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FIXED VECTORS FOR ELEMENTS IN MODULES FOR ALGEBRAIC GROUPS

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To Boris Isakovich Plotkin with best wishes
on the occasion of his 80th birthday

We provide conditions guaranteeing that a given element of a simple algebraic group G has a fixed vector in every nonzero G -module, and deduce similar results for finite Chevalley groups.

Keywords: Algebraic groups; modules; fixed vectors.

1. Introduction

For simple algebraic groups, sufficient conditions which guarantee that a given element has nonzero fixed vectors in all nonzero modules are provided. For groups of different types classes of elements that have such vectors in every nonzero module are found. Similar results on representations of finite Chevalley groups in defining characteristic are obtained.

In what follows G is a simply connected simple algebraic group of rank r over an algebraically closed field F of characteristic $p \geq 0$, ω_i , $1 \leq i \leq r$, are the fundamental weights of G labeled in the standard way as in [1]. We assume that $p \neq 2$ and $r > 2$ for $G = B_r(F)$, $r > 1$ for $G = C_r(F)$, and $r > 3$ for $G = D_r(F)$. All G -modules (and representations) mentioned in the article are supposed to be rational and finite dimensional. Throughout the text "a fixed vector" means "a nonzero fixed vector" when a nonzero module is considered.

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Set $n = r + 1$ for $G = A_r(F)$, $2r + 1$ for $G = B_r(F)$, and $2r$ for $G = C_r(F)$ or $D_r(F)$. For these types G is isomorphic to one of the classical groups of degree n over F ; $G \cong SL_n(F)$ for $G = A_r(F)$, $Spin_n(F)$ for $G = B_r(F)$ and $D_r(F)$, and $Sp_n(F)$ for $G = C_r(F)$. Recall that $Spin_n(F)$ is a central extension of the group $SO_n(F)$ by a subgroup of order 2. The natural n -dimensional $SL_n(F)$ -, $SO_n(F)$ -, or $Sp_n(F)$ -module is called the standard module for the relevant group G .

The conditions meant in the first phrase require an element considered to have fixed vectors in certain special irreducible modules. These modules are indicated below. Define the set $\mathcal{M} = \mathcal{M}(G)$ as follows:

$$\mathcal{M} = \begin{cases} \{\omega_1, \dots, \omega_r\} & \text{for } G = A_r(F); \\ \{\omega_r\} & \text{for } G = B_r(F); \\ \{\omega_1\} & \text{for } G = C_r(F); \\ \{\omega_1, \omega_{r-1}, \omega_r\} & \text{for } G = D_r(F); \\ \{\omega_1, \omega_6\} & \text{for } G = E_6(F); \\ \{\omega_7\} & \text{for } G = E_7(F); \\ \emptyset & \text{for } G = E_8(F), F_4(F), \text{ or } G_2(F). \end{cases} \quad (1)$$

The weights in \mathcal{M} are called microweights (or miniscule weights) (see [2, Ch. VIII, §7, Propositions 5 and 6]). We shall call an irreducible G -module (an irreducible representation of G) a microweight module (representation) if its highest weight is a microweight.

Theorem 1. *Assume that $p \neq 2$ for $G = B_r(F)$, $C_r(F)$, or $G_2(F)$. Let $g \in G$. If $\mathcal{M}(G) \neq \emptyset$, suppose in addition that g has a fixed vector in every microweight module. Then g has such vector in every G -module.*

It is not difficult to find out whether an element of a given Jordan form in the standard G -module fixes vectors in microweight modules since these modules are well known.

A starting point for this theorem is the following observation used in [13]: if $G = E_8(F)$, $F_4(F)$, or $G_2(F)$ with $p \neq 2$ in the latter case, then every element of G fixes a nonzero vector in every nonzero G -module; the same holds for finite groups $E_8(q)$, $F_4(q)$, $G_2(q)$ (with p odd in this case), and ${}^2G_2(q)$. Therefore these groups are not considered in what follows.

We show that elements from certain subgroups of G have fixed vectors in all microweight modules. Then Theorem 1 implies that they have such vectors in all irreducible modules. Arguments of this kind lead to the following results.

Corollary 2. *Let $G = A_r(F)$ and an element $g \in G$ fix every vector in a subspace of dimension $\geq (n - 1)/2$ of the standard G -module. Then g has a fixed vector in every G -module.*

Corollary 3. *Let $p \neq 2$, $G = C_r(F)$ and an element $g \in G$ fix a nonzero vector in the standard module. Then g has a fixed vector in every G -module.*

Observe that Corollary 2 is not valid for the spinor groups $B_r(F)$ and $D_r(F)$ and Corollary 3 does not hold in characteristic 2 (see Section 4). We call a subgroup in G an *SR*-subgroup if it is generated by the root subgroups associated with some simple roots of G and the roots opposite to these roots. Corollary 2 implies that for $G = A_r(F)$ an element lying in an *SR*-subgroup of type A_t with $t \leq (n-1)/2$ fixes a vector in every G -module. Corollary 3 forces that this holds for elements of $C_r(F)$ lying in the *SR*-subgroup of type C_{r-1} (for $r = 2$ we mean the subgroup generated by the root subgroups associated with the roots α_2 and $-\alpha_2$).

Corollary 4. *Let $G = B_r(F)$ and an element $g \in G$ lie in the *SR*-subgroup of type A_{r-1} or $G = D_r(F)$ and g lie in an *SR*-subgroup of type A_{r-2} . Then g has a fixed vector in every G -module.*

Corollary 5. *Let $G = E_6(F)$ and an element $g \in G$ lie in an *SR*-subgroup of type D_5 or $G = E_7(F)$ and g lie in the *SR*-subgroup of type E_6 . Then g has a fixed vector in every G -module.*

A fundamental theorem of Steinberg [11, Theorem 43] connects irreducible representations of finite Chevalley groups in defining characteristic with representations of semisimple algebraic groups. This allows one to obtain analogs of Corollaries 2–5 for these finite groups. In particular, the following is proved.

Corollary 6. *Let $p > 0$, $q = p^s$, and $H = SL_n(q)$ or $SU_n(q^2)$. Assume that an element $h \in H$ fixes every vector in a subspace of dimension $\geq (n-1)/2$ of the natural H -module. Then h has a fixed vector in every H -module over a field of characteristic p .*

Corollary 7. *Let $p > 2$, $q = p^s$, and $H = Sp_n(q)$. Assume that an element $h \in H$ fixes a nonzero vector in the natural H -module. Then h has a fixed vector in every H -module over a field of characteristic p .*

More results of a similar kind are contained in Sections 4 and 5.

The study of fixed points and fixed point free elements for actions of finite groups dates back to C. Jordan. Recently R. Guralnick and P.H. Tiep [4] have investigated the proportions of elements with nonzero fixed points for actions of finite simple groups of Lie type on their modules in defining characteristic. They classified such groups where all elements have fixed points (vectors) in every module in defining characteristic (the unisingular groups in their terminology) ([4, Theorem 1.3]). For a finite simple group Γ of Lie type and a Γ -module M in defining characteristic denote by $s(M)$ the number of semisimple elements in Γ that have a fixed vector in M and set $\lambda(M) = s(M)/|\Gamma|$. Put $\lambda(\Gamma) = \min\{\lambda(M)\}$. In [3] a recognition algorithm for black box groups is proposed that works well for unisingular groups and for groups with big $\lambda(\Gamma)$ and is not very effective for groups with small $\lambda(\Gamma)$. The estimates obtained in [4] show that the latter case is typical. So it makes sense to elaborate some approaches to finding elements having fixed points in all modules in defining characteristic (or large classes of modules) for arbitrary finite Chevalley groups.

Corollary 6 implies a positive answer to a question posed by V.D. Mazurov to the Group Theory Forum in July 2005. This question was an initial motivation for our article. This corollary was used by V.D. Mazurov and A.V. Zavarnitsine in [7] for finding orders of elements in coverings of $L_n(q)$ and $U_n(q)$.

Our work can also be regarded as a contribution to the study of interrelations between actions of special elements or subgroups on various modules for the classical groups and on the standard module. In this connection we would like to mention the following result of Hartley [5]. Let K be a field, V be the standard $GL_n(K)$ -module, and M be an irreducible finite-dimensional $GL_n(K)$ -module over K affording a polynomial representation if K is infinite. Assume that the dimension of M is greater than 1. For a finite subgroup $H \subset GL_n(K)$ suppose that $V \cong KH \oplus V_1$ where V_1 is an H -module. Then M contains a nonzero free KH -module, except some special cases for elementary abelian 2-groups H . In this paper we show that for cyclic subgroups in some cases the existence of a large enough fixed subspace in the standard module guarantees that of nonzero fixed vectors in an arbitrary module.

We wish to make the exposition more accessible to readers who are not deeply familiar with the representation theory of algebraic groups. For this, the case of special linear groups is discussed separately in Section 3 with minimum use of the general theory.

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2. Notation, preliminary results, and the proof of Theorem 1

Throughout the text \mathbb{C} is the complex number field, \mathbb{Z} and \mathbb{Z}^+ are the sets of integers and nonnegative integers, respectively, G , p , and F are such as in the Introduction, \mathbb{F}_q is the finite field of order $q = p^l$, L is the Lie algebra of G , and W is the Weyl group of G . Denote by Ω and R the weight and root systems of G , respectively, Ω^+ is the set of dominant weights (with respect to a fixed maximal torus of G). In what follows Irr is the set of irreducible representations of G (considered up to equivalence), $\Omega(\varphi)$ ($\Omega(M)$) is the weight system of a representation φ (a module M); if φ or M is irreducible, $\omega(\varphi)$ or $\omega(M)$ denotes the highest weight of φ or M ; $M(\lambda)$ is the irreducible G -module with highest weight λ ; $M|S$ is the restriction of a module M to a subgroup $S \subset G$; α_i , $1 \leq i \leq r$, are the simple roots of G labeled in the standard way as in [1], $\mathcal{X}_{\pm i} \subset G$ and $x_{\pm i}(t) \in \mathcal{X}_{\pm i}$, $t \in F$, are the root subgroups and the root elements associated with the roots $\pm\alpha_i$; $G(i_1, \dots, i_k)$ is the subgroup generated by the subgroups $\mathcal{X}_{i_1}, \mathcal{X}_{-i_1}, \dots, \mathcal{X}_{i_k}, \mathcal{X}_{-i_k}$. For a weight $\lambda \in \Omega$ and a root $\alpha \in R$ denote by $\langle \lambda, \alpha \rangle$ the value of λ on α (the canonical pairing in the sense of [11, §3]). One has $\langle \lambda, \alpha \rangle = \lambda(H_\alpha)$ for the element $H_\alpha \in L$ associated with a root α and $\lambda(h_\alpha(t)) = t^{\langle \lambda, \alpha \rangle}$ for a semisimple element $h_\alpha(t) \in G$ associated with α . Naturally, the weight and root systems and root elements are defined with respect to a fixed maximal torus of G . If $p > 0$, denote by Fr the Frobenius morphism of G associated

with raising the elements of F to the p th power. In this case if $\varphi \in \text{Irr}$, then $\varphi_{\mathbb{C}}$ is the irreducible representation with highest weight $\omega(\varphi)$ of the simply connected group over \mathbb{C} with the same root system as G . For an irreducible G -module M define the module $M_{\mathbb{C}}$ in a similar way. Recall that for $p > 0$ a representation $\varphi \in \text{Irr}$ is called p -restricted if $\omega(\varphi) = \sum_{i=1}^r a_i \omega_i$ and all $a_i < p$. Throughout the text $\text{Irr}_p \subset \text{Irr}$ is the set of p -restricted representations (for $p > 0$). All finite Chevalley groups considered in the article are the fixed subgroups for Frobenius morphisms of G that preserve some maximal torus T . One can transfer the definition of an SR -subgroup (with respect to this torus) given in the Introduction to the latter subgroups in the following way: SR -subgroups in a subgroup Γ of this kind are the intersections of such subgroups in G with Γ . This definition may seem strange since for twisted groups Γ some SR -subgroups in Γ cannot be regarded as finite analogs of relevant SR -subgroups in G , but it works well for our purposes. One can identify the weight and root systems of G with such systems for the complex simple Lie algebra of the same type (see [11, §5]). The following theorem allows us to apply well-known facts on the weight structure of irreducible modules for simple algebraic groups (and Lie algebras) in characteristic 0 for analyzing modular representations of such groups in positive characteristic. It is well known that in characteristic 0 the weight multiplicities of irreducible G - and L -modules with the same highest weight coincide.

Theorem 8. [9] *Let $p > 0$ and φ be a p -restricted irreducible representation of G . Assume that $p > 2$ for $G = B_r(F)$, $C_r(F)$, and $F_4(F)$, and $p > 3$ for $G = G_2(F)$. Then $\Omega(\varphi) = \Omega(\varphi_{\mathbb{C}})$.*

For groups of type A_r Theorem 8 was proved earlier in [12] where the relevant conjecture for all types was formulated. Theorem 8 was applied in [13] to obtain an information on eigenvalue 1 in complex representations of finite Chevalley groups, see comments in the Introduction.

Lemma 9. [2, Ch. VIII, §7, Propositions 5 and 6] *The microweights are nonzero weights $\omega \in \Omega^+$ with the following property: there is no dominant weight $\mu \neq \omega$ with $\omega - \mu$ equal to a linear combination of simple roots of G with nonnegative integer coefficients.*

A weight is called radical if it is an integer combination of roots. Let Ω_0 be the set of radical weights and denote by Ω_0^+ the subset of Ω_0 formed by linear combinations of simple roots with non-negative coefficients.

Proposition 10. *Let $p = 0$ or satisfy the assumptions of Theorem 8. Assume that $\varphi \in \text{Irr}_p$ if $p > 0$. Set $\omega = \omega(\varphi)$. Then $0 \in \Omega(\varphi)$ if $\omega \in \Omega_0$, otherwise $\Omega(\varphi)$ contains $\Omega(\rho)$ for a microweight representation $\rho \in \text{Irr}$. If $\omega \in \Omega_0$ and $\omega \neq 0$, the set $\Omega(\varphi)$ contains all short roots (all roots if all roots of G have the same length).*

Proof. By Theorem 8, it suffices to consider the case where $p = 0$. Observe that $\Omega^+ \cap \Omega_0 \subset \Omega_0^+$ since $\langle \alpha_i, \alpha_j \rangle \leq 0$ for $i \neq j$ and $\langle \alpha_i, \alpha_i \rangle > 0$. Choose the minimal

weight μ with $\mu \in \Omega^+$ and $\omega - \mu \in \Omega_0^+$. Then $\mu \in \Omega(\varphi)$ by [2, Ch.VIII, §7.2, Corollary 2 to Proposition 3]. If $\omega \in \Omega_0$, we have $\omega \in \Omega_0^+$ and hence $\mu = 0$ by the choice of μ . This yields the first claim of the proposition. To prove the last one, set $M = M(\omega)$. Let $v \in M$ be a nonzero highest weight vector. Then [2, Ch. VIII, §6.1, Proposition 1] implies that M is generated by vectors of the form $X_{\beta_1} \dots X_{\beta_r} v$ where $X_{\beta_i} \in L$ are the root operators associated with negative roots $\beta_1, \dots, \beta_r \in R$. Since $0 \in \Omega(M)$, for some β_1, \dots, β_r with $\beta_1 + \dots + \beta_r = -\omega$ this vector is nonzero. Then the weight of a nonzero vector $X_{\beta_2} \dots X_{\beta_r} v$ is a positive root α . Then $W\alpha \subset \Omega(\varphi)$ since W preserves $\Omega(\varphi)$. If α is short or all roots in R have the same length, we are done. Let G have roots of different length and α be long. Then the maximal root $\alpha_{max} \in W\alpha \subset \Omega(\varphi)$ since α_{max} is long by [1, Ch. 6, §1.8, Proposition 25]. So $\omega - \alpha_{max} \in \Omega_0^+$. By definition of the maximal root, we have $\alpha_{max} - \alpha_s \in \Omega_0^+$ for the maximal short root α_s . Then $\omega - \alpha_s \in \Omega_0^+$ and therefore $\alpha_s \in \Omega(\varphi)$ by [2, Ch.VIII, §7.2, Corollary 2 to Proposition 3]. Then $W\alpha_s \in \Omega(\varphi)$ as desired.

Now let $\omega \notin \Omega_0$. Then $\mu \neq 0$ as Ω_0 is a \mathbb{Z} -module. Let $\rho \in \text{Irr}$ be the representation with highest weight μ . Set $\Omega' = \Omega(\rho)$. A set $\Lambda \subset \Omega$ is called R -saturated if $\lambda - t\alpha \in \Lambda$ for each $\lambda \in \Lambda$, every $\alpha \in R$ and each integer t that lies between 0 and $\langle \lambda, \alpha \rangle$, see [2, Ch.VIII, §7.2, prior Proposition 4]. By [2, Ch.VIII, §7.2, Propositions 4 and 5 (iv)], Ω' is R -saturated and $\Omega' = W(\Omega' \cap \Omega^+)$. We claim that $\Omega' = W\mu$. Indeed, otherwise Ω' contains a weight $\nu \in \Omega^+$ with $\nu < \mu$ which contradicts the choice of μ . Now $\Omega' \subset \Omega(\varphi)$ and Lemma 9 implies that μ is a microweight. \square

Remark 11. Let $p = 0$ or satisfy the assumptions of Theorem 8. If all roots of G have the same length, regard them as short. Let $\chi \in \text{Irr}$ be the representation whose highest weight is the maximal short root. Then $\Omega(\chi)$ consists of the zero weight and all short roots. Hence Proposition 10 implies that in our assumptions for each $\varphi \in \text{Irr}_p$ the set $\Omega(\varphi)$ contains either $\Omega(\rho)$ for some microweight representation ρ , or $\Omega(\chi)$.

The set $\Omega(\chi)$ is well known. Such representations are described in [8, Subsection 1.4 and Table 2] where they are called the short root representations.

The following well-known fact reduces the proofs of the main results to semi-simple elements.

Lemma 12. *Let $g_s, g_u \in G$, g_s be semisimple, g_u be unipotent, and the elements g_s and g_u commute. Put $g = g_s g_u$. Assume that M_1 and M_2 are the subspaces of fixed vectors for g and g_s , respectively, in a G -module M . Then $M_1 \neq 0$ if and only if $M_2 \neq 0$.*

Proof. Naturally, we can assume that $M \neq 0$. Since g, g_s and g_u commute, g_s and g_u preserve M_1 and g and g_u preserve M_2 . As g_u is unipotent, it is clear that it fixes a nonzero vector in every g_u -invariant subspace. Hence $M_1 \neq 0$ if $M_2 \neq 0$. To complete the proof, we show that $M_1 \subset M_2$. Indeed, if $M_1 \not\subset M_2$, for some $\lambda \neq 1$ the element g_s has a nonzero eigenvector with eigenvalue λ in M_1 . Denote

by $M_\lambda \subset M_1$ the subspace of all such vectors. Then g_u preserves M_λ and has a fixed vector $m \in M_\lambda$ since g_u is a unipotent element commuting with g_s . But then g cannot fix m which yields a contradiction. \square

In positive characteristic we need a reduction to irreducible modules with p -restricted highest weights.

Lemma 13. *Let $p > 0$ and $g \in G$ have a fixed vector in every irreducible G -module with p -restricted highest weight. Then g has such vector in every G -module.*

Proof. It suffices to consider irreducible modules. Let M be an irreducible G -module with $\omega(M) = \sum_{j=0}^s p^j \lambda_j$ where λ_j are p -restricted dominant weights. By [10, Theorem 1.1], M is isomorphic to the tensor product of modules M_j obtained from the modules $M(\lambda_j)$ after applying Fr^j . By our assumptions, g has fixed vectors in every module $M(\lambda_j)$. Now it is clear that it has such vectors in M_j and M . \square

Proof of Theorem 1. Lemmas 12 and 13 reduce the proof to semisimple elements for all p and to irreducible modules with p -restricted highest weights if $p > 0$. So suppose that M is an irreducible G -module with such highest weight and $g \in G$ is a semisimple element satisfying the assumptions of the theorem.

We can assume that $G \neq E_8(F), F_4(F)$ or $G_2(F)$ since for these groups the result was obtained in [13], see comments after the statement of the theorem in the Introduction. Then $\Omega(M)$ contains the zero weight or a microweight by Proposition 10. In the first case it is obvious that every semisimple element has a fixed vector in M .

Now suppose that $\lambda \in \Omega(M)$ is a microweight and put $M' = M(\lambda)$. Theorem 8 and [2, Ch.VIII, §7.3, Proposition 6] imply that $\Omega(M') = W\lambda$. Since W fixes $\Omega(M)$, the set $W\lambda$ is contained in $\Omega(M)$. As g has a fixed vector in M' , there exists $\mu \in W\lambda$ with $\mu(g) = 1$. This yields that g has a fixed vector in M since $\mu \in \Omega(M)$. \square

In the following proposition we use a fundamental theorem of R. Steinberg on the connection between representations of algebraic groups and finite Chevalley groups to extend results on fixed vectors to modules for the latter groups in defining characteristic.

Proposition 14. *Let $p > 0$ and let $H \subset G$ be the group of fixed points of a Frobenius morphism. Assume that $h \in H$ has a fixed vector in every G -module. Then it has such vector in every H -module in characteristic p .*

Proof. We need the following simple assertion.

(*) Let $K \subset E$ be fields, M be a KH -module, and $M' = M \otimes_K E$. Assume that h has a fixed vector in M' . Then h has such vector in M .

Indeed, fix a base in M and observe that it yields a base of M' over E . Let A be the matrix of the operator $h - 1$ in this base. The fixed subspace of h can be identified with the space of solutions of the linear system $Ax = 0$ where x is

an indeterminate row vector of length $\dim M$. As h has a fixed vector in M' , this system has nonzero solutions over E . So A is degenerate and the system has nonzero solutions over K as well. This yields a fixed vector in M .

As before, it suffices to consider irreducible H -modules. So assume that M is an irreducible KH -module. There exists a finite extension $E \supseteq K$ such that $M' = M \otimes_K E$ is a direct sum of absolutely irreducible EH -modules. Let N be one of such summands. By (*), it remains to show that h has a fixed vector in N . It follows from [11, Corollary (c) to Theorems 44 and 45] that there exists a finite subfield $\mathbb{F}_q \subset E$ such that $N = N_q \otimes_{\mathbb{F}_q} E$ for an absolutely irreducible $\mathbb{F}_q H$ -module N_q . Identifying isomorphic finite fields, assume that $\mathbb{F}_q \subset F$. By a theorem of Steinberg [11, Theorem 43], there is a G -module N' such that $N'|_H \cong N_q \otimes_{\mathbb{F}_q} F$. By our assumptions, h has a fixed vector in N' . Now the assertion (*) implies that h has such vectors in N_q and N . This completes the proof. \square

The following lemma is used to show that Corollary 2 cannot be extended to spinor groups and that Corollary 3 does not hold in characteristic 2 (the proofs of Lemma 17 and Corollary 21).

Lemma 15. *Let M be a G -module and $0 \notin \Omega(M)$. If $p > 0$, $l \in \mathbb{Z}^+$, and $q = p^l$, let $H \subset G$ be the subgroup generated by the elements $x_{\pm i}(t)$, $1 \leq i \leq r$, $t \in \mathbb{F}_q$. Then G contains a semisimple element that has no fixed vectors in M . If l is large enough, the group H contains such element.*

Proof. Set $h_i(t) = x_i(t)x_{-i}(-t^{-1})x_i(t)x_i(-1)x_{-i}(1)x_i(-1)$ for $t \in F^*$. It is clear that $h_i(t) \in H$ for $t \in \mathbb{F}_q^*$. By [11, Theorem 6 and Lemma 19 c], the elements $h_i(t)$ generate a maximal torus of G and $\lambda(h_i(t)) = t^{(\lambda, \alpha_i)}$ for $\lambda \in \Omega$. Hence if $\lambda = \sum_{i=1}^r a_i \omega_i$ and $h = \prod_{i=1}^r h_i(t_i)$, we have $\lambda(h) = \prod_{i=1}^r t_i^{a_i}$. Set

$$P_\lambda(t_1, \dots, t_r) = \prod_{i=1}^r (t_i^{a_i} - 1) \quad \text{and} \quad P(t_1, \dots, t_r) = \prod_{\lambda \in \Omega(M)} P_\lambda(t_1, \dots, t_r).$$

Then P is a nonzero polynomial in r indeterminates. Hence there exists an r -tuple (t_1, \dots, t_r) with $t_j \in F^*$ such that $P(t_1, \dots, t_r) \neq 0$. If q is large enough, we can choose t_1, \dots, t_r in \mathbb{F}_q^* . Then $\lambda(\prod_{i=1}^r h_i(t_i)) \neq 1$ for each $\lambda \in \Omega(M)$. This implies that the element $\prod_{i=1}^r h_i(t_i)$ has no fixed vectors in M . This element lies in H if all $t_i \in \mathbb{F}_q^*$. \square

3. The special linear group

In this section $G = A_r(F) \cong SL_n(F)$ with $n = r + 1$. Denote by D the group of diagonal matrices in G and by D_i , $1 \leq i \leq n - 1$, the matrix group $\text{diag}(1, \dots, 1, t, t^{-1}, 1, \dots, 1)$ where the element t at the i th position runs through F^* . Set $d_i(t) = \text{diag}(1, \dots, 1, t, t^{-1}, 1, \dots, 1)$. Obviously, the group D is generated by D_1, \dots, D_{n-1} and each element $d \in D$ can be uniquely represented in the form

$\prod_{i=1}^{n-1} d_i(t_i)$. It is clear that D is a maximal torus in G . By definition, the weights of G (elements of the set Ω) are the homomorphisms of the algebraic groups $D \rightarrow F^*$. Each such homomorphism ρ sends $d_i(t_i)$ to $t_i^{a_i(\rho)}$ with $a_i(\rho) \in \mathbb{Z}$ and hence is completely determined by the sequence of integers $(a_1(\rho), \dots, a_r(\rho))$. Denote by ε_i , $1 \leq i \leq n$, the weight that sends a matrix $d \in D$ to the element of F^* at its i th position. As usually, we write the operation on the set Ω in the additive form. So $(\lambda + \mu)(d) = \lambda(d)\mu(d)$ for $\lambda, \mu \in \Omega$ and $d \in D$. Since the determinant of each element in G is equal to 1, we have $\varepsilon_1 + \dots + \varepsilon_n = 1$. The root system R consists of the weights $\varepsilon_i - \varepsilon_j$, $i \neq j$. One can choose the simple roots α_i such that $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $1 \leq i \leq r$. Then the groups \mathcal{X}_i and \mathcal{X}_{-i} consist of matrices $1 + te_{i,i+1}$ and $1 + te_{i+1,i}$, respectively where e_{ij} is the matrix with 1 at the position (i, j) and zeros at other positions, $t \in F$; the groups D_i are maximal tori in $G(i) = \langle \mathcal{X}_i, \mathcal{X}_{-i} \rangle$ and the elements $d_i(t)$ coincide with the elements $h_i(t)$ defined in [11, §3]. For the fundamental weights we have $\omega_j = \sum_{l=1}^j \varepsilon_l$ ($1 \leq j \leq r$). Indeed, one has $\sum_{l=1}^j \varepsilon_l(d_i(t)) = t^{\delta_{ij}}$ (the Kronecker symbol). So our claim follows from the definition of fundamental weights.

Fix a base v_1, \dots, v_n in the standard G -module V and identify elements of $SL(V)$ with those of $SL_n(F)$ via matrices of operators in this base. Then D is the stabilizer of the n subspaces $\langle v_1 \rangle, \dots, \langle v_n \rangle$ in $SL(V)$, v_i is a weight vector of weight ε_i , and $\varepsilon_i(d) = t_i$ for $d \in D$ if $dv_i = t_i v_i$.

Theorem 16. *Let $K \subset G$ be a diagonalizable subgroup whose fixed point subspace in V is of dimension at least $(n-1)/2$. Then K has a fixed vector in every G -module.*

Proof. The arguments used in the proof of Theorem 1 for individual semisimple elements can be extended to tori (diagonalizable subgroups) as well. They show that it suffices to consider microweight modules. By (1), these are the modules $M_i = M(\omega_i)$, $1 \leq i \leq r$. Denote the dimension of the fixed point subspace mentioned above by m . Choose a base v_1, \dots, v_n in V such that K fixes the vectors v_1, \dots, v_m and the subspaces $\langle v_{m+1} \rangle, \dots, \langle v_n \rangle$. In the arguments at the beginning of this section take for D the subgroup in $SL(V)$ consisting of operators with diagonal matrices in this base. Then $K \subset D$ and for every $d \in K$ one has $\varepsilon_i(d) = 1$ for $i \leq m$ and $(\sum_{t=m+1}^n \varepsilon_t)(d) = 1$. Set $\lambda_i = \omega_i$ for $i \leq m$ and

$$\lambda_i = \left(\sum_{l=1}^{i+m-n} \varepsilon_l \right) + \sum_{t=m+1}^n \varepsilon_t$$

for $m < i \leq r$. Observe that $i \geq n - m$ if $i > m$ since $m \geq (n-1)/2$. It is well known that $W\omega_i = \{\varepsilon_{j_1} + \dots + \varepsilon_{j_i} \mid 1 \leq j_1 < \dots < j_i \leq n\}$. Hence in all cases $\lambda_i \in \Omega(M_i)$. As $\lambda_i(d) = 1$ for $d \in K$, we conclude that K has a fixed vector in M_i as desired. This completes the proof. \square

Proof of Corollary 2. Using Lemma 12, we can assume that g is semisimple. So it is clear that g lies in a subgroup K satisfying the assumptions of Theorem 16. It

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remains to apply that theorem. \square

Proof of Corollary 6. Recall that $SU_n(q^2)$ is a twisted Chevalley group ${}^2A_{n-1}(q)$. Now the corollary follows from Corollary 2 and Proposition 14. \square

4. Other classical groups

In this section $G = B_r(F)$, $C_r(F)$, or $D_r(F)$. Recall that $p \neq 2$ for $G = B_r(F)$. As before, denote by V the standard G -module. Let Φ be a nondegenerate bilinear form on V preserved by G . Fix a base

$$v_1, v_2, \dots, v_n \tag{2}$$

of V such that $\Phi(v_i, v_j) = 0$ if $i + j \neq n + 1$. Let $D \subset G$ be the set of operators that have diagonal matrices in this base. Then D is a maximal torus of G . For $1 \leq i \leq r$ define the weights ε_i as follows: $\varepsilon_i(d) = d(v_i)$ for $d \in D$. Then $\omega_i = \sum_{j=1}^i \varepsilon_j$ for $G = C_r(F)$, for $G = B_r(F)$ with $i < r$, and for $G = D_r(F)$ with $i < r - 1$; $\omega_r = (\sum_{j=1}^r \varepsilon_j)/2$ for $G = B_r(F)$ or $D_r(F)$; and $\omega_{r-1} = ((\sum_{j=1}^{r-1} \varepsilon_j) - \varepsilon_r)/2$ for $G = D_r(F)$; $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $i < r$, $\alpha_r = \varepsilon_r$ for $G = B_r(F)$, $2\varepsilon_r$ for $G = C_r(F)$, and $\varepsilon_{r-1} + \varepsilon_r$ for $G = D_r(F)$. For $G = B_r(F)$ the vector v_{r+1} has weight 0. Furthermore, the subgroup $G(i)$ with $i < r$ fixes all vectors of the base (2), except v_i, v_{i+1}, v_{n+1-i} , and v_{n-i} ; the subgroup $G(r)$ fixes all these vectors, except v_r, v_{r+1} , and v_{r+2} , for $G = B_r(F)$, all base vectors, except v_r and v_{r+1} , for $G = C_r(F)$, and all base vectors, except v_{r-1}, v_r, v_{r+1} , and v_{r+2} , for $G = D_r(F)$. This implies that in all cases the subgroup $S = G(m, m + 1, \dots, r)$ with $1 < m < r$ and also $m < r - 1$ for $G = D_r(F)$ fixes the vectors $v_1, \dots, v_{m-1}, v_{n+2-m}, \dots, v_n$. Let V' be the linear span of other vectors in (2). Then V' is isomorphic to the standard $G(m, m + 1, \dots, r)$ -module (here we assume that the standard module for the group $B_2(F)$ has dimension 5 and such module for $D_3(F)$ has dimension 6). It occurs that $D \cap S$ is a maximal torus in S . Denote by ε_i^S the restriction of the weight ε_i to $D \cap S$. Then $\varepsilon_i^S = 0$ for $i < m$ and the weights ε_i^S with $m \leq i \leq r$ are linearly independent.

For zero characteristic the action of the relevant group in such base is described in [2, §13]. Using the construction of irreducible G -modules described in [11, §3 and §12], one easily concludes (and this is well known) that a desired base exists for the FG -module V as well (here for $G = B_r(F)$ it is essential that $p \neq 2!$). Notice that for $p = 2$ the group $D_r(F)$ preserves an alternating form on V , but this does not influence our arguments.

Proof of Corollary 3. Apply Theorem 1 and take into account (1). \square

Proof of Corollary 7. The result follows from Corollary 3 and Proposition 14. \square

Lemma 17. *Let $G = B_r(F)$ or $D_r(F)$ and $S = G(m, m + 1, \dots, r)$ with $m < r$ for $G = B_r(F)$ and $m < r - 1$ for $G = D_r(F)$. Assume that $M = M(\omega_r)$ for*

$G = B_r(F)$ and $M = M(\omega_{r-1})$ or $M(\omega_r)$ for $G = D_r(F)$. Then there exists a semisimple element $s \in S$ that has no fixed vectors in M . If q is large enough, the same holds for the groups $B_r(q)$ and $D_r(q)$ and their subgroups generated by the elements $x_{\pm i}(t)$, $m \leq i \leq r$, $t \in \mathbb{F}_q$.

Proof. It is well known that $\Omega(M)$ consists of all possible combinations of the form $(\pm \varepsilon_1 \pm \dots \pm \varepsilon_r)/2$ for $G = B_r(F)$, of all such combinations with odd number of "the minus" signs for $G = D_r(F)$ and $\omega(M) = \omega_{r-1}$, and of all such combinations with even number of "the minus" signs for $G = D_r(F)$ and $\omega(M) = \omega_r$. Let M' be the restriction $M|_S$. Naturally, $\Omega(M') = \Omega(M)$ for $m = 1$. Otherwise $\Omega(M')$ has the same form as $\Omega(M)$ for $G = B_r(F)$, but one has to replace the weights ε_i by ε_i^S . Hence $0 \notin \Omega(M')$. It remains to apply Lemma 15. \square

Remark 18. The arguments in the description of the base (2) yield that for $m > (r + 2)/2$ each element of the subgroup S of Lemma 17 has a fixed subspace of dimension $\geq r + 1 > n/2$ in the module V .

Proof of Corollary 4. Since $r > 2$ for $G = B_r(F)$, the subgroup $G(1, \dots, r-1)$ is the unique SR -subgroup of type A_{r-1} for $G = B_r(F)$. Observe that for $G = D_r(F)$ each SR -subgroup of type A_{r-2} is conjugate to $G(1, \dots, r-2)$. So set $\Gamma = G(1, \dots, r-1)$ for $G = B_r(F)$ and $\Gamma = G(1, \dots, r-2)$ for $G = D_r(F)$. Then $D \cap \Gamma$ is a maximal torus in Γ . Denote by ε_i^Γ the restriction of the weight ε_i to $D \cap \Gamma$. One has

$$\sum_{i=1}^r \varepsilon_i^\Gamma = 0 \quad \text{for } G = B_r(F) \quad \text{and} \quad \varepsilon_r^\Gamma = \sum_{i=1}^{r-1} \varepsilon_i^\Gamma = 0 \quad \text{for } G = D_r(F).$$

This implies that the restriction of the weight ω_{r-1} to $D \cap \Gamma$ is zero for $G = D_r(F)$ and the restriction of ω_r to this subgroup is zero in both cases. Put $\omega = \omega_r$ for $G = B_r(F)$ and $\omega = \omega_{r-1}$ or ω_r for $G = D_r(F)$ and set $M_\Gamma = M(\omega)|_\Gamma$. Then $0 \in \Omega(M_\Gamma)$. If $G = D_r(F)$, the weight $0 \in \Omega(V|_\Gamma)$ as well since $\varepsilon_r^\Gamma = 0$. Now (1) implies that a semisimple element of Γ has a fixed vector in every microweight module. To complete the proof, apply Theorem 1 and Lemma 12. \square

Corollary 19. *Corollary 4 remains valid if one replaces G by the group $G_q = B_r(\mathbb{F}_q)$, $D_r(\mathbb{F}_q)$, ${}^2D_r(\mathbb{F}_q)$ or ${}^3D_4(q)$ and Γ by $\Gamma \cap G_q$.*

Proof. This follows immediately from Corollary 4 and Proposition 14. \square

The following lemma and corollary show that Corollary 3 cannot be extended to characteristic 2.

Lemma 20. *Let $p = 2$, $G = C_r(F)$, $l < r/2$, and $S = G(r-l+1, \dots, r)$. Assume that $\omega = \omega_{i_1} + \dots + \omega_{i_l} + \omega_r \in \Omega$ with $i_1 < \dots < i_l$ and $i_l(i_l + 1)/2 < l$. Put $M = M(\omega)$. Then $0 \notin \Omega(M|_S)$.*

Proof. Set $M_1 = M(\omega_{i_1} + \dots + \omega_{i_t})$ and $M_2 = M(\omega_r)$. Then $M \cong M_1 \otimes M_2$ by [11, Corollary to Theorem 41]. It is well known (and can be deduced with the use of the description of the isomorphism between the groups $B_r(F)$ and $C_r(F)$ in characteristic 2, see [11, Theorem 28]) that the set $\Omega(M_2) = W\omega_r$ and consists of all possible combinations of the form $\pm\varepsilon_1 \pm \dots \pm \varepsilon_r$. Naturally, $\Omega(M_1)$ lies in the weight system of the irreducible $C_r(\mathbb{C})$ -module $M_{\mathbb{C}}$ with highest weight $\omega_{i_1} + \dots + \omega_{i_t}$. Put $m = i_t(i_t + 1)/2$ and observe that $m = 1 + 2 + \dots + i_t$. Denote by N_j the irreducible $C_r(\mathbb{C})$ -module with highest weight ω_{i_j} , $1 \leq j \leq t$. Several applications of [2, Ch. VIII, §7.4, Proposition 10] yield that $\Omega(M_{\mathbb{C}}) = \{\mu_1 + \dots + \mu_t \mid \mu_j \in \Omega(N_j)\}$. Now one can deduce from the description of the weight systems of the fundamental $C_r(\mathbb{C})$ -modules in [2, Ch. VIII, §13.3] that weights in $\Omega(M_{\mathbb{C}})$ are linear combinations of the weights ε_i with at most m nonzero coefficients. Hence each weight $\lambda \in \Omega(M)$ is a linear combination of these weights with at least $r - m > r - l$ nonzero coefficients. So the restriction of λ to $D \cap S$ is a nonzero linear combination of linearly independent weights ε_i^S with $i > r - l$. This completes the proof. \square

Corollary 21. *Let G , S , and M be as in Lemma 20. Then some elements of S have no fixed vectors in M . If $q = 2^s$ is large enough, the same holds for the subgroup generated by $x_{\pm i}(t)$ with $r - l + 1 \leq i \leq r$ and $t \in \mathbb{F}_q$.*

Proof. This follows immediately from Lemmas 20 and 15. \square

5. Exceptional groups

Theorem 1, the condition (1) and Proposition 14 imply that it suffices to consider the groups $E_6(F)$ and $E_7(F)$.

Proof of Corollary 5. The group $E_6(F)$ has two conjugate SR -subgroups of type D_5 : $G(1, \dots, 5)$ and $G(2, \dots, 6)$. The group $E_7(F)$ has the unique SR -subgroup of type E_6 : $G(1, \dots, 6)$. So set $S = G(1, \dots, 5)$ for $G = E_6(F)$ and $S = G(1, \dots, 6)$ for $G = E_7(F)$. For $G = E_6(F)$ it is clear that highest weight vectors in the G -module $M(\omega_6)$ and lowest weight vectors in $M(\omega_1)$ (of weight $-\omega_6$) are fixed by S . For $G = E_7(F)$ the same holds for S and highest weight vectors in the G -module $M(\omega_7)$. To complete the proof, apply Theorem 1 and Lemma 9 as for other groups. \square

Corollary 22. *Corollary 5 remains valid for the finite groups $E_6(\mathbb{F}_q)$, ${}^2E_6(\mathbb{F}_q)$, and $E_7(\mathbb{F}_q)$.*

Proof. This follows from Proposition 14. \square

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