mention the recent paper [JM 1] of James and Mathas where one can find a description of complex representations of $GL_n(q)$ which remain irreducible modulo a prime r with $(r, q) = 1$. Also, a class of complex representations of $GL_n(q)$ which remain irreducible modulo every prime r not dividing q is given in [GT 1].

Very recently, the authors of this survey have developed another approach which enables one to prove the reducibility of a certain class of complex representations modulo the defining characteristic $p > 3$. The results obtained will be reported elsewhere. Observe that at present this new approach does not cover all complex representations.

There is another problem which is also related to globally irreducible representations. Let G be a finite subgroup of $GL_n(\mathbf{O_K})$, where $\mathbf{O_K}$ denotes the ring of integers of an algebraic number field $\mathbf{K}$. Under which condition can one guarantee that the $\mathbf{O_K}$-span of G coincides with the matrix ring $M_n(\mathbf{O_K})$? An analogue of the classical Burnside's Theorem has been proved in [OZ], which shows that this happens if and only if the natural representation of G on $\mathbf{K}^n$ is globally irreducible.

13 Some other questions

1. Let G be a finite subgroup of $GL_n(\mathbf{Q})$, or, more generally, let G be a finite group with a faithful irreducible $\mathbf{Q}$-valued character of degree n. Then the Minkowski-Schur Theorem (cf. p. 129 of [Sch]) states that the order of G divides $\prod_{p \text{ primes}} p^{M_p(n)}$, where $M_p(n) = \sum_{i \geq 0} [n/(p^i(p-1))]$. This result has been strengthened by Feit in [F3].

More generally, let G be a finite subgroup of $GL_n(\mathbf{C})$. A classical theorem of Jordan asserts that there is a function $f : \mathbf{N} \to \mathbf{R}$ such that G has a normal abelian subgroup A with $|G/A| \leq f(n)$. Schur [Sch] gave the bound $f(n) = (49n)^{n^2}$. A better bound $f(n) = n! \cdot n \cdot 12^{\pi(n+1)+1}$ was found by Frobenius, where $\pi(n+1)$ is the number of prime divisors of n. A much better bound, using the CFSG, was obtained in an unfinished manuscript of Weisfeiler. Namely, he proved that one can take $f = f_W$ with $f_W(n) = (n+2)!$ if $n \geq 64$ and $n^4 \cdot (n+2)!$ if $n < 64$. Very recently Collins [C] has established the best possible result, again using the CFSG.

Theorem 13.1 [C] *If n is big enough and G is a finite subgroup of $GL_n(\mathbf{C})$, then G has a normal abelian subgroup A with $|G/A| \leq (n+1)!$. Moreover, if $|G/A| = (n+1)!$, then $G/Z(G) \simeq \mathbf{S}_{n+1}$.*

Weisfeiler also proved that if G is a finite subgroup of $GL_n(\mathbf{F})$, $\mathbf{F}$ an algebraically closed field of characteristic p, then G has a normal series $G \rhd L \rhd T \rhd O_p(G)$, such that L/T is a direct product of finite groups of Lie type in characteristic p, $T/O_p(G)$ is an abelian p'-group, and $|G/L| \leq f_W(n)$. Improving this result to the best possible, Collins [C] shows that if n is big enough then one can replace f_W by f_C, where $f_C(n) = (n+1)!$ if $p \not| (n+2)$ and $f_C(n) = (n+2)!$ if $p | (n+2)$. By "big enough" Collins anticipates establishing that $n \geq 45$ will suffice for Theorem 13.1 and also for the modular case if $p > 5$, and $n \geq 70$ for $p \leq 5$.

2. About 60 years ago Zassenhaus [Za] classified near-fields. In doing so he also gave a complete description of those finite groups which have complex representations Φ such that $\Phi(g)$ has no eigenvalue 1 for any nontrivial element $g \in G$ (nowadays such a group is called semiregular). One of the results in [Za] characterizes $SL_2(5)$ as the only perfect semiregular group. A generalization of this result was obtained in [FLT 1]. Let p be a prime. An $\mathbf{F}G$-representation Φ is said to be *p'-semiregular*, if $\Phi(g)$ has no eigenvalue 1 for any nontrivial p'-element of G. The finite p'-semiregular groups are classified in [FLT 1] (a part of this classification was obtained in [GW]). This result has an interesting application in the theory of permutation groups. A finite permutation group G is said to be *p-Frobenius*, if every 2-point stabilizer in G is a p-group. This is clearly a generalization of Frobenius groups. Also, if G is a primitive permutation group with elementary abelian socle V, then G is p-Frobenius if and only if the action of G_1 (a point stabilizer in G) on V (via conjugation) is p'-semiregular. Using the results of [FLT 1] and [FLT 2], the paper [FLT 3] classifies all finite primitive p-Frobenius groups.

References

[An] H. H. Andersen, Modular representations of algebraic groups, Proc. Symp. Pure Math. **47** (1987), pp. 23 – 36.

[AJS] H. H. Andersen, J. C. Jantzen, and W. Soergel, "Representations of Quantum Groups at a p-th Root of Unity and of Semisimple Groups in Characteristic p: Independence of p", Astérisque **220** (1994).

[As 1] M. Aschbacher, On the maximal subgroups of the finite classical groups, Invent. Math. **76** (1984), 469 – 514.

[As 2] M. Aschbacher, $GF(2)$-representations of finite groups, *Amer. J. Math.* **104** (1982), 683 – 771.

[Atlas] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, and R. A. Wilson, "An ATLAS of Finite Groups", Clarendon Press, Oxford, 1985.

[BK] C. Bessenrodt and A. S. Kleshchev, On Kronecker products of complex representations of the symmetric and alternating groups, Preprint, 1997.

[BZ] H. I. Blau and J. Zhang, Linear groups of small degree over fields of finite characteristic, *J. Algebra* **159** (1993), 358 – 386.

[BKS] J. Brundan, A. S. Kleshchev, and I. D. Suprunenko, Semisimple restrictions from $GL(n)$ to $GL(n-1)$, *J. für die reine und angew. Math.* (to appear).

[C] M. Collins, (in preparation).

[Ca] R. Carter, "Finite Groups of Lie Type: Conjugacy Classes and Complex Characters", Wiley, Chichester et al, 1985.

[Cas] L. Casian, Kazhdan-Lusztig conjecture in the negative level case (Kac-Moody algebras of affine type), Preprint.

[De] U. Dempwolff, Linear groups with large cyclic subgroups and translation planes, *Rend. Sem. Mat. Univ. Padova* **77** (1987), 69 – 113.

[DZ] J. Dixon and A. E. Zalesskii, Finite primitive linear groups of prime degree, *J. London Math. Soc.* (2) **57** (1998), 126 – 134.

[El] N. D. Elkies, Mordell-Weil lattices in characteristic 2: I. Construction and first properties, *Int. Math. Res. Notices* (1994) **8**, 343 – 360.

[F1] W. Feit, The current situation in the theory of finite simple groups, Proc. Int. Congr. Math., (1970), vol. I, pp. 55 – 93.

[F2] W. Feit, On finite linear groups in dimension at most 10, in: Proceedings of the Conference on Finite Groups (Univ. Utah, Park City, Utah, 1975), Academic Press, 1976, pp. 397 – 407.

[F3] W. Feit, Finite linear groups and theorems of Minkowski and Schur, *Proc. Amer. Math. Soc.* **125** (1997), 1259 – 1262.

[Fer] P. Ferguson, Complex linear groups of degree at most $v-3$, *J. Algebra* **93** (1985), 246 – 252.

[FLT 1] P. Fleischmann, W. Lempken, and Pham Huu Tiep, Finite p'-semiregular groups, *J. Algebra* **188** (1997), 547 – 579.

[FLT 2] P. Fleischmann, W. Lempken, and Pham Huu Tiep, The p-intersection subgroups in quasi-simple groups and almost simple groups, *J. Algebra* (to appear).

[FLT 3] P. Fleischmann, W. Lempken, and Pham Huu Tiep, The primitive p-Frobenius groups, *Proc. Amer. Math. Soc.* (to appear).

[FLZ] P. Fleischmann, W. Lempken, and A. E. Zalesskii, Finite linear groups over a field of characteristic 2 containing an element of order 3 with 2 distinct eigenvalues (in preparation).

[Gow] R. Gow, Unimodular integral lattices associated with the basic spin representations of $2A_n$ and $2S_n$, *Bull. London Math. Soc.* **21** (1989), 257 – 262.

[GK] R. Gow and A. S. Kleshchev, Connections between the representations of the symmetric group and the symplectic group in characteristic 2, (in preparation).

[Gr] B. H. Gross, Group representations and lattices, *J. Amer. Math. Soc.* **3** (1990), 929 – 960.

[GMS] R. M. Guralnick, K. Magaard, and J. Saxl, Small cross characteristic representations for odd characteristic symplectic groups, Preprint, 1997.

[GPPS] R. M. Guralnick, T. Pentilla, C. Praeger, and J. Saxl, Linear groups with orders having certain primitive prime divisors, *Proc. London Math. Soc.* (to appear).

[GT 1] R. M. Guralnick and Pham Huu Tiep, Low-dimensional representations of special linear groups in cross characteristics, *Proc. London Math Soc.* (to appear).

[GT 2] R. M. Guralnick and Pham Huu Tiep, (in preparation).

[GW] R. M. Guralnick and R. Wiegand, Galois groups and the multiplicative structure of field extensions, *Trans. Amer. Math. Soc.* **331** (1992), 563 – 574.

[He 1] C. Hering, Transitive linear groups and linear groups which contain irreducible subgroups of prime order, *Geometriae Dedicata* **2** (1974), 425 – 460.

[He 2] C. Hering, Transitive linear groups and linear groups which contain irreducible subgroups of prime order. II, *J. Algebra* **93** (1985), 151 – 164.

[HM] G. Hiss and G. Malle, (in preparation).

[Ho 1] C. Y. Ho, Chevalley groups of odd characteristic as quadratic pairs, *J. Algebra* **41** (1976), 202 – 211.

[Ho 2] C. Y. Ho, On the quadratic pairs, *J. Algebra* **43** (1976), 338 – 358.

[Hof 1] C. Hoffman, Projective representations in cross characteristics for some classical groups, Preprint, 1997.

[Hof 2] C. Hoffman, Low dimensional projective representations for some exceptional finite groups of Lie type, Preprint, 1997.

[Is] I. M. Isaacs, (in preparation).

[Ja 1] J. C. Jantzen, Zur Reduktion modulo p der Charaktere von Deligne und Lusztig, *J. Algebra* **70** (1981), 452 – 474.

[Ja 2] J. C. Jantzen, "Representations of Algebraic Groups", Academic Press, Boston et al, 1987.

[JM 1] G. James and A. Mathas, A q-analogue of the Jantzen-Schaper theorem, *Proc. London Math. Soc.* (3) **74** (1997), 1 – 37.

[JM 2] G. James and A. Mathas, The irreducible Specht modules in characteristic 2, Preprint, 1998.

[JLPW] C. Jansen, K. Lux, R. A. Parker, and R. A. Wilson, "An ATLAS of Brauer Characters", Oxford University Press, Oxford, 1995.

[Ka] W. M. Kantor, Subgroups of classical groups generated by long root elements, *Trans. Amer. Math. Soc.* **248** (1979), 347 – 379.

[KT] M. Kashiwara and T. Tanisaki, Kazhdan-Lusztig conjecture for affine Lie algebras I, II, *Duke Math. J.* **77** (1995), 21 – 62; **84** (1996), 771 – 813,

[KaL] D. Kazhdan and G. Lusztig, Tensor structures arising from affine Lie algebras. I, II, III, IV, J. Amer. Math. Soc. **6** (1993), 905 – 947, 949 – 1011; **7** (1994), 335 – 381, 383 – 453.

[Kl] P. B. Kleidman, The subgroup structure of some finite simple groups, Ph. D. Thesis, Imperial College, London, 1986.

[KL] P. B. Kleidman and M. W. Liebeck, "The Subgroup Structure of the Finite Classical Groups", London Math. Soc. Lecture Note Ser. no. 129, Cambridge University Press, 1990.

[KP] A. S. Kleshchev and A. A. Premet, The globally irreducible representations of symmetric groups, *Proc. Amer. Math. Soc.* (to appear).

[Ko 1] A. S. Kondrat'ev, Modular representations of small degrees of finite groups of Lie type, Preprint, 1989.

[Ko 2] A. S. Kondrat'ev, Modular representations of degree ≤ 27 of finite quasi-simple groups of alternating and sporadic types, *Proc. Inst. Math. Mech. Ural Branch Acad. Sci. Russia* **1** (1992), 20 – 49.

[Kor 1] A. V. Korljukov, Finite linear groups generated by quadratic elements of order 4, *Vestsi Akad. Navuk BSSR, Ser. Fiz.-Mat. Navuk* (1986), no. 4, 38 – 40.

[Kor 2] A. V. Korljukov, Finite linear groups that are generated by quadratic elements of order 4 and 3, *Problems in Algebra* (1989), no. 4, 146 – 151.

[LS] V. Landázuri and G. Seitz, On the minimal degrees of the projective representations of the finite Chevalley groups, *J. Algebra* **32** (1974), 418 – 443.

[Le] H. S. Leonard, Finite complex linear groups of small degree, In: "Finite Simple Groups", Academic Press, N. Y., 1977, pp. 432 – 445.

[Li] M. W. Liebeck, The affine permutation groups of rank three, *Proc. London Math. Soc.* **54** (1987), 477 – 516.

[Lu] F. Lübeck, Smallest degrees of complex characters of exceptional groups of Lie type, Preprint, 1997.

[Lus 1] G. Lusztig, Some problems in the representation theory of finite Chevalley groups, Proc. Symp. Pure Math. **37** (1980), pp. 313 – 318.

[Lus 2] G. Lusztig, "Characters of Reductive Groups over a Finite Field", Annals of Math. Studies, vol. 107, Princeton University Press, 1984.

[Lus 3] G. Lusztig, Monodromic systems on affine flag manifolds, *Proc. Roy. Soc. London Ser. A* **445** (1994), no. 1923, 231 – 246.

[M1] J. McLaughlin, Some subgroups of $SL_n(2)$, *Illinois J. Math.* **13** (1969), 108 – 115.

[M2] J. McLaughlin, Some groups generated by transvecions, *Arch. Math.* **18** (1967), 364 – 368.

[Me] U. Meierfrankenfeld, A characterization of the spin module for $2 \cdot A_n$, *Arch. Math.* **57** (1991), 238 – 246.

[MS 1] U. Meierfrankenfeld and G. Stroth, Quadratic GF(2)-modules for sporadic simple groups and alternating groups, *Comm. Algebra* **18** (1990), 2099 – 2139.

[MS 2] U. Meierfrankenfeld and G. Stroth, On quadratic GF(2)-modules for Chevalley groups in odd characteristic, *Arch. Math.* **55** (1990), 105 – 110.

[MT] K. Magaard and Pham Huu Tiep, Irreducible tensor products of representations of finite groups of Lie type, (in preparation).

[MP] O. Mathieu and O. Papadopoulo, (in preparation).

[Mu] G. Mullineux, Representations of A_n in characteristic 2 with the 3-cycles fixed-point-free, *Quart. J. Math.* **29** (1978), 199 – 212.

[OZ] F. van Oystaeyen and A. E. Zaleskii, Finite groups over arithmetic rings and globally irreducible representations, *J. Algebra* (to appear).

[Ra] R. Rasala, On the minimal degrees of characters of $\mathbf{S}_n$, *J. Algebra* **45** (1977), 132 – 181.

[Ro 1] G. R. Robinson, Some remarks on reduction (mod p) of complex linear groups, *J. Algebra* **83** (1983), 477 – 483.

[Ro 2] G. R. Robinson, On elements with restricted eigenvalues in linear groups, *J. Algebra* **178** (1995), 635 – 642.

[Sch] I. Schur, "Collected Works", vol. 1, Springer-Verlag, Berlin et al, 1957.

[Se] G. M. Seitz, The maximal subgroups of classical algebraic groups, Memoirs Amer. Math. Soc. **67** (1987), no. 365.

[SZ] G. M. Seitz and A. E. Zalesskii, On the minimal degrees of projective representations of the finite Chevalley groups. II, *J. Algebra* **158** (1993), 233 – 243.

[Sh] T. Shioda, Mordell-Weil lattices and sphere-packings, *Amer. J. Math.* **113** (1991), 931 – 948.

[Su 1] I. D. Suprunenko, Irreducible representations of simple algebraic groups containing matrices with big Jorgan blocks. II, *Proc. London Math. Soc.* (3) **71** (1998), 281 – 332.

[Su 2] I. D. Suprunenko, The minimal polynomials of elements of order p in irreducible representations of Chevalley groups over a field of characteristic p, *Siber. Advances in Math.* **6** (1996), no. 4, 97 – 150.

[Su 3] I. D. Suprunenko, p-large representations and asymptotics, this volume.

[SZ 1] I. D. Suprunenko and A. E. Zalesskii, Representations of dimensions $(p^n \pm 1)/2$ of the symplectic group of degree $2n$ over a field of characteristic p, *Vestsi Acad. Sci. BSSR, Ser. Fiz.-Mat. Navuk*, no. 6 (1987), 9 – 15.

[SZ 2] I. D. Suprunenko and A. E. Zalesskii, Irreducible representations of finite classical groups containing matrices with simple spectra, *Comm. Algebra* **26** (1998), 863 – 888.

[SZ 3] I. D. Suprunenko and A. E. Zalesskii, Irreducible representations of finite exceptional groups of Lie type containing matrices with simple spectra, (in preparation).

[Sw] R. G. Swan, Induced representations and projective modules, Ann. of Math. **71** (1960), 552 – 578.

[Ta] T. Tanisaki, (in preparation).

[Th 1] J. G. Thompson, Quadratic pairs, in: Actes du Congrès International des Mathématiciens (Nice 1970), Gauthier-Villars, Paris (1971), Tome 1, pp. 375 – 376.

[Th 2] J. G. Thompson, Finite groups and even lattices, *J. Algebra* **38** (1976), 523 – 524.

[Th 3] J. G. Thompson, A simple subgroup of $E_8(3)$, in: *Finite Groups Symposium*, N. Iwahori ed., Japan Soc. for Promotion of Science, 1976, pp. 113 – 116.

[T1] Pham Huu Tiep, Globally irreducible representations of the finite symplectic group $Sp_4(q)$, *Comm. Algebra* **22** (1994), 6439 – 6457.

[T2] Pham Huu Tiep, Basic spin representations of $2\mathbf{S}_n$ and $2\mathbf{A}_n$ as globally irreducible representations, *Archiv Math.* **64** (1995), 103 – 112.

[T3] Pham Huu Tiep, Globally irreducible representations of finite groups and integral lattices, *Geometriae Dedicata* **64** (1997), 85 – 123.

[T4] Pham Huu Tiep, Weil representations as globally irreducible representations, *Math. Nachr.* **184** (1997), 313 – 327.

[T5] Pham Huu Tiep, Globally irreducible representations of $SL_3(q)$ and $SU_3(q)$, *Israel J. Math.* (to appear).

[TZ 1] Pham Huu Tiep and A. E. Zalesskii, Minimal characters of the finite classical groups, *Comm. Algebra* **24** (1996), 2093 – 2167.

[TZ 2] Pham Huu Tiep and A. E. Zalesskii, Some characterizations of the Weil representations of the symplectic and unitary groups, *J. Algebra* **192** (1997), 130 – 165.

[Wa] D. B. Wales, Some projective representations of S_n, *J. Algebra* **61** (1979), 37 – 57.

[Wag 1] A. Wagner, Groups generated by elations, *Abh. Math. Sem. Univ. Hamburg* **41** (1974), 190 – 205.

[Wag 2] A. Wagner, The faithful linear representations of least degree of S_n and A_n over a field of characteristic 2, *Math. Z.* **151** (1976), 127 – 137.

[Wag 3] A. Wagner, The faithful linear representations of least degree of S_n and A_n over a field of odd characteristic, *Math. Z.* **154** (1977), 103 – 114.

[Wag 4] A. Wagner, An observation on the degrees of projective representations of the symmetric and alternating groups over an arbitrary field, *Arch. Math.* **29** (1977), 583 – 589.

[Wag 5] A. Wagner, Determination of the finite primitive reflection groups over an arbitrary field of characteristic not 2. I, II, III, *Geom. Dedicata* **9** (1980), 239 – 253; **10** (1981), 191 – 203, 475 – 523.

[Wi] R. Wilson, Certain representations of Chevalley groups over $GF(2^n)$, *Comm. Algebra* **3** (1975), 319 – 364.

[Z1] A. E. Zaleskiĭ, Linear groups, *Uspekhi Mat. Nauk* **36** (1981), 57 – 107 (Russian); English transl.: *Russ. Math. Surv.* **36** (1981), 63 – 128.

[Z2] A. E. Zaleskiĭ, Spectra of elements of order p in representations of Chevalley groups of characteristic p *Vestsi Akad. Navuk BSSR, Ser. Fiz.-Mat. Navuk*, 1986, no. 6, 20 – 25 (Russian).

[Z3] A. E. Zalesskiĭ, Eigenvalues of matrices of complex representations of finite groups of Lie type, in: "Algebra, Some Current Trends", Lecture Notes in Math., vol. **1352** (1988), pp. 206 – 218.

[Z4] A. E. Zalesskiĭ, The eigenvalue 1 of matrices in complex representations of finite Chevalley groups, in: "Galois Theory, Rings, Algebraic Groups and Their Applications", *Trudy Mat. Inst. AN SSSR*, **183**, (1990), 96 – 105 (Russian); English transl.: *Proc. Steklov Inst. Math.* **183** (1990).

[Z5] A. E. Zalesskiĭ, Linear groups, in: "Encyclopaedia of Mathematical Sciences", vol. 37: Algebra IV, Springer-Verlag, Berlin-Heidelberg, 1993.

[Z6] A. E. Zalesskiĭ, Minimal polynomials and eigenvalues of p-elements in representations of quasi-simple groups with a cyclic Sylow p-subgroup, *J. London Math. Soc.* (to appear).

[Z7] A. E. Zalesskiĭ, The eigenvalues of matrices of projective complex representations of alternating groups, *Vestsi Akad. Navuk Belarusi, Ser. Fiz.-Mat.-Inform. Navuk*, 1996, no. 3, 41 – 43.

[Z8] A. E. Zalesskiĭ, Minimal polynomials of elements of prime order in complex irreducible representations of quasi-simple groups, (in preparation).

[ZS] A. E. Zalesskiĭ and V. S. Serezjkin, Finite linear groups generated by reflections, *Izv. Akad. Nauk SSSR, Ser. Mat.*, **44** (1980), 1279 – 1307.

[Za] H. Zassenhaus, Über endliche Fastkörper, *Abd. Math. Sem. Univ. Hamburg* **11** (1936), 187 – 220.

[Zh 1] J. Zhang, Complex linear groups of degree at most $p - 1$, *Contemp. Math.* **82** (1989), 243 – 254.

[Zh 2] J. Zhang, On linear groups of degree at most $|P| - 1$, *J. Algebra* **143** (1991), 307 – 314.

[Zs] K. Zsigmondy, Zur Theorie der Potenzreste, *Monatsh. Math. Phys.* **3** (1892), 265 – 284.

TABLE III. The refined Landázuri-Seitz lower bounds for $\ell(G(q))$

$G(q)$	Bound	Exceptions
$PSL_2(q)$	$(q-1)/\gcd(2,q-1)$	$\begin{cases} \ell(PSL_2(4)) = 2, \\ \ell(PSL_2(9)) = 3. \end{cases}$
$PSL_n(q)$, $n \geq 3$	$(q^n-1)/(q-1)-2$	$\begin{cases} \ell(PSL_3(2)) = 2, \\ \ell(PSL_3(4)) = 4, \\ \ell(PSL_4(2)) = 7, \\ \ell(PSL_4(3)) = 26. \end{cases}$
$PSp_{2n}(q)$, $n \geq 2$	$\begin{cases} (q^n-1)/2, & 2 \not\| q \\ (q^n-1)(q^n-q)/(2(q+1)), & 2 \| q \end{cases}$	$\ell(Sp_4(2)') = 2$
$PSU_n(q)$, $n \geq 3$	$\begin{cases} (q^n-q)/(q+1), & 2 \not\| n \\ (q^n-1)/(q+1), & 2 \| n \end{cases}$	$\begin{cases} \ell(PSU_4(2)) = 4, \\ \ell(PSU_4(3)) = 6. \end{cases}$
$P\Omega^+_{2n}(q)$, $n \geq 4$	$\begin{cases} (q^n-1)(q^{n-1}+q)/(q^2-1)-2, & q > 3 \\ (q^n-1)(q^{n-1}-1)/(q^2-1), & q \leq 3 \end{cases}$	$\ell(P\Omega^+_8(2)) = 8$
$P\Omega^-_{2n}(q)$, $n \geq 4$	$(q^n+1)(q^{n-1}-q)/(q^2-1)-1$	$\begin{cases} \ell(P\Omega^-_8(2)) \geq 32 \\ \ell(P\Omega^-_8(4)) \geq 1026 \\ \ell(P\Omega^-_{10}(2)) \geq 151 \\ \ell(P\Omega^-_{10}(3)) \geq 2376 \end{cases}$
$\Omega_{2n+1}(q)$, $n \geq 3, 2 \not\| q$	$\begin{cases} (q^{2n}-1)/(q^2-1)-2, & q \neq 3 \\ (q^n-1)(q^n-q)/(q^2-1), & q = 3 \end{cases}$	$\ell(\Omega_7(3)) = 27$
$E_6(q)$	$q(q^6+q^3+1)(q^4+1)-1$	
$E_7(q)$	$q(q^{14}-1)(q^4-q^2+1)/(q^2-1)-2$	
$E_8(q)$	$q(q^{12}+1)(q^{10}+1)(q^6+1)-3$	
$F_4(q)$	$\begin{cases} q^6(q^2-1), & 2 \not\| q \\ q^7(q^3-1)(q-1)/2, & 2 \| q \end{cases}$	$\ell(F_4(2)) \geq 44$
${}^2E_6(q)$	$q^9(q^2-1)$	
$G_2(q)$	$\begin{cases} q(q^2-1), & 3 \not\| q \\ q^2(q^2+1), & 3 \| q \end{cases}$	$\begin{cases} \ell(G_2(3)) = 14, \\ \ell(G_2(4)) = 12. \end{cases}$
${}^3D_4(q)$	$q^3(q^2-1)$	
${}^2F_4(q)$	$q^4(q-1)\sqrt{q/2}$	
${}^2B_2(q)$	$(q-1)\sqrt{q/2}$	$\ell({}^2B_2(8)) = 8$
${}^2G_2(q)$	$q(q-1)$	

TABLE IV. Modular characters of the first three degrees of small groups $PSL_n(q)$

L	(d_1, N_1)	(d_2, N_2)	(d_3, N_3)	Conditions
$PSL_2(q)$, $2 \not| q$, $q \neq 3, 5, 9$	$\left(\frac{q-1}{2}, 2\right)$ $\left(\frac{q-1}{2}, 2\right)$ $\left(\frac{q-1}{2}, 2\right)$ $\left(\frac{q-1}{2}, 2\right)$ $\left(\frac{q-1}{2}, 2\right)$	$\left(q-1, \frac{q-1}{4}\right)$ $\left(q-1, \frac{s-1}{2}\right)$ $\left(\frac{q+1}{2}, 2\right)$ $\left(\frac{q+1}{2}, 2\right)$ $\left(\frac{q+1}{2}, 2\right)$	$\left(q+1, \frac{s-1}{2}\right)$ $\left(q+1, \frac{q-3}{4}\right)$ $\left(q-1, \frac{q-1}{2}\right)$ $\left(q-1, \frac{s}{2}\right)$ $\left(q-1, \frac{q-1}{2}\right)$	$2 = p \mid \frac{q-1}{2}$ $2 = p \mid \frac{q+1}{2}$ $2 \neq p \mid (q-1)$ $2 \neq p \mid (q+1)$ $p = 0$
$PSL_2(q)$, $2 \mid q$, $q \geq 8$	$\left(q-1, \frac{q}{2}\right)$ $\left(q-1, \frac{s+1}{2}\right)$ $\left(q-1, \frac{q}{2}\right)$	$(q, 1)$ $\left(q+1, \frac{q}{2}-1\right)$ $(q, 1)$	$\left(q+1, \frac{s-1}{2}\right)$ $-$ $\left(q+1, \frac{q-2}{2}\right)$	$p \mid (q-1)$ $p \mid (q+1)$ $p = 0$
$PSL_2(5)$	$(2, 2)$	$(4, 1)$	$-$	$p = 2$
$PSL_2(4)$	$(2, 2)$ $(2, 1)$ $(2, 2)$	$(3, 2)$ $(3, 1)$ $(3, 2)$	$(4, 1)$ $(4, 1)$ $(4, 2)$	$p = 3$ $p = 5$ $p = 0$
$PSL_2(9)$	$(3, 4)$ $(3, 2)$ $(3, 4)$	$(4, 2)$ $(4, 2)$ $(4, 2)$	$(8, 2)$ $(5, 2)$ $(5, 2)$	$p = 2$ $p = 5$ $p = 0$
$PSL_3(2)$	$(3, 2)$ $(2, 1)$	$(4, 2)$ $(3, 1)$	$(6, 3)$ $(4, 1)$	$p = 0, 3$ $p = 7$
$PSL_3(4)$	$(4, 6)$ $(6, 2)$ $(6, 2)$ $(6, 2)$	$(6, 3)$ $(8, 6)$ $(8, 12)$ $(8, 12)$	$(8, 12)$ $(10, 6)$ $(10, 3)$ $(10, 6)$	$p = 3$ $p = 5$ $p = 7$ $p = 0$
$PSL_4(2)$	$(7, 1)$ $(7, 1)$	$(8, 1)$ $(8, 1)$	$(13, 1)$ $(14, 1)$	$p = 3, 5$ $p = 0, 7$
$PSL_4(3)$	$(26, 2)$ $(26, 2)$ $(26, 2)$	$(38, 1)$ $(38, 1)$ $(39, 1)$	$(208, 2)$ $(40, 1)$ $(40, 1)$	$p = 2$ $p = 5$ $p = 0, 13$
$PSL_6(2)$	$(62 - \kappa_6, 1)$ $(62 - \kappa_6, 1)$	$(217, 1)$ $(217, 1)$	$(526 + \kappa_6, 1)$ $(588 - \kappa_5, 1)$	$p = 3, 5$ $p = 0, 7, 31$
$PSL_6(3)$	$(363 - \kappa_6, 1)$	$(364, 1)$	$(6292, 2)$	

In Table IV, d_i is the ith smallest degree of nontrivial projective irreducible representations of $PSL_n(q)$ in characteristic r, and N_i is the total number of such representations of degree d_i. Furthermore, s is defined to be $(q-1)_{2'}$ if $2 = r \mid \frac{q-1}{2}$, $(q+1)_{2'}$ if $2 = r \mid \frac{q+1}{2}$, $(q-1)_{r'}$ if $2 \neq r \mid (q-1)$, and $(q+1)_{r'}$ if $2 \neq r \mid (q+1)$.