ON RESTRICTING REPRESENTATIONS OF SIMPLE ALGEBRAIC GROUPS TO SEMISIMPLE SUBGROUPS WITH TWO SIMPLE COMPONENTS

©2005 I. Suprunenko, A. Zalesskii

Institute of Mathematics
National Academy of Sciences of Belarus
e-mail: suprunenko@im.bas-net.by
Received by the editors on September 2, 2005

Let G be a simple algebraic group over an algebraically closed field K of any characteristic, and let G_1 and G_2 be non-trivial commuting subgroups of G (that is, $[G_1, G_2] = 1$). Let ρ be an irreducible representation of G. In numerous situations one needs an information on the restriction of ρ to G_1G_2 rather than to G_1 or G_2. An essential question is whether the restriction has a composition factor nontrivial on both G_1, G_2. In this paper we consider a key case where G_1 and G_2 are subsystem subgroups (see below) that are simple as algebraic groups. As soon as this case is settled, one can deduce results of similar kind for subgroups $H_1 \subset G_1$ and $H_2 \subset G_2$.

Our main motivation for this work is concerned with applications to the study of eigenvalue multiplicities, minimal polynomials and Jordan block structure of elements in irreducible representations of G (see [1]). However, we are quite sure that the results of this paper can be used for solving other problems as well. For instance, they imply that the multiplicity of some non-trivial factor in the restriction to G_1 is at least the dimension of a nontrivial G_2-module (and vice versa). Arguments of a similar nature play a crucial role in getting some lower bounds for the maximal weight multiplicities in representations of symplectic groups (Baranov, Osinovskaya, and Suprunenko, in preparation).

To be precise, throughout the text G_1 and G_2 are assumed to be "subsystem subgroups". Recall that a nonempty subset S of the root system of G is called a subsystem if $\alpha+\beta \in S$ for all roots α and $\beta \in S$ such that $\alpha+\beta$ is a root and $-\alpha \in S$ for every $\alpha \in S$. By a subsystem subgroup we mean a subgroup generated by all root subgroups associated with the roots of a fixed subsystem.

We call a representation of G_1G_2 **solid** if it has a composition factor non-trivial on both G_1 and G_2. For example, let $G = SL(n, K)$ and $G_i = SL(n_i, K)$ for $i = 1, 2$ with $n = n_1 + n_2$ and the natural embedding of $G_1 \times G_2$ into G. Let ρ and ρ_i be the natural representations of G and G_i, respectively, and let 1_{G_i} denote the trivial representation of G_i. Then $\rho|G_1G_2 \cong \rho_1 \otimes 1_{G_2} \oplus 1_{G_1} \otimes \rho_2$. Therefore, $\rho|G_1G_2$ is not solid. This example can make an impression that the restriction $\rho|G_1G_2$ is rarely solid. However, in reality the opposite is true.

Our main result describes all triples ρ, G_1, G_2 such that ρ is an irreducible representation of G, G_1 and G_2 are as above and $\rho|G_1G_2$ is not solid. For technical reasons, if G is classical, we also assume that the dimension of ρ differs from that of the natural representation of G. (The case excluded by the latter assumption is quite clear.) To state the result, we need some notation. We say that ρ is a twist of a representation φ if $\rho(g) = \varphi(\alpha(g))$ for some algebraic group endomorphism α of G and $g \in G$. Denote by R the root system of G (with respect to a fixed maximal torus) and by $\mathcal{X}_\beta \subset G$ the root subgroup associated with a root $\beta \in R$. For a subsystem $S \subset R$ let $G(S) \subset G$ be the subgroup generated by all subgroups $\mathcal{X}_\beta$ with $\beta \in S$. We assume that the field characteristic $p \neq 2$ if $G = B_r(K)$ and $r > 3$ for $G = D_r(K)$ (unless otherwise stated). For the classical groups ε_j are weights of the standard G-module described in [2, ch. VIII, §13].

Theorem 1. *Let G be a simply connected simple algebraic group of rank $r > 1$ over an algebraically closed field K of characteristic $p \geq 0$ and φ be a non-trivial irreducible representation of G. Assume that R_1 and $R_2 \subset R$ are subsystems and the groups $G(R_1)$ and $G(R_2)$ are simple and commute. Let $\varphi|G(R_1)G(R_2)$ be not solid. Then G is classical and one of the following holds:*

1) φ is a twist of the natural representation of G;

2) $G = B_r(K)$, $C_r(K)$ with $p = 2$ or $D_r(K)$ with $r > 4$, $R_1 \cup R_2 = \{\pm\varepsilon_i \pm \varepsilon_j\}$ with fixed $1 \leq i < j \leq r$, and φ is a twist of a spinor representation of G.

One easily observes that the natural representations of the classical groups yield many nonsolid restrictions to subsystem subgroups with two components, see more details in Lemma 9.

Exceptions connected with the spinor representations do exist. In what follows $\omega(\varphi)$ denotes the highest weight of an irreducible representation φ, ω_i, $1 \leq i \leq r$, are the fundamental weights of G.

Lemma 2. *Let $G = B_r(K)$ or $D_r(K)$ or $p = 2$ and $G = C_r(K)$. Assume that R_1 and R_2 are irreducible subsystems and $R_1 \cup R_2 = \{\pm\varepsilon_i \pm \varepsilon_j\}$ with fixed $1 \leq i < j \leq r$. Let φ be an irreducible representation of G and $\omega = \omega(\varphi)$. Suppose that $\omega = \omega_r$ for $p = 0$ and $G = B_r(K)$, $p^j\omega_r$ for $p > 0$ and $G = B_r(K)$, $2^j\omega_r$ for $G = C_r(K)$, ω_{r-1} or ω_r for $p = 0$ and $G = D_r(K)$, and $p^j\omega_{r-1}$ or $p^j\omega_r$ for $p > 0$ and $G = D_r(K)$ (here j is a nonnegative integer). Then $\varphi|G(R_1)G(R_2)$ is not solid.*

These results can be immediately transferred to representations of finite Chevalley groups in defining characteristic.

Throughout the text the following **notation** is used. $\mathbb{Z}$ and $\mathbb{Z}^+$ are the sets of integers and nonnegative integers, respectively. If $S_1, \dots, S_k$ are subgroups of a group S, then $\langle S_1, \dots, S_k \rangle$ is the subgroup generated by them. For a semisimple algebraic group Γ the symbols $W(\Gamma)$, $\mathbf{X}(\Gamma)$, $\mathbf{X}^+(\Gamma)$, $\mathbf{X}(\varphi)$, and $\mathrm{Irr}(\Gamma)$ denote its Weyl group, the set of weights, the subset of dominant weights, the set of weights of a fixed representation φ, and the set of irreducible rational representations of Γ, respectively. If $\Gamma = G$, we omit the indication of the group in this notation and use the symbol $W\lambda$ to denote the W-orbit of a weight $\lambda \in \mathbf{X}$. The symbols α_i, $1 \leq i \leq r$, denote the simple roots of G, the fundamental weights and the simple roots are labeled as in [3]. Recall that $\alpha_j = \varepsilon_j - \varepsilon_{j+1}$ for $j < r$, $\alpha_r = \varepsilon_r$ for $G = B_r(K)$, $2\varepsilon_r$ for $G = C_r(K)$, and $\varepsilon_{r-1} + \varepsilon_r$ for $G = D_r(K)$. If $\lambda \in \mathbf{X}^+$, then $\mathbf{X}(\lambda)$ is the set of weights of the irreducible representation with highest weight λ for a simple simply connected algebraic group with the root system R in characteristic 0. In what follows $\varphi|H$ is the restriction of a representation φ of Γ to a subgroup H and $\mathrm{Irr}(\varphi|H)$ is the set of composition factors of this restriction, $\varphi(\lambda)$ is the representation in $\mathrm{Irr}(\Gamma)$ with highest weight λ. For $\lambda \in \mathbf{X}^+$ we say that a weight $\mu \preceq \lambda$ if μ belongs to a W-orbit of a weight $\nu \in \mathbf{X}^+$ and $\nu \leq \lambda$ (with respect to the standard order on $\mathbf{X}$). Denote by $\langle \mu, \beta \rangle$ the value of a weight $\mu \in \mathbf{X}$ on a root $\beta \in R$. If $\beta_1, \dots, \beta_k$ is a base of a subsystem $R_1 \subset R$, set $R(\beta_1, \dots, \beta_k) = R_1$ and $G(\beta_1, \dots, \beta_k) = G(R_1)$. If some $\beta_j = \alpha_i$, we replace β_j by i in this notation and use symbols $G(i_1, \dots, i_m, \beta)$, etc. If $H \subset G$ is a subgroup of the form $G(\beta_1, \dots, \beta_k)$, the symbol μ_H means the restriction of a weight $\mu \in \mathbf{X}$

to the intersection of a fixed maximal torus in G used to determine the root system with H. For $R_1, R_2 \subset R$ denote by $\mathbf{X}(R_1, R_2)$ the set of weights $\lambda \in \mathbf{X}$ such that $\langle \lambda, \alpha \rangle$, $\langle \lambda, \beta \rangle > 0$ for some $\alpha \in R_1 \cap R^+$ and some $\beta \in R_2 \cap R^+$. Throughout the text in all cases where we mention a triple (φ, R_1, R_2), etc., we mean that $\varphi \in \mathrm{Irr}$, R_1 and $R_2 \subset R$ are irreducible subsystems, $R_1 \cap R_2 = \emptyset$, and the subgroups $G(R_1)$ and $G(R_2)$ commute. If $p > 0$, denote by $\mathrm{Irr}_p \subset \mathrm{Irr}$ the subset of p-restricted representations. Recall that $\varphi \in \mathrm{Irr}$ is p-restricted if $\omega(\varphi) = \sum_{i=1}^{r} a_i \omega_i$ with all $a_i < p$. Let $R_1, R_2 \subset R$ be distinct irreducible subsystems and the groups $G(R_1)$ and $G(R_2)$ commute. For a nontrivial irreducible representation ρ of G we call a triple (ρ, R_1, R_2) special if $\rho | G(R_1)G(R_2)$ is not solid. Otherwise call such triple nonspecial.

We can and shall assume that $r > 2$ for $G = A_r(K)$ since the group $A_2(K)$ has no commuting subsystem subgroups. Set

$$\mathbf{X}_0 = \begin{cases} \{\omega_1, \omega_r\} & \text{for } G = A_r(K), \\ \{\omega_1\} & \text{for } G = B_r(K),\ C_r(K), \text{ or } G = D_r(K) \text{ and } r > 4, \\ \{\omega_1, \omega_3, \omega_4\} & \text{for } G = D_4(K), \\ \emptyset & \text{for exceptional types.} \end{cases} \tag{1}$$

If $p > 0$, put $\mathbf{X}_1 = \{p^j \omega \mid \omega \in \mathbf{X}_0, j \in \mathbb{Z}^+\}$.

The following technical lemmas will be used in the proofs of the main results.

Lemma 3. *The following assertions are equivalent.*

(1) *A triple (φ, R_1, R_2) is nonspecial.*

(2) $\mathbf{X}(R_1, R_2) \cap \mathbf{X}(\varphi) \neq \emptyset$.

(3) *$\langle \mu, \alpha \rangle \neq 0$ and $\langle \mu, \beta \rangle \neq 0$ for some $\mu \in \mathbf{X}(\varphi)$ and some $\alpha \in R_1$, $\beta \in R_2$.*

Proof. Set $H = G(R_1)G(R_2)$. The claim of the lemma is obvious since the set of weight restrictions $\{\mu_H \mid \mu \in \mathbf{X}(\varphi)\}$ coincides with the union of the sets of weights of the composition factors of the restriction $\varphi | H$ and every nontrivial representation of $G(R_i)$ $(i = 1, 2)$ has a weight ν with $\langle \nu, \alpha \rangle > 0$ for a fixed root $\alpha \in R_i \cap R^+$.

Lemma 4. *Let $R_1, R_2, R_3, R_4 \subset R$ be irreducible subsystems. Assume that $R_1 \cap R_2 = R_3 \cap R_4 = \emptyset$, $R_3 \subseteq R_1$, and $R_2 \subseteq R_4$. Then for every $\varphi \in \mathrm{Irr}$ the triples (φ, R_1, R_2) and (φ, R_3, R_4) are special or nonspecial simultaneously.*

Proof. Obviously, it suffices to prove that (φ, R_3, R_4) is nonspecial if (φ, R_1, R_2) is such (one can interchange subsystems in a triple and pairs (R_1, R_2) and (R_3, R_4)). For $1 \leq i \leq 4$ set $H_i = G(R_i)$. If $R_i \cap R_j = \emptyset$, put $H_{ij} = H_i H_j$ and $\mathrm{Irr}_{ij} = \mathrm{Irr}(\varphi | H_{ij})$. Assume that Irr_{12} contains a composition factor $\psi \cong \psi_1 \otimes \psi_2$ where $\psi_i \in \mathrm{Irr}(H_i)$, $i = 1, 2$, and both ψ_1 and ψ_2 are nontrivial. Since H_3 is simple, the restriction $\psi_1 | H_3$ has a nontrivial composition factor ρ. Hence $\rho \otimes \psi_2 \in \mathrm{Irr}(\psi | H_{32}) \subset \mathrm{Irr}_{32}$. So there exists $\tau \in \mathrm{Irr}_{34}$ such that $\rho \otimes \psi_2 \in \mathrm{Irr}(\tau | H_{32})$. It is clear that $\tau | H_3$ and $\tau | H_4$ are nontrivial irreducible representations of relevant groups. This completes the proof.

Lemma 5. *Let $R_1, R_2 \subset R$ be irreducible subsystems. Assume that $R_1 \cap R_2 = \emptyset$. Let $w \in W$. Then for every representation $\varphi \in \mathrm{Irr}$ the triples (φ, R_1, R_2) and (φ, wR_1, wR_2) are special or nonspecial simultaneously.*

Proof. This follows immediately from Lemma 3 since the set $\mathbf{X}(\varphi)$ and the pairing $\langle \mu, \alpha \rangle$ with $\mu \in \mathbf{X}$ and $\alpha \in R$ are invariant under the action of W.

Lemma 6. *Let $\varphi = \varphi_1 \otimes \varphi_2 \in \mathrm{Irr}$. Assume that the representations φ_1 and φ_2 are nontrivial. Then there are no special triples for φ.*

Proof. Let $R_1, R_2 \subset R$ be irreducible subsystems with $R_1 \cap R_2 = \emptyset$. Assume that $G(R_1)$ and $G(R_2)$ commute. Set $H_i = G(R_i)$ and $H = H_1 H_2$. Since the groups H_i are simple, there exist composition factors $\rho \in \mathrm{Irr}(\varphi_1 | H)$ and $\tau \in \mathrm{Irr}(\varphi_2 | H)$ such that $\rho = \rho_1 \otimes \rho_2$, $\tau = \tau_1 \otimes \tau_2$, $\rho_i, \tau_i \in \mathrm{Irr}(H_i)$ for $i = 1, 2$, and ρ_1 and τ_2 are nontrivial. Since ρ_1 and τ_2 are nontrivial, the representations $\rho_i \otimes \tau_i$ have nontrivial composition factors $\nu_i \in \mathrm{Irr}(H_i)$. It is clear that $\nu_1 \otimes \nu_2 \in \mathrm{Irr}(\rho \otimes \tau) \subset \mathrm{Irr}(\varphi|H)$. This yields the lemma.

Lemma 7. *Let $p > 0$, $\varphi_1, \varphi_2 \in \mathrm{Irr}$ and $\omega(\varphi_2) = p^j \omega(\varphi_1)$ for some nonnegative integer j. Assume that R_1 and $R_2 \subset R$ are irreducible subsystems, $R_1 \cap R_2 = \emptyset$, and the groups $G(R_1)$ and $G(R_2)$ commute. Then the triples (φ_1, R_1, R_2) and (φ_2, R_1, R_2) are special or nonspecial simultaneously.*

Proof. By the Steinberg tensor product theorem [4, Theorem 1.1], $\varphi_2 = \varphi_1 \mathrm{Fr}^j$ where Fr is the Frobenius morphism of G determined by raising the elements of K to the pth power. Hence for $H = G(R_1)$, $G(R_2)$, or $G(R_1)G(R_2)$ the set $\mathrm{Irr}(\varphi_2|H)$ can be obtained from $\mathrm{Irr}(\varphi_1|H)$ by twisting the composition factors in the latter set by the jth power of the similar morphism for H. It is clear that this operation preserves trivial representations and sends nontrivial to nontrivial ones.

For $\alpha \in R$ set $C_\alpha = \{\beta \in R \mid \mathcal{X}_{\pm\beta}$ commutes with $\mathcal{X}_\alpha\}$. Table 1 below is based on commutator relations for root elements in G (see [5, Lemma 15] and [6, Propositions 33.4 and 33.5]) and describes C_α for some fixed α. Denote the maximal root in R^+ by α_{max}. It is well known (and can be extracted from [3, Tables II–IV]) that $\alpha_{max} = \varepsilon_1 + \varepsilon_2$ for $G = B_r(K)$ or $D_r(K)$ and $\alpha_{max} = 2\varepsilon_1$ for $G = C_r(K)$. For $G = F_4(K)$ or $G_2(K)$ let α_{sh} be the maximal short root in R^+.

Table 1

G	p	α	C_α
$A_r,\ r > 2$		α_1	$R(3, \dots, r)$
B_r	$p \neq 2$	α_1	$R(\alpha_{max}, 3, \dots, r)$
B_r	$p \neq 2$	α_r	$R(1, \dots, r-2, \varepsilon_{r-2} + \varepsilon_{r-1})$
$C_r,\ r > 2$	$p \neq 2$	α_1	$R(3, \dots, r)$
C_r	$p = 2$	α_1	$R(\varepsilon_1 + \varepsilon_2, 3, \dots, r)$
C_r		α_r	$R(1, \dots, r-2, 2\varepsilon_{r-1})$
D_r		α_1	$R(\alpha_{max}, 3, \dots, r)$
E_6		α_{max}	$R(1, 3, \dots, 6)$
E_7		α_{max}	$R(2, \dots, 7)$
E_8		α_{max}	$R(1, \dots, 7)$
F_4		α_{max}	$R(2, 3, 4)$
F_4	$p \neq 2$	α_{sh}	$R(1, 2, \alpha_2 + 2\alpha_3)$
F_4	$p = 2$	α_{sh}	$R(1, 2, 3)$
G_2		α_{max}	$R(1)$
G_2		α_{sh}	$R(2)$

Proposition 8. *Assume that* $\lambda \in \mathbf{X}^+ \setminus \{0\}$ *and* $\lambda \notin \mathbf{X}_0$. *Let* α *be one of the roots in Table 1,* $R_1 = R(\alpha)$ *and* R_2 *be an irreducible component of the subsystem* C_α. *If* $G = B_r(K)$ *or* $G = D_r(K)$ *with* $r > 4$, $\alpha = \alpha_1$, *and* $R_2 = R(\varepsilon_1 + \varepsilon_2)$, *suppose also that* $\lambda \neq \omega_r$ *for* $G = B_r(K)$ *and* $\lambda \notin \{\omega_{r-1}, \omega_r\}$ *for* $G = D_r(K)$. *Assume that* $p \neq 2$ *and* $r > 2$ *if* $G = C_r(K)$ *and* $\alpha = \alpha_1$. *Then there exists a weight* $\mu \in \mathbf{X}$ *such that* $\mu \preceq \lambda$ *and* $\mu \in \mathbf{X}(R_1, R_2)$.

Proof. We shall call dominant weights that satisfy the assertion of the proposition typical. (Strictly speaking, we should consider typical weights for the pair (R_1, R_2), but there will be no confusion since all our arguments concern a fixed pair from a well-defined list.) It is clear that λ is typical if $\delta < \lambda$ and δ is typical. It is well known that $\mathbf{X}(\lambda) = \{\mu \in \mathbf{X} \mid \mu \preceq \lambda\}$ ([2, ch. VIII, §7, Proposition 5]). Since for dominant λ_1 and λ_2 the set $\mathbf{X}(\lambda_1 + \lambda_2) = \{\mu_1 + \mu_2 \mid \mu_j \in \mathbf{X}(\lambda_j),\ j = 1, 2\}$ by [2, ch. VIII, §7, Proposition 10], we have $\mu + \lambda_2 \preceq \lambda_1 + \lambda_2$ if $\mu \preceq \lambda_1$. It is clear that $\langle \lambda_2, \beta \rangle \geq 0$ for all $\beta \in R^+$. Hence $\mu + \lambda_2 \in \mathbf{X}(R_1, R_2)$ if $\mu \in \mathbf{X}(R_1, R_2)$. This yields the following

Assertion (*). For dominant λ_1 and λ_2 the weight $\lambda_1 + \lambda_2$ is typical if λ_1 is typical.

For $\lambda = \sum_{i=1}^r m_i \omega_i$ one has

$$\langle \lambda, \alpha_{max} \rangle = \begin{cases} m_1 + m_r + 2\sum_{i=2}^{r-1} m_i & \text{for } G = B_r(K), \\ \sum_{i=1}^r m_i & \text{for } G = C_r(K), \\ m_1 + m_{r-1} + m_r + 2\sum_{i=2}^{r-2} m_i & \text{for } G = D_r(K). \end{cases}$$

First assume that $\lambda = \omega_i$.

1). Let $G = A_r(K)$. Then $\alpha = \alpha_1$, $R_2 = R(3, \ldots, r)$, and $1 < i < r$. Set $\mu = \omega - \sum_{j=2}^{i} \alpha_j$. Then $\mu \in W\lambda$ and $\langle \mu, \alpha_1 \rangle = \langle \mu, \alpha_{i+1} \rangle = 1$. Hence $\mu \preceq \lambda$, $\mu \in \mathbf{X}(R_1, R_2)$, and so λ is typical.

2). Now let $G = B_r(K)$ and $\alpha = \alpha_1$. Recall that $i > 1$ and that by the assumptions of the lemma, $R_2 = R(3, \ldots, r)$ if $i = r$. Choose μ as in Case 1). One can conclude that $\langle \mu, \alpha_1 \rangle = 1$, $\langle \mu, \alpha_{i+1} \rangle > 0$ and $\langle \mu, \alpha_{max} \rangle = 1$ if $i < r$, and $\langle \mu, \alpha_r \rangle = 1$ for $i = r$. Therefore $\mu \in \mathbf{X}(R_1, R_2)$ and λ is typical in all possible situations.

3). Next, assume that $G = B_r(K)$ and $\alpha = \alpha_r$. As before, $i > 1$. Observe that $\alpha_j \in R_2$ for all $j < r - 1$ and $\alpha_{max} \in R_2$. Set $\mu = \lambda - \sum_{j=i}^{r-1} \alpha_j$ if $i < r$ and $\mu = \lambda$ for $i = r$. Then in all cases $\mu \in W\lambda$ and $\langle \mu, \alpha \rangle > 0$. Furthermore, one has $\langle \mu, \alpha_{i-1} \rangle > 0$ if $i < r$ and $\langle \mu, \alpha_{max} \rangle > 0$ for $i = r$. Hence $\mu \preceq \lambda$, $\mu \in \mathbf{X}(R_1, R_2)$, and λ is typical.

4). Now suppose that $G = C_r(K)$ and $\alpha = \alpha_1$. Recall that due to our assumptions $i > 1$, $r > 2$ and $R_2 = R(3, \ldots, r)$. If $i < r$, choose μ as in Items 1) and 2). For $i = r$ put $\mu = \lambda - \alpha_r - 2\sum_{j=2}^{r-1} \alpha_j$. Then in both cases $\mu \in W\lambda$ and $\langle \mu, \alpha \rangle > 0$. We get $\langle \mu, \alpha_{i+1} \rangle > 0$ if $i < r$ and $\langle \mu, \alpha_r \rangle > 0$ for $i = r$. Hence μ is a required weight and λ is typical.

5). Next, let $G = C_r(K)$ and $\alpha = \alpha_r$. Argue as in Item 3).

6). Now assume that $G = D_r(K)$. Then $\alpha = \alpha_1$. As before, $i > 1$. Recall that $r > 4$ and $R_2 = R(3, \ldots, r)$ for $i = r - 1$ or r. If $i < r$, choose μ as in Items 1) and 2). For $i = r$ put $\mu = \lambda - \alpha_r - \sum_{j=2}^{r-2} \alpha_j$. Then $\mu \in W\lambda$ and $\langle \mu, \alpha \rangle > 0$. One also has $\langle \mu, \alpha_{max} \rangle = 1$ if $i < r - 1$, $\langle \mu, \alpha_{i+1} \rangle = 1$ if $i < r$, and $\langle \mu, \alpha_{r-1} \rangle = 1$ for $i = r$. Hence in all cases $\mu \in \mathbf{X}(R_1, R_2)$ and λ is typical.

7). Next let $G = E_r(K)$ with $6 \leq r \leq 8$. Then $\alpha = \alpha_{max}$. Since $\lambda \neq 0$, we have $\langle \lambda, \alpha \rangle > 0$. Furthermore, either $\langle \lambda, \alpha_j \rangle > 0$ for some $\alpha_j \in R_2$ and hence $\lambda \in \mathbf{X}(R_1, R_2)$ or one of the following holds: a) $r = 6$ and $i = 2$; b) $r = 7$ and $i = 1$; c) $r = 8$ and $i = 8$. In these exceptional cases put $\mu = \lambda - \alpha_i$ and observe that $\lambda = \alpha_{max}$, $\mu \in W\lambda$, $\langle \alpha_i, \alpha_{max} \rangle = 1$, $\langle \mu, \alpha_{max} \rangle = 1$, and $\langle \mu, \alpha_j \rangle = 1$ for some $\alpha_j \in R_2$. Hence $\mu \in \mathbf{X}(R_1, R_2)$ and λ is typical in all possible situations.

8). Now assume that $G = F_4(K)$. As before, we conclude that $\langle\lambda, \alpha\rangle > 0$ as $\lambda \neq 0$. Observe that one of the following holds: a) $\lambda \in \mathbf{X}(R_1, R_2)$; b) $\alpha = \alpha_{max}$ and $i = 1$; c) $\alpha = \alpha_{sh}$ and $i = 4$. In Cases b) and c) we have $\lambda = \alpha$ and $\langle\alpha_i, \alpha\rangle = 1$. In these cases put $\mu = \lambda - \alpha_i$ and complete the proof as in Item 7).

9) Finally, suppose that $G = G_2(K)$. Then either $\lambda \in \mathbf{X}(R_1, R_2)$ or one of the following holds: a) $\alpha = \alpha_{max}$ and $i = 2$; b) $\alpha = \alpha_{sh}$ and $i = 1$. Set $\mu = \lambda - \alpha_2$ in Case a) and $\mu = \lambda - \alpha_1$ in Case b). Observe that $\langle\alpha_2, \alpha_{max}\rangle = \langle\alpha_1, \alpha_{sh}\rangle = 1$. In both cases $\mu \in W\lambda$, $\lambda = \alpha$, and $\mu \in \mathbf{X}(R_1, R_2)$. This completes the proof for the fundamental weights.

Now consider nonfundamental weights λ that cannot be represented in the form $\lambda = \lambda_1 + \lambda_2$ with typical λ_1 for fixed R_1 and R_2 and dominant λ_2. Recall that $r > 2$ and $\alpha = \alpha_1$ for $G = A_r(K)$.

10). First assume that $G = A_r(K)$ with $\lambda = 2\omega_1$ or $2\omega_r$ or $G \in \{B_r(K), C_r(K), D_r(K)\}$ with $\lambda = 2\omega_1$. If $\lambda = 2\omega_1$, the weight $\omega_2 < \lambda$. Hence λ is typical as ω_2 is. Similar arguments with the weight ω_{r-1} are applied to show that $2\omega_r$ is typical for $G = A_r(K)$.

11). Next, let $G = A_r(K)$ and $\lambda = \omega_1 + \omega_r$. Since $r > 2$, it is clear that $\lambda \in \mathbf{X}(R_1, R_2)$.

12) Now suppose that $G = B_r(K)$ or $D_r(K)$, $R_1 = R(\alpha_1)$, $R_2 = R(\alpha_{max})$, $\lambda = 2\omega_r$ for $G = B_r(K)$, and $\lambda \in \{2\omega_{r-1}, 2\omega_r\}$ for $G = D_r(K)$. Set $\delta = \omega_{r-1}$ for $G = B_r(K)$ and $\delta = \omega_{r-2}$ for $G = D_r(K)$. In both cases $\delta < \lambda$. Complete the proof as in Case 10).

13). Finally, let $G = D_r(K)$ and $\lambda \in \{\omega_1 + \omega_{r-1}, \omega_1 + \omega_r, \omega_{r-1} + \omega_r\}$. It is clear that $\lambda \in \mathbf{X}(R_1, R_2)$ or $\lambda = \omega_{r-1} + \omega_r$. In the latter case set $\mu = \lambda - \sum_{j=2}^{r-1} \alpha_j$. Then $\mu \in W\lambda$, $\langle\mu, \alpha_1\rangle = \langle\mu, \alpha_{max}\rangle = 1$, and $\langle\mu, \alpha_r\rangle = 2$. Hence $\mu \in \mathbf{X}(R_1, R_2)$ and we are done.

In all other cases one can write $\lambda = \nu_1 + \nu_2$ where ν_1 is one of the weights considered in Items 1)–13) and ν_2 is dominant. To complete the proof, it remains to apply the Assertion $(*)$.

Proof of Theorem 1. Let φ, ω, R_1, and R_2 satisfy the assumptions of Theorem 1. First assume that $p = 0$ or $\varphi \in \mathrm{Irr}_p$ and that $\omega \notin \mathbf{X}_0$. We claim that assertion 2) of the theorem holds. Without loss of generality we may suppose that $|R_1| \leq |R_2|$. Lemma 4 permits one to reduce the problem to the case where $R_1 = R(\alpha)$ and R_2 is an irreducible component of $C(\alpha)$. Furthermore, one can assume that $|R_2|$ has not decreased in the result of this reduction. Since roots of the same length belong to the same W-orbit, using Lemma 5, we can suppose that α is one of the roots that occur in Table 1. Observe that for $G = B_r(K)$, $C_r(K)$, or $D_r(K)$ we get the pair $(R(\alpha_1), R(\alpha_{max}))$ after these changes if and only if for the initial pair (R_1, R_2) the union $R_1 \cup R_2 = \{\pm\alpha_i \pm \alpha_j\}$ for some i and j, $1 \leq i < j \leq r$. Next, we suppose that R_1, R_2, and φ do not satisfy assertion 2) of the theorem, analyze all rows of Table 1 and for each possible pair $(R_1 = R(\alpha), R_2)$ find a weight $\mu \in \mathbf{X}(\varphi) \cap \mathbf{X}(R_1, R_2)$. Then Lemma 3 completes the proof. By Premet's theorem [7], $\mathbf{X}(\varphi) = \mathbf{X}(\omega)$ unless $p = 2$ and $G = C_r(K)$, $F_4(K)$, or $G_2(K)$ or $p = 3$ and $G = G_2(K)$ (recall that $p \neq 2$ for $G = B_r(K)$ by our assumptions). As we have mentioned before, $\mathbf{X}(\omega) = \{\mu \mid \mu \preceq \omega\}$. Now Proposition 8 implies that it suffices to consider the special cases for p and G indicated above. By Lemma 6, one can assume that φ is tensor indecomposable. By [5, Corollary of Theorem 41], for $G = C_r(K)$ or $F_4(K)$ with $p = 2$ and for $G = G_2(K)$ with $p = 3$ the representation φ is tensor indecomposable if and only if all the coefficients $m_i = 0$ either for the long, or for the short roots α_i. So we have to handle only such weights ω and suppose that the coefficients m_i satisfy the relevant condition.

Now assume that $p = 2$ and $G = C_r(K)$.

1). Suppose that $\alpha = \alpha_1$ and $R_2 = R(3, \ldots, r)$. Let $\omega \neq \omega_r$. Then due to our assumptions $m_i \neq 0$ for some i with $1 < i < r$. Fix minimal such i and set $\mu = \omega - \sum_{j=2}^{i} \alpha_j$. Then $\mu \in W\omega \subset \mathbf{X}(\varphi)$. Since $\langle \mu, \alpha_1 \rangle \neq 0$ and $\langle \mu, \alpha_{i+1} \rangle \neq 0$, we conclude that $\mu \in \mathbf{X}(R_1, R_2)$.

If $\omega = \omega_r$, argue as in Item 4) of the proof of Proposition 8 (for the case $i = r$). Notice that the weight μ constructed in that item lies in the W-orbit of the highest weight and so we actually do not need Premet's theorem for this particular case.

2) Next, let $\alpha = \alpha_1$ and $R_2 = R(\varepsilon_1 + \varepsilon_2)$. Recall that $\omega \neq \omega_r$ in this case. Observe that for $\delta = \sum_{i=1}^{r} b_i \omega_i \in \mathbf{X}$ one has $\langle \delta, \varepsilon_1 + \varepsilon_2 \rangle = b_1 + 2 \sum_{i=2}^{r} b_i$. Choose μ as in Item 1 for $\omega \neq \omega_r$ and check that $\mu \in \mathbf{X}(R_1, R_2)$.

3). Finally, let $\alpha = \alpha_r$. Then $R_2 = R(1, \ldots, r-2, 2\varepsilon_{r-1})$. If $\omega = \omega_r$, then $\omega \in \mathbf{X}(R_1, R_2)$. Now assume that $\omega \neq \omega_r$. Since $\omega \neq \omega_1$, the coefficient $m_i = 1$ for some i with $1 < i < r$. Fix maximal such i and set $\mu = \omega - \sum_{j=i}^{r-1} \alpha_j$. Then $\mu \in W\omega$ and hence $\mu \in \mathbf{X}(\varphi)$. We have $\langle \mu, \alpha_{i-1} \rangle > 0$ and $\langle \mu, \alpha_r \rangle > 0$. Therefore $\mu \in \mathbf{X}(R_1, R_2)$. This completes the proof for $G = C_r(K)$.

Now assume that $p = 2$ and $G = F_4(K)$. Then one of the following holds: a) $\omega \in \mathbf{X}(R_1, R_2)$; b) $\alpha = \alpha_{max}$ and $\omega = \omega_1$; c) $\alpha = \alpha_{sh}$ and $\omega = \omega_4$. To complete the proof, argue as in Item 8) of the proof of Proposition 8 taking into account that the weights μ considered in that item lie in the W-orbit of the highest weight.

Finally, let $p = 2$ or 3 and $G = G_2(K)$. Then either $\omega \in \mathbf{X}(R_1, R_2)$, or one of the following holds: a) $\alpha = \alpha_{max}$ and $\omega = \omega_1$; b) $\alpha = \alpha_{sh}$ and $\omega = \omega_2$; c) $p = 3$, $\alpha = \alpha_{max}$ and $\omega = 2\omega_1$; d)$p = 3$, $\alpha = \alpha_{sh}$ and $\omega = 2\omega_2$. Set $\mu = \omega - \alpha_1$ in Case a), $\omega - \alpha_2$ in Case b), $\omega - 2\alpha_1$ in Case c), and $\omega - 2\alpha_2$ in Case d). Then $\mu \in W\omega$ and hence $\mu \in \mathbf{X}(\varphi)$. Arguments similar to those in Item 9 of the proof of Proposition 8 show that $\mu \in \mathbf{X}(R_1, R_2)$.

Now all the possibilities have been considered. This completes the proof for $\varphi \in \mathrm{Irr}_p$ and for $p = 0$.

The proof of the theorem for $p > 0$ and $\varphi \notin \mathrm{Irr}_p$ follows immediately from the Steinberg tensor product theorem [4, Theorem 1.1], Lemmas 6 and 7, and the result just proven for p-restricted representations.

Proof of Lemma 2. We keep the notation of that lemma. Lemma 4 reduces the question to the following cases: a) $G = B_r(K)$, $\omega = \omega_r$; b) $p = 2$, $G = C_r(K)$, $\omega = \omega_r$; c) $G = D_r(K)$, $\omega = \omega_{r-1}$ or ω_r. Set $S = G(R_1)G(R_2)$. Then commutator relations in G imply that $S \cong A_1(K) \times A_1(K) \cong D_2(K)$. One has $\mathbf{X}(\varphi) = W\omega$; this is well known for $G = B_r(K)$ or $D_r(K)$ and can be deduced for $G = C_r(K)$ and $p = 2$ from the description of the canonical isomorphism between $B_r(K)$ and $C_r(K)$ in characteristic 2. Next, it is also well known and can be deduced by analyzing the action of root subgroups on an irreducible G-module with highest weight ω that $\varphi|S$ is a direct sum of spinor representations of S for $G = B_r(K)$ and $D_r(K)$ and of spinor representations twisted by the Frobenius morphism for $G = C_r(K)$. Hence all composition factors of $\varphi|S$ have dimension 2 and are trivial either for $G(R_1)$ or for $G(R_2)$.

One easily observes that there are many special triples with $\omega \in \mathbf{X}_0$ or $\mathbf{X}_1$.

Lemma 9. *For $\omega \in \mathbf{X}_0$ or $\mathbf{X}_1$ the number of pairs (R_1, R_2) of irreducible root subsystems such that $(\varphi(\omega), R_1, R_2)$ is a special triple tends to infinity when $r \to \infty$.*

Proof. Recall that now G is one of the classical groups. Lemma 7 reduces the question to the case where $\omega \in \mathbf{X}_0$. Fix an integer l with $1 < l < r$ for $G \neq C_r(K)$

and $0 < l < r$ for $G = C_r(K)$. Set

$$R_1 = \begin{cases} R(1, 2, \dots, l-1) & \text{for } G = A_r(K), \\ R(1, 2, \dots, l-1, \varepsilon_{l-1} + \varepsilon_l) & \text{for } G = B_r(K) \text{ or } D_r(K), \\ R(1, 2, \dots, l-1, 2\varepsilon_l) & \text{for } G = C_r(K); \end{cases}$$

and $R_2 = R(l+1, \dots, r-1, r)$ in all cases. One easily observes that R_1 and R_2 are irreducible subsystems and that $G(R_1)$ and $G(R_2)$ commute. Obviously, $R_1 \cap R_2 = \emptyset$. Set $H = G(R_1)G(R_2)$. Analyzing the action of H on the standard G-module and on its dual as well for $G = A_r(K)$, we can conclude that a triple (φ, S_1, S_2) is special if $S_i \subset R_i$ are irreducible subsystems ($i = 1, 2$), $\omega(\varphi) \in \mathbf{X}_0$ for $G \neq D_4(K)$ and $\omega(\varphi) = \omega_1$ for $G = D_4(K)$. This implies the lemma.

Remark 10. *The arguments in the proof of Lemma 9 yield that for $G = D_4(K)$, $\varphi = \varphi(\omega_1)$, $R_1 = R(1, \varepsilon_1 + \varepsilon_2)$, and $R_2 = R(3, 4)$ the triple (φ, R_1, R_2) is special. Since the representations $\varphi(\omega_3)$ and $\varphi(\omega_4)$ can be obtained from φ with the help of graph automorphisms, this forces that special triples exist for them as well.*

The first author has been supported by the Institute of Mathematics of the National Academy of Sciences of Belarus in the framework of the program "Mathematical structures". Part of this research was done during the first author's visit to the University of Leicester in the framework of the project "Asymptotic Problems in Representation Theory" supported by a grant from the Royal Society for Excellence in Science (2004/R1-FSU scheme).

References

1. *Suprunenko I.D.* Minimal polynomials of elements of order p in irreducible representations of Chevalley groups over fields of characteristic p// Siberian Advances in Mathematics. 1996. V.4. 6. P.97-150.
2. *Bourbaki N.* Groupes et algebres de Lie. Ch. VII–VIII. Hermann: Paris. 1975.
3. *Bourbaki N.* Groupes et algebres de Lie. Ch. IV–VI. Hermann: Paris. 1968.
4. *Steinberg R.* Representations of algebraic groups // Nagoya Math. J. 1963. V. 22. P.33-56.
5. *Steinberg R.* Lectures on Chevalley groups. Yale University: New Haven. 1968.
6. *Humphreys J.E.* Linear algebraic groups. Springer-Verlag. New York-Heidelberg-Berlin. 1975.
7. *Premet A.A.* Weights of infinitesimally irreducible representations of Chevalley groups over a field of prime characteristic // Math USSR-Sb. 1988. V. 61. 1. P. 167-183.

Summary

Restrictions of irreducible representations of simple algebraic groups to semisimple subsystem subgroups with two simple components are studied. It is proved that the restrictions almost always have a composition factor that is nontrivial for both components. All exceptions are explicitly described.