INVARIANT IDEALS OF ABELIAN GROUP ALGEBRAS UNDER THE MULTIPLICATIVE ACTION OF A FIELD, I

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Abstract. Let D be a division ring and let $V = D^n$ be a finite-dimensional D-vector space, viewed multiplicatively. If $G = D^\bullet$ is the multiplicative group of D, then G acts on V and hence on any group algebra $K[V]$. Our goal is to completely describe the semiprime G-stable ideals of $K[V]$. As it turns out, this result follows from the corresponding results for the field of rational numbers (due to Brookes and Evans) and for infinite locally-finite fields. Part I of this work is concerned with the latter situation, while Part II deals with arbitrary division rings.

Introduction I

In a long series of papers (see [Z]), the second author studied the ideal structure of various complex group algebras $\mathbb{C}[\mathfrak{H}]$, with $\mathfrak{H}$ an infinite locally-finite simple group. It now appears that the next family of groups to be considered will have the form $\mathfrak{H} = V \rtimes \mathfrak{G}$, where V is an elementary abelian group and $\mathfrak{G}$ is an infinite locally-finite "almost" simple group (see [PZ1]). For example, $\mathfrak{G}$ might be the group $\mathrm{GL}_n(F)$ where F is an infinite locally-finite field, and V could be a suitable finite-dimensional F-vector space viewed multiplicatively. Note that a field is *locally finite* or *absolute* if every finite subset generates a finite subfield. In other words, F is locally finite precisely when it is a subfield of the algebraic closure of a finite field. Now $\mathfrak{G}$ acts on V, so it acts on $\mathbb{C}[V]$. Thus, a necessary first ingredient in the ideal structure of $\mathbb{C}[\mathfrak{H}]$ is a description of the $\mathfrak{G}$-stable ideals of $\mathbb{C}[V]$. Indeed, since $\mathfrak{G}$ might contain an isomorphic copy of $F^\bullet$, the multiplicative group of F acting naturally on the F-vector space V, it is appropriate to consider the $F^\bullet$-stable ideals of $\mathbb{C}[V]$. For this problem, there is really no need to restrict our attention to the

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complex field. Thus, we let K be any field, although we usually require that its characteristic be different from that of F.

Let V be a multiplicative abelian group and let $K[V]$ denote its group algebra over the field K. If A is a subgroup of V, then there exists a natural epimorphism $K[V] \to K[V/A]$ and we let $\omega(A;V) = \omega_K(A;V)$, the *augmentation ideal* of A in V, denote its kernel. Thus, $\omega(A;V)$ is the K-linear span of all elements of the form $(1-a)v$ with $a \in A$ and $v \in V$. If G is a group which acts as automorphisms on V, then G also acts on $K[V]$, and it is clear that A is a G-stable subgroup of V if and only if $\omega(A;V)$ is a G-stable ideal of $K[V]$. The main result of this paper is

Theorem A. *Let F be an infinite locally-finite field and let $V = F^n$ be a finite-dimensional F-vector space, viewed multiplicatively. If $G = F^\bullet$, then G acts on V and hence on the group algebra $K[V]$. Suppose, in addition, that* $\operatorname{char} K \neq \operatorname{char} F$. *Then every G-stable ideal of $K[V]$ can be written uniquely as a finite irredundant intersection $\bigcap_{i=1}^k \omega(A_i;V)$ of augmentation ideals, where each A_i is an F-subspace of V. As a consequence, the set of these G-stable ideals is Noetherian.*

Note that, if V is a torsion abelian group having no elements of order equal to the characteristic of K, then $K[V]$ is a commutative von Neumann regular algebra (see [P, Theorem 1.1.5]). It follows that if $I, J \triangleleft K[V]$, then $I \cap J = IJ$. In particular, finite products and finite intersections of ideals coincide here. Furthermore, every ideal of $K[V]$ is semiprime. In view of this latter comment, it is clear that Theorem A is the locally-finite analog (with some enhancements) of [BE, Proposition 6] which studies finite-dimensional rational vector spaces.

§1. Generalities

Let G be a group of operators on the abelian group V. We start with a simple observation.

Lemma 1.1. *Let W be a subgroup of V.*

i. $K[W] \cap \omega(A;V) = \omega(W \cap A; W)$.
ii. *If $I = \bigcap_{i\in\mathcal{I}} \omega(A_i;V)$, then $K[W] \cap I = \bigcap_{i\in\mathcal{I}} \omega(W \cap A_i; W)$.*

Proof. Part (i) follows since the restriction to $K[W]$ of the map $K[V] \to K[V/A]$ is clearly $K[W] \to K[W/(W \cap A)]$. Part (ii) is immediate from (i). □

Now we say that $\{\, (V_i, G_i) \mid i = 1, 2, \dots \}$ is a *local system* for (V, G) if

(1) $V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n \subseteq \cdots$ and $\bigcup_1^\infty V_i = V$.
(2) $G_1 \subseteq G_2 \subseteq \cdots \subseteq G_n \subseteq \cdots$ and $\bigcup_1^\infty G_i = G$.
(3) G_i stabilizes V_i and both these subgroups are finite.

The next lemma shows that the property of being an intersection of augmentation ideals can be lifted from a local system to the entire group.

Lemma 1.2. *Let $\{ (V_i, G_i) \mid i = 1, 2, \dots \}$ be a local system for (V, G), and let I be a G-stable ideal of $K[V]$. Suppose that, for each i, the ideal $K[V_i] \cap I$ of $K[V_i]$ is an intersection of augmentation ideals of G_i-stable subgroups of V_i. Then I is an intersection of augmentation ideals of G-stable subgroups of V.*

Proof. For each i, let $\mathbb{A}_i$ be the set of all families $\mathfrak{A}$ of G_i-stable subgroups of V_i such that $I_i = K[V_i] \cap I = \bigcap_{A \in \mathfrak{A}} \omega(A; V_i)$. By assumption, each $\mathbb{A}_i$ is nonempty. Furthermore, since V_i is a finite group, it is clear that each $\mathfrak{A}$, as above, is finite, and so also is each $\mathbb{A}_i$. Since

$$I_i = K[V_i] \cap I = K[V_i] \cap (K[V_{i+1}] \cap I) = K[V_i] \cap I_{i+1},$$

Lemma 1.1(ii) implies that there is an intersection map $\mathrm{Int}_i \colon \mathbb{A}_{i+1} \to \mathbb{A}_i$ given by

$$\{ A_1, A_2, \dots , A_t \} \mapsto \{ V_i \cap A_1, V_i \cap A_2, \dots , V_i \cap A_t \}.$$

Thus $\mathbb{T} = \{ \mathbb{A}_i, \mathrm{Int}_i \mid i = 1, 2, \dots \}$ can be viewed as a tree with $\mathbb{A}_i$ being the set of elements at level i and with Int_i describing the branches from level $i + 1$ to level i. Since each $\mathbb{A}_i$ is finite and nonempty, there exists an infinite branch in this tree. In other words, we can choose fixed $\mathfrak{A}_i \in \mathbb{A}_i$ such that $\mathrm{Int}_i \colon \mathfrak{A}_{i+1} \mapsto \mathfrak{A}_i$ for all i. Say $\mathfrak{A}_i = \{ A_{i,1}, A_{i,2}, \dots , A_{i,t_i} \}$.

Since $\mathrm{Int}_i \colon \mathfrak{A}_{i+1} \mapsto \mathfrak{A}_i$, there is a second intersection map $\mathrm{int}_i \colon \mathfrak{A}_{i+1} \to \mathfrak{A}_i$ given by $\mathrm{int}_i \colon A_{i+1,j} \mapsto V_i \cap A_{i+1,j}$. Thus $\mathcal{T} = \{ \mathfrak{A}_i, \mathrm{int}_i \mid i = 1, 2, \dots \}$ can be viewed as a tree with $\mathfrak{A}_i$ being the set of elements at level i and with int_i describing the branches from level $i + 1$ to level i. Again, each $\mathfrak{A}_i$ is finite and nonempty. Furthermore, since $\mathrm{Int}_i(\mathfrak{A}_{i+1}) = \mathfrak{A}_i$, this tree has the additional property that each node at level i is the image of a node at level $i+1$. Equivalently, each $A_{i,j}$ is equal to $V_i \cap A_{i+1,j'}$ for some subscript j'.

Now let $\mathfrak{B}$ be any full branch of the tree $\mathcal{T}$. Then $\mathfrak{B}$ has nodes

$$A_{1,j_1} \subseteq A_{2,j_2} \subseteq \cdots \subseteq A_{n,j_n} \subseteq \cdots$$

for suitable subscripts $j_1, j_2, \dots , j_n, \dots$, and we let $B_{\mathfrak{B}} = \bigcup_i A_{i,j_i}$. We claim that each $B_{\mathfrak{B}}$ is a G-stable subgroup of V and that I is equal to the ideal $J = \bigcap_{\mathfrak{B}} \omega(B_{\mathfrak{B}}; V)$, where the intersection is over all such branches $\mathfrak{B}$. This will certainly yield the result.

To start with, it is clear that each $B_{\mathfrak{B}}$ is a subgroup of V. Furthermore, for any $k \geq i$, we know that $G_k \supseteq G_i$ and that A_{k,j_k} is G_k-stable. Thus $B_{\mathfrak{B}}$ is G_i-stable for all i, and hence it is G-stable. Next, by definition of the map int_i, if $B_{\mathfrak{B}}$ is defined as above, then $V_i \cap A_{k,j_k} = A_{i,j_i}$ for all $k \geq i$ and hence $V_i \cap B_{\mathfrak{B}} = A_{i,j_i}$. Thus, by Lemma 1.1(ii), and the fact that each $A_{i,j}$ is a member of some full branch $\mathfrak{B}$, we have

$$J_i = K[V_i] \cap J = \bigcap_{\mathfrak{B}} \omega(V_i \cap B_{\mathfrak{B}}; V_i) = \bigcap_{j=1}^{t_i} \omega(A_{i,j}; V_i) = K[V_i] \cap I = I_i,$$

where the latter uses the fact that $\{A_{i,1}, A_{i,2}, \dots, A_{i,t_i}\} = \mathfrak{A}_i \in \mathbb{A}_i$. Since V is the ascending union of the subgroups V_i, it now follows from $J_i = I_i$ that $J = I$, and the lemma is proved. □

As is readily apparent, the preceding argument only shows that I is an infinite intersection of augmentation ideals. Even so, this turns out to be a fairly powerful conclusion. For example, if I is properly smaller than $\omega(V; V)$, then it follows from the above that $I \subseteq \omega(A; V)$ for some proper G-stable subgroup A of V. To proceed further, it is necessary to recall some standard notation.

First, a G-stable ideal I of $K[V]$ is said to be *G-prime* if the inclusion $J_1 J_2 \subseteq I$ implies that one of the G-stable ideals J_1 or J_2 is contained in I. Second, $K[V]$ is a *G-prime algebra* if $I = 0$ is a G-prime ideal. In other words, $K[V]$ is G-prime if and only if the product of any two nonzero G-stable ideals is again nonzero. Finally, B/A is a *G-section* of V, if $A \subset B$ are any two distinct G-stable subgroups of V.

Lemma 1.3. *Suppose that all G-sections of V are infinite.*

i. *$K[V]$ is a G-prime algebra.*
ii. *If A is a G-stable subgroup of V, then $\omega(A; V)$ is a G-prime ideal of $K[V]$.*

Proof. (i) Form the group $H = V \rtimes G$, and suppose that I and J are G-stable ideals of $K[V]$ with $IJ = 0$. Then $I' = I \cdot K[H]$ and $J' = J \cdot K[H]$ are two-sided ideals of $K[H]$ with $I'J' = 0$. In particular, if $\Delta(H)$ is the f.c. center of H and if $\theta \colon K[H] \to K[\Delta(H)]$ denotes the natural projection, then [P, Theorem 4.2.9] implies that $\theta(I')\theta(J') = 0$ and hence that $\theta(I)\theta(J) = 0$. But all G-sections of V are infinite, so it follows from [P, Lemma 4.1.8] that $V \cap \Delta(H)$ must be torsion-free abelian. In particular, $K[V \cap \Delta(H)]$ is a domain, so $\theta(I)\theta(J) = 0$ implies that either $\theta(I) = 0$ or $\theta(J) = 0$, and consequently that either $I = 0$ or $J = 0$.

(ii) Since $K[V]/\omega(A; V) = K[V/A]$ and V/A has the same G-structure as V, the result follows from part (i). □

As a consequence, we can now easily describe the uniqueness aspects of finite intersections of augmentation ideals. If $I = \bigcap_1^n \omega(A_i; V)$, and if $A_1 \subseteq A_2$, then $\omega(A_1; V) \subseteq \omega(A_2; V)$ and $\omega(A_2; V)$ is not needed. Thus, we say that the intersection is *irredundant* if $A_i \subseteq A_j$ implies that $i = j$.

Lemma 1.4. *Suppose that all G-sections of V are infinite. Let $A_1, A_2, \dots, A_n$ and $B_1, B_2, \dots, B_m$ be G-stable subgroups of V and assume that*

$$I = \bigcap_1^n \omega(A_i; V) \quad \text{and} \quad J = \bigcap_1^m \omega(B_j; V)$$

are irredundant intersections. Then $I \supseteq J$ if and only if each A_i contains some B_j. In particular, if $I = J$, then $n = m$ and, by reordering the B_j's if necessary, we have $A_i = B_i$ for all i.

Proof. Suppose first that $I \supseteq J$. Then, for each subscript i, we have

$$\omega(A_i; V) \supseteq I \supseteq J \supseteq \prod_1^m \omega(B_j; V).$$

Thus, since $\omega(A_i; V)$ is a G-prime ideal of $K[V]$ by Lemma 1.3(ii), we conclude that $\omega(A_i; V) \supseteq \omega(B_j; V)$ for some j. In particular, $A_i \supseteq B_j$. Conversely, suppose that each A_i contains some B_j. Then, for each i, we have $\omega(A_i; V) \supseteq \omega(B_j; V) \supseteq J$ and hence $I = \bigcap_i \omega(A_i; V) \supseteq J$.

Finally, suppose that $I = J$ and let i be given. Then $I \supseteq J$, so $A_i \supseteq B_j$ for some j. On the other hand, $J \supseteq I$, so $B_j \supseteq A_{i'}$ for some i'. Thus $A_i \supseteq B_j \supseteq A_{i'}$ and the irredundancy of the I-intersection implies that $i = i'$ and $A_i = B_j$. We have therefore shown that

$$\{ A_1, A_2, \dots, A_n \} \subseteq \{ B_1, B_2, \dots, B_m \}.$$

By symmetry, we get the reverse inclusion, and the result follows. □

We will also need the following variant of the above.

Lemma 1.5. *Assume that all G-sections on V are infinite. Let I be a G-stable ideal of $K[V]$, and let $X \subseteq Y$ be G-stable subgroups of V. Suppose that $I \cap K[X] = \bigcap_{i=1}^n \omega(A_i; X)$ and $I \cap K[Y] = \bigcap_{j=1}^m \omega(B_j; Y)$ are finite irredundant intersections, where the A_i and B_j are G-stable subgroups of X and Y, respectively. Then each B_j contains some $A_{j'}$, and each A_i can be written as $A_i = B_{i'} \cap X$ for some $B_{i'}$.*

Proof. By assumption and Lemma 1.1(ii), we have

$$\bigcap_{i=1}^n \omega(A_i; X) = I \cap K[X] = (I \cap K[Y]) \cap K[X] = \bigcap_{j=1}^m \omega(B_j \cap X; X).$$

Since $\omega(B_j \cap X; X) \supseteq \bigcap_{i=1}^n \omega(A_i; X)$, it follows from Lemma 1.3(ii) that $B_j \supseteq B_j \cap X \supseteq A_{j'}$ for some j'. Conversely, if we eliminate unnecessary terms from $\bigcap_{j=1}^m \omega(B_j \cap X; X)$, then we have a finite irredundant intersection, so Lemma 1.4 implies that $A_i = B_{i'} \cap X$ for some i'. □

Next, we introduce an assumption to guarantee that any arbitrary intersection of augmentation ideals can be reduced to a finite intersection.

Lemma 1.6. *Assume that V has a finite-length composition series as a G-module. Let $\mathfrak{A}$ be a nonempty family of G-stable subgroups of V and let $I = \bigcap_{A \in \mathfrak{A}} \omega(A; V)$. Then I can be written as a finite intersection of augmentation ideals $\omega(B; V)$ with each B a G-stable subgroup of V.*

Proof. For convenience we can assume that $V \in \mathfrak{A}$ since $\omega(V; V) \supseteq \omega(A; V)$ for all A. We proceed by induction on the G-composition length of V, the result being trivial if the length is 0. Now if $I = 0$, then $I = \omega(1; V)$ and we are done. So assume that $I \neq 0$ and choose $0 \neq \gamma = \sum_0^n k_i x_i \in I$ with $x_0 = 1$ and $0 \neq k_0 \in K$. Let X_i be the G-stable subgroup of V generated by x_i for $i = 1, 2, \ldots, n$, and let $\mathfrak{A}_i = \{ A \in \mathfrak{A} \mid A \supseteq X_i \}$. Note that each $\mathfrak{A}_i$ is nonempty since $V \in \mathfrak{A}_i$. We claim that $\mathfrak{A} = \bigcup_1^n \mathfrak{A}_i$.

To this end, let $A \in \mathfrak{A}$. Then

$$\gamma = \sum_0^n k_i x_i \in I \subseteq \omega(A; V),$$

so $x_0 = 1$ and $k_0 \neq 0$ implies that $x_i \in A$ for some $i \neq 0$. But A is G-stable, so $X_i \subseteq A$ and hence $A \in \mathfrak{A}_i$, as required. Now set $I_i = \bigcap_{A \in \mathfrak{A}_i} \omega(A; V)$, so that $I = I_1 \cap I_2 \cap \cdots \cap I_n$. It clearly suffices to show that each I_i is a finite intersection of suitable $\omega(B; V)$.

For this, note that $A \supseteq X_i$ implies that $\omega(A; V) \supseteq \omega(X_i; V)$ and that

$$\omega(A; V)/\omega(X_i; V) = \omega(A/X_i; V/X_i) \subseteq K[V/X_i].$$

Thus

$$I_i/\omega(X_i; V) = \bigcap_{A \in \mathfrak{A}_i} \omega(A/X_i; V/X_i).$$

Now, $X_i \neq 1$, so V/X_i has smaller G-composition length than V. Thus, by induction, $I_i/\omega(X_i; V)$ can be written as a finite intersection $\bigcap_j \omega(B_{i,j}/X_i; V/X_i)$ where each $B_{i,j}$ is a G-stable subgroup of V containing X_i. Thus $I_i = \bigcap_j \omega(B_{i,j}; V)$ and we have the finite intersection

$$I = \bigcap_i I_i = \bigcap_{i,j} \omega(B_{i,j}; V),$$

as required. □

A slight generalization of part of the preceding argument yields the following result which requires no assumption on the G-module structure of V.

Lemma 1.7. *Let $\mathfrak{A}$ be a nonempty family of subgroups of V, let $I = \bigcap_{A \in \mathfrak{A}} \omega(A; V)$, and let $\alpha \in I$. Then we can write $\mathfrak{A}$ as a finite union $\mathfrak{A} = \mathfrak{A}_1 \cup \mathfrak{A}_2 \cup \cdots \cup \mathfrak{A}_m$, depending upon α, so that if $A_i = \bigcap_{A \in \mathfrak{A}_i} A$, then $\alpha \in \bigcap_1^m \omega(A_i; V)$.*

Proof. Let S be the *support* of α, namely the finite set of elements of V which appear in the representation of $\alpha \in K[V]$. Note that each $A \in \mathfrak{A}$ partitions V into disjoint A-cosets, and hence each such A gives rise to a partition of S. But S is finite, so there are only finitely many possible partitions. Thus, we can write $\mathfrak{A} = \mathfrak{A}_1 \cup \mathfrak{A}_2 \cup \cdots \cup \mathfrak{A}_m$ as a finite union of nonempty sets, where all As in a fixed

$\mathfrak{A}_i$ yield the same partition $\mathcal{P}_i$ of S. If $A_i = \bigcap_{A\in\mathfrak{A}_i} A$, it suffices to show that $\alpha \in \omega(A_i;V)$. To this end, fix i and write $\alpha = \alpha_1 + \alpha_2 + \cdots + \alpha_k$ where the α_js have support corresponding to the various subsets of S in the partition $\mathcal{P}_i$. Now for any $A \in \mathfrak{A}_i$, we have $\alpha = \sum_j \alpha_j \in \omega(A;V)$. Thus, since each α_j has support in precisely one coset of A and since these cosets are disjoint, we conclude that $\alpha_j \in \omega(A;V)$. In particular, the sum of the coefficients of each α_j is 0, and since the support of α_j is contained in precisely one coset of $A_i = \bigcap_{A\in\mathfrak{A}_i} A$, we conclude that $\alpha_j \in \omega(A_i;V)$. It follows that $\alpha \in \omega(A_i;V)$, as required. □

Next, we prove the expected Noetherian result.

Lemma 1.8. *Assume that V has a finite-length composition series as a G-module and that all composition factors are infinite. If $\mathcal{I}$ is the set of ideals of $K[V]$ which can be written as intersections of augmentation ideals of G-stable subgroups of V, then $\mathcal{I}$ satisfies the ascending chain condition.*

Proof. It is clear from the assumption that every G-section of V is infinite. Furthermore, Lemma 1.6 implies that every ideal in $\mathcal{I}$ can be written as a finite intersection of augmentation ideals of G-stable subgroups of V. Indeed, we can then assume that these intersections are irredundant and hence satisfy the uniqueness condition of Lemma 1.4. In other words, each $I \in \mathcal{I}$ corresponds uniquely to a finite irredundant set $\{A_1, A_2, \dots\}$ of G-stable subgroups of V. Moreover, if J corresponds to $\{B_1, B_2, \dots\}$, then $I \supseteq J$ if and only if each A_i contains some B_j.

Let $\mathbb{S}$ denote the collection of all finite irredundant sets $\{A_1, A_2, \dots\}$ of G-stable subgroups of V, and define $\{B_1, B_2, \dots\} \preceq \{A_1, A_2, \dots\}$ if and only if each A_i contains some B_j. In view of the comments of the preceding paragraph, it suffices to show that $\mathbb{S}, \preceq$ satisfies the ascending chain condition. To this end, for every G-stable subgroup A of V, define the *depth* of A, $d(A)$, to equal the G-composition length of V/A, and note that this parameter is at most equal to the G-composition length of V. Now let $\mathcal{T}_1 \preceq \mathcal{T}_2 \preceq \mathcal{T}_3 \preceq \cdots$ be an ascending sequence of elements of $\mathbb{S}$ and let $d(\mathcal{T})$, the depth of this sequence $\mathcal{T}$, equal the largest depth of all the G-stable subgroups which occur as members of the various $\mathcal{T}_i$s. We prove by induction on $d(\mathcal{T})$ that the sequence terminates, the result being trivial if $d(\mathcal{T}) = 0$ since V is the only G-stable subgroup of V having depth ≤ 0.

Now suppose that $d(\mathcal{T}) = n$ and that the result holds for all sequences of smaller depth. Write $\mathcal{T}_i = \mathcal{T}_i' \cup \mathcal{T}_i''$, where $\mathcal{T}_i'$ contains the G-stable subgroups of depth smaller than n and $\mathcal{T}_i''$ contains those of depth precisely n. Suppose $r \leq s$ and let $A \in \mathcal{T}_s'' \subseteq \mathcal{T}_s$. Then $\mathcal{T}_r \preceq \mathcal{T}_s$ implies that A contains some $B \in \mathcal{T}_r$. But $B \subseteq A$ implies that $d(B) \geq d(A) = n$, so the definition of $n = d(\mathcal{T})$ implies that $d(B) = n$ and $A = B$. In other words, if $r \leq s$ then $\mathcal{T}_r'' \supseteq \mathcal{T}_s''$ and we obtain the decreasing sequence of finite sets $\mathcal{T}_1'' \supseteq \mathcal{T}_2'' \supseteq \mathcal{T}_3'' \supseteq \cdots$ which clearly terminates. By deleting the first few terms if necessary, we can now assume that all $\mathcal{T}_i''$ are equal.

Again let $r \leq s$ and now take $A \in \mathcal{T}_s' \subseteq \mathcal{T}_s$. Then $\mathcal{T}_r \preceq \mathcal{T}_s$ implies that A contains some $B \in \mathcal{T}_r = \mathcal{T}_r' \cup \mathcal{T}_r''$. If $B \in \mathcal{T}_r''$, then $B \in \mathcal{T}_s''$ and $A \supseteq B$ violates the irredundancy of the set $\mathcal{T}_s$. Thus $B \in \mathcal{T}'_r$ and we conclude that $\mathcal{T}'_r \preceq \mathcal{T}'_s$. Since each $\mathcal{T}'_i$ is obviously irredundant, we now have a new ascending sequence $\mathcal{T}'_1 \preceq \mathcal{T}'_2 \preceq \mathcal{T}'_3 \preceq \cdots$ in $\mathbb{S}$, and this one has depth smaller than n. By induction, this new sequence terminates, and since all $\mathcal{T}''_i$ are equal, the original sequence $\mathcal{T}$ also terminates. □

Finally, we show that the property of being an intersection of augmentation ideals can be lifted from finitely generated submodules to the entire group.

Lemma 1.9. *Let G act on V in such a way that all G-sections are infinite, and let I be a G-stable ideal of the group algebra $K[V]$. Assume that, for every finitely generated G-submodule W of V, the G-stable ideal $I \cap K[W]$ is a finite intersection of augmentation ideals of G-stable subgroups of W. Then I is an intersection of augmentation ideals of G-stable subgroups of V.*

Proof. Let $\mathbb{S}$ denote the set of all finitely generated G-submodules of V. If $X \in \mathbb{S}$ then, by assumption and Lemma 1.4, $I \cap K[X] = \bigcap_{i=1}^{n} \omega(A_i; X)$ is uniquely a finite irredundant intersection of augmentation ideals with each A_i a G-submodule of X. For convenience, write $\mathbb{A}_X = \{ A_1, A_2, \dots, A_n \}$.

For each $X \in \mathbb{S}$ and $A \in \mathbb{A}_X$, let $\mathbb{B}_{X,A}$ denote the set of G-submodules B of V satisfying

i. $B \cap X \subseteq A$, and
ii. $B \cap Y$ contains a member of $\mathbb{A}_Y$ for each $Y \in \mathbb{S}$ with $Y \supseteq X$.

We first prove that each $\mathbb{B}_{X,A}$ is nonempty. To this end, let $\mathbb{S}_X$ denote the subset of $\mathbb{S}$ consisting of all Y with $Y \supseteq X$, and consider all "choice" functions f defined on subsets $\mathcal{D}$ of $\mathbb{S}_X$ satisfying

$$f\colon \mathcal{D} \to \bigcup_{Y \in \mathbb{S}_X} \mathbb{A}_Y = \mathbb{T}_X \quad \text{and} \quad f(Y) \in \mathbb{A}_Y .$$

Let us say that such a function f is "good" if for all finite subsets $\{ Y_1, Y_2, \dots, Y_m \}$ of $\mathcal{D}$, we have

$$X \cap \langle f(Y_1), f(Y_2), \dots, f(Y_m) \rangle \subseteq A.$$

It is clear that if $f\colon \mathcal{D} \to \mathbb{T}_X$ is good, then so is the restriction of f to any subset of $\mathcal{D}$. Next, suppose that $Y_1, Y_2, \dots, Y_n \in \mathbb{S}_X$ and let $Z = \langle Y_1, Y_2, \dots, Y_n \rangle$. By assumption and Lemma 1.5, there exists some $E \in \mathbb{A}_Z$ with $E \cap X = A$. Furthermore, by Lemma 1.5 again, $E \cap Y_i \supseteq C_i$ for some $C_i \in \mathbb{A}_{Y_i}$. Thus, if we define

$$f : \{ Y_1, Y_2, \dots, Y_m \} \to \mathbb{T}_X$$

by $f(Y_i) = C_i$, then

$$X \cap \langle f(Y_1), f(Y_2), \dots, f(Y_m) \rangle = X \cap \langle C_1, C_2, \dots, C_m \rangle \subseteq X \cap E = A$$

and f is a good function.

Since each $\mathbb{A}_Y$ is finite, the Compactness Theorem (see [P, Theorem 6.3.1] for a slightly weaker version of this result) implies that there exists a good function $g\colon \mathbb{S}_X \to \mathbb{T}_X$, and we set $B = \langle g(Y) \mid Y \in \mathbb{S}_X \rangle$. Then B is certainly a G-stable subgroup of V and, for all $Y \in \mathbb{S}_X$, we have $B \cap Y \supseteq g(Y)$, so $B \cap Y$ contains a member of $\mathbb{A}_Y$. In addition, if $v \in X \cap B$, then there exist $Y_1, Y_2, \ldots, Y_m \in \mathbb{S}_X$ with $v \in X \cap \langle g(Y_1), g(Y_2), \ldots, g(Y_m)\rangle \subseteq A$. Thus $X \cap B \subseteq A$, so $B \in \mathbb{B}_{X,A}$ and $\mathbb{B}_{X,A}$ is indeed a nonempty set.

It is now a simple matter to prove that I is equal to $J = \bigcap \omega(B; V)$, where the latter intersection is over all $X \in \mathbb{S}$, $A \in \mathbb{A}_X$, and $B \in \mathbb{B}_{X,A}$. For this, first note that if $B \in \mathbb{B}_{X,A}$ and if $Y \in \mathbb{S}_X$, then $B \cap Y \supseteq C$ for some $C \in \mathbb{A}_Y$ by condition (ii). Thus

$$\omega(B; V) \cap K[Y] = \omega(B \cap Y; Y) \supseteq \omega(C; Y) \supseteq I \cap K[Y]$$

and, since this holds for all such Y, it follows that $\omega(B; V) \supseteq I$. Consequently, $J = \bigcap \omega(B; V) \supseteq I$. Conversely, let $X \in \mathbb{S}$ and let $\mathbb{A}_X = \{ A_1, A_2, \ldots, A_n \}$. Then, for each i, $\mathbb{B}_{X,A_i} \neq \emptyset$, so we can choose some B_i in this set, and condition (i) yields

$$\begin{aligned} K[X] \cap J &\subseteq K[X] \cap \bigcap_{i=1}^{n} \omega(B_i; V) \\ &= \bigcap_{i=1}^{n} \omega(B_i \cap X; X) \subseteq \bigcap_{i=1}^{n} \omega(A_i; X) = K[X] \cap I. \end{aligned}$$

Since this holds for all $X \in \mathbb{S}$, we have the reverse inclusion, and therefore $I = J$ is a suitable intersection. □

§2. Locally Finite Fields

We continue to assume that K is an arbitrary field. In addition, we take F to be a locally-finite field and we let V be an F-vector space, viewed multiplicatively. Then V is an elementary abelian p-group, where $p = \operatorname{char} F$, and $G = F^\bullet$ acts on V in a natural manner. We begin with the case where both F and V are finite.

Lemma 2.1. *Let F be a finite field and let V be a finite-dimensional F-vector space, viewed multiplicatively. Assume that $\operatorname{char} F \neq \operatorname{char} K$. Then $G = F^\bullet$ acts on V and every G-stable ideal of $K[V]$ contained in $\omega(V; V)$ is a finite intersection of augmentation ideals $\omega(A; V)$ with A a G-stable subgroup of V.*

Proof. Suppose first that K is algebraically closed or at least that it contains a primitive pth root of unity for $p = \operatorname{char} F$. Let $\Lambda = \Lambda(V)$ be the set of nonprincipal irreducible representations of $K[V]$ and let $\{ e_\lambda \mid \lambda \in \Lambda \}$ be the corresponding set of primitive idempotents in $K[V]$. Then we know that the latter are orthogonal idempotents and that $\omega(V;V) = \oplus\sum_{\lambda\in\Lambda} Ke_\lambda$. Furthermore, if I is an ideal of $K[V]$ contained in $\omega(V;V)$, then $I = \oplus\sum_{\lambda\in\Lambda} Ie_\lambda$ and, for each λ, Ie_λ is either 0 or Ke_λ. Thus I is uniquely determined by its *support*, namely $\operatorname{supp} I = \{\, \lambda \in \Lambda \mid e_\lambda \in I \,\}$. Indeed, $I = \oplus\sum_{\lambda\in\operatorname{supp} I} Ke_\lambda$, and I is G-stable if and only if $\operatorname{supp} I$ is G-stable.

Let $\mathcal{O}_1, \mathcal{O}_2, \dots, \mathcal{O}_m$ be the orbits of the natural action of G on Λ, and let I_i be the ideal of $K[V]$ contained in $\omega(V;V)$ and satisfying $\operatorname{supp} I_i = \Lambda \setminus \mathcal{O}_i$. Then, the I_is are clearly the maximal G-stable ideals of $K[V]$ properly contained in $\omega(V;V)$. Furthermore, if I is a G-stable ideal of $K[V]$ with $I \subseteq \omega(V;V)$ and $\operatorname{supp} I = \Lambda \setminus \bigcup_{i\in\mathcal{I}} \mathcal{O}_i$, then clearly $I = \bigcap_{i\in\mathcal{I}} I_i$. Thus, it suffices to show that each I_i is a suitable augmentation ideal.

To this end, let $|F| = q$ and $|V| = q^n$. Since the split extension $V \rtimes G$ is a Frobenius group, it follows that G acts in a semiregular fashion, that is with full orbit sizes, on Λ. Thus, since $|\Lambda| = q^n - 1$, we see that the number of orbits $\mathcal{O}_i$ is precisely equal to $m = (q^n - 1)/(q - 1)$ and this is the same as the number of F-subspaces of V of codimension 1. In particular, if $n = 1$, then $m = 1$ and we see that $\omega(V;V)$ is the unique nonzero G-stable ideal of $K[V]$ contained in $\omega(V;V)$. For the general case, let $A_1, A_2, \dots, A_m$ be the m subspaces of V of codimension 1. Then each $\omega(A_i;V)$ is G-stable and

$$\omega(V;V)/\omega(A_i;V) = \omega(V/A_i;V/A_i) \triangleleft K[V/A_i].$$

But $\dim_F V/A_i = 1$, so the above $n = 1$ observation implies that $\omega(V/A_i;V/A_i)$ is a minimal nonzero G-stable ideal of $K[V/A_i]$. Consequently, each $\omega(A_i;V)$ is a G-stable ideal of $K[V]$ maximal subject to being properly contained in $\omega(V;V)$. In other words, we have constructed m distinct G-stable ideals $\omega(A_i;V)$, all maximal subject to being properly contained in $\omega(V;V)$. Since this obviously accounts for all the ideals $I_1, I_2, \dots, I_m$, the algebraically closed case is proved..

Finally, let K be arbitrary, subject to $\operatorname{char} K \neq \operatorname{char} F$, and let K' be its algebraic closure. Note that, if J is any ideal of $K[V]$, then $J' = J{\cdot}K' = J{\cdot}K'[V]$ is an ideal of $K'[V]$. Furthermore, since K is a K-module direct summand of K', it follows that $J = J' \cap K[V]$. In particular, since $\omega_K(A;V)' = \omega_{K'}(A;V)$, we see that $\omega_{K'}(A;V) \cap K[V] = \omega_K(A;V)$. Now I is a G-stable ideal of $K[V]$ contained in $\omega_K(V;V)$, so I' is a G-stable ideal of $K'[V]$ contained in $\omega_{K'}(V;V)$. Thus, by the algebraically closed result, $I' = \bigcap_1^t \omega_{K'}(B_i;V)$ for suitable G-stable subgroups B_i of V, and consequently

$$I = I' \cap K[V] = \bigcap_1^t \bigl(\omega_{K'}(B_i;V) \cap K[V]\bigr) = \bigcap_1^t \omega_K(B_i;V),$$

as required. □

It is now a simple matter, using the results of the preceding section, to lift this lemma to the infinite situation. Note that, if V is an F-vector space and if $G = F^\bullet$, then the G-stable subgroups of V are precisely the F-subspaces of V. In particular, when F is infinite, each G-section of V is a nontrivial F-vector space and hence is also infinite.

Lemma 2.2. *Let F be an infinite-locally finite field and suppose that V is an F-vector space, viewed multiplicatively. Assume that* $\operatorname{char} K \neq \operatorname{char} F$. *Then $G = F^\bullet$ acts on V and every G-stable ideal of $K[V]$ is an intersection of augmentation ideals $\omega(A; V)$ with A a G-stable subgroup of V.*

Proof. Assume first that $V = F^n$ is finite dimensional, and let $F_1 \subseteq F_2 \subseteq \cdots$ be an ascending union of finite subfields of F with $\bigcup_1^\infty F_i = F$. If $G_i = F_i^\bullet$ and $V_i = F_i^n$, then it is easy to see that $\{ (V_i, G_i) \mid i = 1, 2, \dots \}$ is a local system for (V, G). Now if I is any G-stable ideal of $K[V]$ contained in $\omega(V; V)$, then Lemma 1.1(i) implies that, for each i, $K[V_i] \cap I$ is a G_i-stable ideal of $K[V_i]$ contained in $\omega(V_i; V_i)$. Hence, by Lemma 2.1, each $K[V_i] \cap I$ is a finite intersection of augmentation ideals $\omega(A; V_i)$ with A a G_i-stable subgroup of V_i, and it follows from Lemma 1.2 that I is an intersection of augmentation ideals of G-stable subgroups of V.

Next, let I be an arbitrary G-stable ideal of $K[V]$. If $I \supseteq \omega(V; V)$, then either $I = \omega(V; V)$ or $I = K[V]$, and the latter is an empty intersection of augmentation ideals. On the other hand, if $I \not\supseteq \omega(V; V)$, then $J = I \cap \omega(V; V)$ is a G-stable ideal of $K[V]$ properly contained in $\omega(V; V)$. By the above, J is contained in some augmentation ideal $\omega(B; V)$ with B a G-stable subgroup of V properly smaller than V. In particular, $\omega(B; V) \not\supseteq \omega(V; V)$. But $\omega(B; V) \supseteq J = I \cap \omega(V; V)$ and $\omega(B; V)$ is a G-prime ideal by Lemma 1.3(ii). Hence $I \subseteq \omega(B; V) \subseteq \omega(V; V)$ and the result of the preceding paragraph applies to I.

Finally, if V is an arbitrary F-vector space, then the finitely generated G-submodules of V are precisely the finite-dimensional F-subspaces of V. With this, the preceding remarks and Lemma 1.9 yield the result. □

We can now offer the

Proof of Theorem A. If V is a finite-dimensional F-vector space, then it is clear that V has G-composition length equal to $\dim_F V < \infty$. Now, by Lemma 2.2, every G-stable ideal of $K[V]$ is an intersection of augmentation ideals $\omega(A; V)$ with A a G-stable subgroup of V. Next, Lemma 1.6 implies that every such ideal is a finite intersection of suitable augmentation ideals. We can then assume that these intersections are irredundant, and conclude from Lemma 1.4 that the corresponding G-stable subgroups are unique. Finally, by Lemma 1.8, the set of all such ideals satisfies the ascending chain condition. □

Let $\mathfrak{G}$ be a group acting on a set Λ. Then we recall that an element $\lambda \in \Lambda$ is said to be $\mathfrak{G}$-*orbital* if it has finitely many $\mathfrak{G}$-conjugates or equivalently if the stabilizer in $\mathfrak{G}$ of λ has finite index in the group.

Lemma 2.3. *Let F be an infinite locally-finite field, let V be a finite-dimensional F-vector space, viewed multiplicatively, and let $\mathfrak{G}$ be a group which acts on V and* *contains $F^\bullet$, in its natural action, as a normal subgroup. If* $\operatorname{char} K \neq \operatorname{char} F$, *then every $\mathfrak{G}$-stable ideal of $K[V]$ is uniquely an irredundant intersection $\bigcap_{A\in\mathfrak{A}} \omega(A;V)$, where $\mathfrak{A}$ is a finite $\mathfrak{G}$-stable set of F-subspaces of V. In particular, each $A \in \mathfrak{A}$ is an $F^\bullet$-stable, $\mathfrak{G}$-orbital subgroup of V.*

Proof. If I is $\mathfrak{G}$-stable, then it is $F^\bullet$-stable. Hence by Theorem A, I can be written uniquely as the irredundant intersection $I = \bigcap_{i=1}^m \omega(A_i;V)$ where each A_i is $F^\bullet$-stable. In particular, if $g \in \mathfrak{G}$, then since I is $\mathfrak{G}$-stable, we have

$$\bigcap_{i=1}^m \omega(A_i;V) = I = I^g = \bigcap_{i=1}^m \omega(A_i^g;V).$$

But $F^\bullet$ is normal in $\mathfrak{G}$, so each A_i^g is certainly an $F^\bullet$-stable subgroup of V. Hence, by uniqueness, $\{A_1^g, A_2^g, \ldots, A_m^g\} = \{A_1, A_2, \ldots, A_m\}$ and consequently $\mathfrak{G}$ permutes the finite set $\mathfrak{A} = \{A_1, A_2, \ldots, A_m\}$. □

With this, we can consider a question posed in [HZ]. Namely, let F be a locally-finite field and suppose that F is generated by two infinite subfields F_1 and F_2. If V is an F-vector space and if I is an ideal of $K[V]$ stable under both $F_1^\bullet$ and $F_2^\bullet$, must I also be stable under $F^\bullet$? We obtain an affirmative answer, at least when $F_1 \cap F_2$ is infinite.

Lemma 2.4. *Let F be a locally finite field generated by the two subfields F_1 and F_2 with $F_1 \cap F_2$ infinite. Let V be an F-vector space, viewed multiplicatively, and let* $\operatorname{char} K \neq \operatorname{char} F$. *If I is an ideal of $K[V]$ stable under both $F_1^\bullet$ and $F_2^\bullet$, then I is stable under $F^\bullet$.*

Proof. Let I be as above and choose $\alpha \in I$ and $f \in F^\bullet$. The goal is to show that $\alpha^f \in I$. Now $f \in \langle F_1, F_2\rangle$, so there exists a subfield $\tilde{F}_2$ of F_2 with $f \in \langle F_1, \tilde{F}_2\rangle = \tilde{F}$ and with degree $(\tilde{F}_2 : F_1 \cap F_2) < \infty$. Furthermore, there exists a finite-dimensional $\tilde{F}$-subspace $\tilde{V}$ of V with $\alpha \in I \cap K[\tilde{V}] = \tilde{I}$. Since $(\tilde{F} : F_1) \leq (\tilde{F}_2 : F_1 \cap F_2) < \infty$ and $\dim_{\tilde{F}} \tilde{V} < \infty$, we see that $\tilde{V}$ is also finite dimensional as an F_1-space. Now $\tilde{I}$ is certainly both $F_1^\bullet$- and $\tilde{F}_2^\bullet$-stable and F_1 is an infinite field, so Lemma 2.3, applied to the group $F_1^\bullet \tilde{F}_2^\bullet$, implies that $\tilde{I} = \bigcap_{A\in\mathfrak{A}} \omega(A;\tilde{V})$ where each $A \in \mathfrak{A}$ is an $F_1^\bullet$-stable, $\tilde{F}_2^\bullet$-orbital subgroup of $\tilde{V}$. By assumption, $F_1 \cap F_2$ is infinite, so $\tilde{F}_2$ is infinite, and therefore any subgroup of finite index in $\tilde{F}_2^\bullet$ must additively generate the field $\tilde{F}_2$. In particular, since each $A \in \mathfrak{A}$ is $\tilde{F}_2^\bullet$-orbital, it follows that each such A is $\tilde{F}_2^\bullet$-stable. Furthermore, since each A is $F_1^\bullet$-stable and since $\tilde{F} = \langle F_1, \tilde{F}_2\rangle$, we see that each $A \in \mathfrak{A}$ is $\tilde{F}^\bullet$-stable. Thus, $\tilde{I}$ is an intersection of the $\tilde{F}^\bullet$-stable augmentation ideals $\omega(A;\tilde{V})$, so $\tilde{I}$ is also $\tilde{F}^\bullet$-stable. Since $\alpha \in \tilde{I}$ and $f \in \tilde{F}^\bullet$, we conclude that $\alpha^f \in \tilde{I} \subseteq I$, as required. □

It follows from the above and a trivial induction that if $F = \langle F_1, F_2, \ldots\rangle$ is locally finite and if $F_{n+1} \cap \langle F_1, F_2, \ldots, F_n\rangle$ is infinite for all $n \geq 1$, then an ideal of $K[V]$ is $F^\bullet$-stable if and only if it is $F_n^\bullet$-stable for all n.

Finally, as we observed in the introduction, Theorem A is essentially the locally-finite analog of [BE, Proposition 6], a result on the field of rational numbers. Both of these facts will be used in Part II of this work to handle arbitrary division rings.

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