1

Prime power degree representations of quasi-simple groups

G. Malle and A. E. Zalesskiĭ

October 26, 2007

1 Introduction

The problem of classification of complex finite linear groups of particular degree attracts attention since the beginning of the century. In the 60s a large contribution to the topic was made by R. Brauer, W. Feit and their successors. After the classification of finite simple groups and huge advances in the character theory for groups of Lie type, this problem becomes much more accessible. The list of primitive finite irreducible linear groups G of prime degree was obtained in [6], see also [12, Corollary on page 420]. In [21] for quasi-simple groups G and primes p dividing $|G|$ there were determined all irreducible characters of degree $\leq 2p$. In this paper we list quasi-simple irreducible linear groups G (with $G/Z(G)$ not alternating) of degree p^k for any k. In other words we list the characters whose degrees are divisible by a single prime.

We make heavy use of earlier computations of one of the authors [13] performed for a different purpose. There are hints that the prime divisor structure of degrees of irreducible representations of quasi-simple groups may be useful in various situations and hence deserves further attention. See also [14], [15]. The result is also connected with the problem raised by W. Feit [8] on the existence of p-Steinberg characters for arbitrary finite groups as the degree of such a character is a p-power. For simple groups the problem of Feit has been solved by Tiep [20] who also announced (without a proof) that

---

[*] We thank the referee for a careful reading of the manuscript.

for simple groups of Lie type $G(q)$ with $q > 8$ for p being a non-describing characteristic only Weil characters can be of p-power degree. Abdukhalikov [1] observed that if G is a simple group of Lie type in characteristic p then the Steinberg representation is the only non-trivial one of p-power degree unless the Schur multiplier of G is divisible by p.

The main result of the present paper is the following:

Theorem 1.1 *Let G be a quasi-simple group such that $S := G/Z(G)$ is not an alternating group $\mathfrak{A}_n$ with $n > 18$, and χ a faithful irreducible complex character of G. Suppose that $\chi(1) = p^d$ where p is a prime. Then one of the following holds:*

(1) *$G = S$ is a simple group of Lie type of characteristic p and χ is the Steinberg character of G (so $\chi(1) = |G|_p$);*

(2) *$S = \mathrm{L}_2(q)$ and $\chi(1) \in \{q \pm 1\}$, or q is odd and $\chi(1) \in \{(q \pm 1)/2\}$;*

(3) *$S = \mathrm{L}_n(q)$, $q > 2$, n is an odd prime, $(n, q-1) = 1$, $\chi(1) = (q^n - 1)/(q-1)$;*

(4) *$S = \mathrm{U}_n(q)$, n is an odd prime, $(n, q+1) = 1$, $\chi(1) = (q^n + 1)/(q+1)$;*

(5) *$S = \mathrm{PSp}_{2n}(q)$, $n > 1$, $q = r^k$ with r an odd prime, kn is a 2-power and $\chi(1) = (q^n + 1)/2$;*

(6) *$S = \mathrm{PSp}_{2n}(3)$, $n > 1$ is a prime, $\chi(1) = (3^n - 1)/2$;*

(7) *$G = \mathfrak{A}_{p^d+1}$, $\chi(1) = p^d$;*

(8) $G = 2 \cdot \mathfrak{A}_n$, $\chi(1) = \begin{cases} 2^{(n-3)/2} & \text{if } n \text{ is odd,} \\ 2^{(n-2)/2} & \text{if } n \text{ is even;} \end{cases}$

(9) *$G = 2 \cdot \mathfrak{A}_n$ where $n = 2^{m+1} + 2$ with $m \in \mathbb{N}$ and $\chi(1) = 2^{m+2^m}$.*

(10) *$p^d = 3$, $G = 3 \cdot \mathfrak{A}_6$;*

(11) *$p^d = 7$, $G = \mathrm{Sp}_6(2)$;*

(12) *$p^d = 8$, $G \in \{\mathfrak{A}_6, 4_1 \cdot \mathrm{L}_3(4), 2 \cdot \mathrm{Sp}_6(2), 2 \cdot \Omega_8^+(2)\}$;*

(13) *$p^d = 9$, $G = 3 \cdot \mathfrak{A}_6$;*

(14) *$p^d = 11$, $G \in \{M_{11}, M_{12}\}$;*

(15) *$p^d = 16$, $G \in \{M_{11}, M_{12}, \mathrm{L}_3(3)\}$;*

(16) *$p^d = 23$, $G \in \{M_{24}, Co_2, Co_3\}$;*

(17) $p^d = 27$, $G \in \{\mathfrak{A}_9, \mathrm{Sp}_6(2), 3 \cdot \Omega_7(3), 3 \cdot G_2(3), {}^2F_4(2)'\}$;

(18) $p^d = 32$, $G \in \{2 \cdot M_{12}, \mathrm{U}_3(3) \cong G_2(2)'\}$;

(19) $p^d = 64$, $G \in \{2 \cdot \mathrm{Sp}_6(2), 2 \cdot \mathrm{U}_4(2), 2 \cdot \mathrm{SL}_4(2), 2 \cdot \mathrm{L}_3(4), 4 \cdot \mathrm{L}_3(4), 2 \cdot Sz(8), G_2(3), 2 \cdot J_2\}$;

(20) $p^d = 2^9$, $G = 2 \cdot \mathrm{Sp}_6(2)$;

(21) $p^d = 3^6$, $G \in \{3 \cdot \mathrm{U}_4(3), 3 \cdot G_2(3)\}$;

(22) $p^d = 2^{12}$, $G \in \{2 \cdot G_2(4), 2 \cdot \Omega_8^+(2)\}$;

(23) $p^d = 2^{13}$, $G = 2 \cdot Ru$;

(24) $p^d = 3^9$, $G = 3 \cdot \Omega_7(3)$;

(25) $p^d = 2^{15}$, $G = 2 \cdot \mathrm{U}_6(2)$;

(26) $p^d = 2^{24}$, $G = 2 \cdot F_4(2)$;

(27) $p^d = 2^{36}$, $G = 2 \cdot {}^2E_6(2)$.

Remarks. In (2) – (6) not each value for $\chi(1)$ is a prime power. For example in case (2) this leads to the question of determining Fermat and Mersenne primes. In all of the cases (2) – (6) it is explicitly known for which covering group G of S the character χ is faithful.

We have made use of the isomorphisms $\mathfrak{A}_6 \cong \mathrm{L}_2(9)$, $\mathfrak{A}_8 \cong \mathrm{SL}_4(2)$, $\mathrm{U}_4(2) \cong \mathrm{PSp}_4(3)$ to shorten the list.

The case where $G/Z(G) \cong \mathfrak{A}_n$ seems to be non-obvious, although the degrees of irreducible characters are known explicitly by the hook formula. We exclude this case from our analysis and content ourselves with stating the following conjecture which has been checked by computer for $n < 19$.

Conjecture. Let $G/Z(G) \cong \mathfrak{A}_n$ with $n > 8$, and $\chi \neq 1_G$ an irreducible complex character of G. Suppose that $\chi(1) = p^d$ where p is a prime. Then one of the following holds:

(1) $G = \mathfrak{A}_{p^d+1}$, $\chi(1) = p^d$;

(2) $G = 2 \cdot \mathfrak{A}_n$, $\chi(1) = \begin{cases} 2^{(n-3)/2} & \text{if } n \text{ is odd,} \\ 2^{(n-2)/2} & \text{if } n \text{ is even} \end{cases}$
(χ is the character of a so called basic representation of G);

(3) $G = 2 \cdot \mathfrak{A}_n$ where $n = 2^{m+1} + 2$ with $m \in \mathbb{N}$ and $\chi(1) = 2^{m+2^m}$.

After the completion of this manuscript the case $G = \mathfrak{A}_n$ of the preceding conjecture has been settled by Bessenrodt, Balog, Olsson and Ono [2].

2 The groups of Lie type

We first give an outline of the argument. Generically, the covering groups of simple groups of Lie type are the groups of fixed points of simple simply-connected algebraic groups over the algebraic closure of a finite field under a Frobenius morphism. As such, their ordinary characters were classified by G. Lusztig. In particular, the degrees of all irreducible characters are known; they are products of cyclotomic polynomials. Using this information we are going to show that most character degrees are divisible by at least two Zsigmondy primes and hence cannot equal the power of a prime. This part of the argument closely follows [13]. The remaining characters either provide examples or can be ruled out by ad hoc arguments.

We are then left to consider the groups of Lie type with exceptional covering group, as well as those groups which do not have two different Zsigmondy primes.

We first collect some well-known lemmas.

Lemma 2.1 (Zsigmondy's theorem) *Let* q *be a prime power. For each* $n \in \mathbb{N}$ *there is a prime* l *such that* l *divides* $q^n - 1$ *and does not divide* $q^m - 1$ *for* $m < n$*, except when* $q = 2, n = 6$ *or* $n = 2$*,* q *is a* 2*-power.*

Such an l is called a *Zsigmondy prime*; the smallest Zsigmondy prime for fixed n and q will be denoted by $l_n(q)$. (A new proof of Zsigmondy's theorem and some related results can be found in [16].)

Lemma 2.2 [11, Ch. IX, Lemma 2.7] *Let* q *be a prime power such that* $q + 1 = 2^s$ *for some* $s \in \mathbb{N}$*. Then* q *is a prime.*

Let G be an algebraic group over the algebraic closure of a finite field. Then there exists an *order polynomial* $|G|(x) = x^k \Pi_m \Phi_m(x)^{r_m}$, with $k, r_m \in \mathbb{N}$, such that, whenever F is a Frobenius morphism defining a rational structure over $\mathbb{F}_q$ then $|G^F| = |G|(q)$ (see for example [3], [10]). Here, $\Phi_m(x)$ is the m-th cyclotomic polynomial. We will need the following criterion for cyclic Sylow subgroups.

Lemma 2.3 *Let* $G = G(q)$ *be a group of Lie type other than* ${}^2B_2(q^2)$*,* ${}^2G_2(q^2)$*,* ${}^2F_4(q^2)$*, with order polynomial* $|G|(x) = x^k \Pi_m \Phi_m(x)^{r_m}$*. Let* m *with* $r_m = 1$ *and let* l *be a prime dividing* $\Phi_m(q)$ *and not dividing any* $\Phi_n(q)$ *for any* $n \neq m$ *with* $r_n \neq 0$*. Then the Sylow* l*-subgroups of* $G(q)$ *are cyclic.*

This is shown, for example, in [10], §10.

Remark. The lemma above is true for $^2B_2(q^2)$, $^2G_2(q^2)$, $^2F_4(q^2)$ as well if q is understood as in [10], but we will not need this assertion.

2.1 The classical groups

We start with the classical groups of Lie type.

We denote by $\mathcal{L}$ the set consisting of the covering groups of the following simple groups:

$\mathrm{L}_2(q)$, $\mathrm{L}_3(q)$, $\mathrm{L}_4(2)$, $\mathrm{L}_6(2)$, $\mathrm{L}_7(2)$, $\mathrm{U}_3(q)$, $\mathrm{U}_4(2)$, $\mathrm{U}_4(3)$, $\mathrm{U}_6(2)$,

$\mathrm{PSp}_4(q)$, $\mathrm{Sp}_4(2)'$, $\mathrm{Sp}_6(2)$, $\mathrm{Sp}_8(2)$, $\Omega_7(3)$, $\Omega_8^+(2)$, $\Omega_8^-(2)$. $G_2(3)$, $G_2(4)$.

Lemma 2.4 *Theorem* 1.1 *is true for the groups in the list* $\mathcal{L}$.

Proof. One only needs to check the character tables of these groups which are available in the explicit form, see [17], [18], [7], [4].

The following table is taken from Table 3.5 in [13]. For each group of classical Lie type it specifies two (conjugacy classes of) maximal tori T_1, T_2 by giving their orders (which determines them uniquely), and two Zsigmondy primes l_1, l_2 such that l_i divides $|T_i|$. Note that, the groups from $\mathcal{L}$ being excluded, both Zsigmondy primes exist in all cases by Lemma 2.1.

TABLE 1: Two tori for classical groups
(The groups from $\mathcal{L}$ are excluded)

G	$\|T_1\|$	$\|T_2\|$	l_1	l_2
$A_n(q), n > 2$	$(q^{n+1}-1)/(q-1)$	q^n-1	$l_{n+1}(q)$	$l_n(q)$
$^2A_n(q)$, $n \equiv 0 (\mathrm{mod}\, 4)$	$(q^{n+1}+1)/(q+1)$	q^n-1	$l_{2n+2}(q)$	$l_n(q)$
$^2A_n(q)$, $n \equiv 1 (\mathrm{mod}\, 4)$	$(q^{n+1}-1)/(q+1)$	q^n+1	$l_{(n+1)/2}(q)$	$l_{2n}(q)$
$^2A_n(q)$, $n \equiv 2 (\mathrm{mod}\, 4)$	$(q^{n+1}+1)/(q+1)$	q^n-1	$l_{2n+2}(q)$	$l_{n/2}(q)$
$^2A_n(q)$, $n \equiv 3 (\mathrm{mod}\, 4)$	$(q^{n+1}-1)/(q+1)$	q^n+1	$l_{n+1}(q)$	$l_{2n}(q)$
B_n, C_n, n odd	q^n+1	q^n-1	$l_{2n}(q)$	$l_n(q)$
B_n, C_n, n even	q^n+1	$(q^{n-1}+1)(q+1)$	$l_{2n}(q)$	$l_{2n-2}(q)$
D_n, n odd	$(q^{n-1}+1)(q+1)$	q^n-1	$l_{2n-2}(q)$	$l_n(q)$
D_n, n even	$(q^{n-1}+1)(q+1)$	$(q^{n-1}-1)(q-1)$	$l_{2n-2}(q)$	$l_{n-1}(q)$
^2D_n	q^n+1	$(q^{n-1}+1)(q-1)$	$l_{2n}(q)$	$l_{2n-2}(q)$

Lemma 2.5 *Let $G = G(q) \notin \mathcal{L}$ be a quasi-simple group of classical Lie type, and let l_i, T_i be the primes and tori given by Table 1. Then the Sylow l_i-subgroups of G are cyclic and conjugate to a subgroup of T_i.*

Proof. The order polynomials $|G|(x)$ for the classical groups are explicitly given in [3] or [10], for example. From the property of being Zsigmondy primes and the formulae for $|G|(x)$ it is easy to check that l_i divides exactly one of the cyclotomic polynomials Φ_m appearing in $|G|(x)$, and that $r_m = 1$ for this one. We conclude with Lemma 2.3 that the Sylow l_i-subgroups are cyclic. They must even be conjugate to a subgroup of T_i since $\Phi_m(q)$, which divides $|T_i|$, is the only factor in the generic order of G divisible by l.

Corollary 2.6 *Let $G = G(q) \notin \mathcal{L}$ be a quasi-simple group of classical Lie type. Let l_1, l_2, T_1, T_2 be the primes and the tori given by Table 1. Let $\chi \neq 1_G$ be a complex irreducible character of G. Suppose that $\chi(1) \in \{l_1^a, l_2^a\}_{a\in\mathbb{N}}$. Then $\chi(1) \leq \max\{|T_1|, |T_2|\}$.*

Proof. By Lemma 2.5 the torus T_i contains a Sylow l_i-subgroup of G. Since the degree of any complex irreducible character divides $|G|$ the assertion follows.

Lemma 2.7 *Let $G = G(q) \notin \mathcal{L}$ be a quasi-simple group of classical Lie type, and let T_i be the tori given by Table 1. Let $1_G \neq \chi \in \mathrm{Irr}(G)$ with $\chi(1) \leq \max(|T_1|, |T_2|)$. Then one of the following holds:*

(1) *$G = \mathrm{SL}_n(q)$, $n > 2$, $\chi(1) \in \{(q^n-1)/(q-1)\ (q \neq 2), (q^n-q)/(q-1)\}$;*

(2) *$G = \mathrm{SU}_n(q)$, $n > 2$, $\chi(1) \in \{(q^n–(-1)^n)/(q+1), (q^n+q(-1)^n)/(q+1)\}$;*

(3) *$G = \mathrm{Sp}_{2n}(q)$, $n > 2$, q odd, $\chi(1) = (q^n \pm 1)/2$;*

If, moreover, $\chi(1)$ is a prime power then in (1) $\chi(1) = (q^n-1)/(q-1)$, $q \neq 2$ and n is a prime, in (2) $\chi(1) = (q^n+1)/(q+1)$ and n is a prime; in (3) $\chi(1) = (q^n+1)/2$, $q = r^k$ and nk is a 2-power, or $\chi(1) = (3^n-1)/2$ with prime n.

Proof: The first claim follows from the results in [21]. The second one is elementary. For example, in case (3), if $\chi(1) = (q^n-1)/2$ then the numerator is equal to $(q-1)(q^{n-1} + \ldots + q + 1)$. The second factor, but not the first, is divisible by the Zsigmondy prime $l_n(q)$. It follows that $\chi(1)$ cannot be a prime power unless $q = 3$.

We now treat the remaining characters. See Carter's book [3] for the classification of unipotent characters and the notion of symbols.

Lemma 2.8 [13] *Let $G = G(q) \notin \mathcal{L}$ be a simply connected group of classical Lie type of rank > 2, and let l_1, l_2 be the primes from Table 1. Let $\chi \neq 1_G$ be an irreducible complex character of G. Suppose that $(\chi(1), l_1 l_2) = 1$. Then χ is either the Steinberg character of G (of degree $|G|_p$) or one of the following holds:*

(1) *G is of type B_n or C_n, χ is unipotent and is determined by the symbol*

$$\begin{pmatrix} 0 & 1 & n \\ & - & \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 1 & \dots & n-2 & n-1 & n \\ & 1 & \dots & n-2 & & \end{pmatrix}.$$

(2) *G is of type D_n, χ is unipotent and is determined by the symbol*

$$\begin{pmatrix} n-1 \\ 1 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 2 & \dots & n-3 & n-1 & n \\ 1 & 3 & \dots & n-2 & n-1 & \end{pmatrix}.$$

(3) *G is of type $C_n(q)$ with q odd, and*

$$\chi(1) = (q^n - 1)/2 \text{ or } \chi(1) = q^{n(n-2)/4}(q^n - 1)/2.$$

The proof was already given as part of the argument in [13], see sections 3B–3G there. It uses Lusztig's parametrization of irreducible characters of G in terms of the Jordan decomposition and explicit formulae for the degrees of unipotent characters.

Proof of Theorem 1.1 for groups of classical Lie type: We may assume that $G \notin \mathcal{L}$ since the groups in $\mathcal{L}$ were treated in Lemma 2.4. Suppose that $\chi \in \mathrm{Irr}(G)$ is not the Steinberg character of G and $\chi(1) > 1$.

Let l_1, l_2 be the primes given by Table 1 for $G = G(q)$. Then the Sylow l_i-subgroups of G are cyclic (Lemma 2.5). Clearly we may suppose for our purpose that $\chi(1)$ is divisible by at most one of l_1, l_2. Suppose first that $\chi(1)$ is divisible by l_i with $i \in \{1, 2\}$. By Corollary 2.6 we have $\chi(1) \leq \max\{|T_1|, |T_2|\}$, with T_1, T_2 as in Table 1. But then by Lemma 2.7 it follows that for this case Theorem 1.1 is true.

Next, suppose that $(\chi(1), l_1 l_2) = 1$. By Lemma 2.8 either χ is the Steinberg character of G or one of the characters indicated in (1), (2) or (3). Note that in all three cases, the second character given is the Alvis-Curtis dual of the first, thus its degree is a positive power of q times the degree of the first. It hence suffices to rule out that the first possibility in each of (1), (2) and (3) gives an example.

In case (3) of Lemma 2.8 the first possibility was already considered in Lemma 2.7 and we arrive at case (6) of Theorem 1.1. Thus we are left with the unipotent characters in cases (1) and (2) of Lemma 2.8. By [3], 13.8, the degree of the unipotent character of $G = B_n(q)$ or $G = C_n(q)$ labeled by the symbol $\begin{pmatrix} 0 & 1 & n \\ & - & \end{pmatrix}$ is equal to $(q^n - 1)(q^n - q)/2(q + 1)$ which is not a prime power since $n \geq 3$, and $n \geq 5$ if $q = 2$. The degree of the unipotent character of $G = D_n(q)$ labelled by the symbol $\begin{pmatrix} n-1 \\ 1 \end{pmatrix}$ is equal to $q(q^n - 1)(q^{n-2} + 1)/(q^2 - 1)$ which is not a prime power either. This completes the proof of Theorem 1.1 for classical groups.

2.2 The exceptional groups

Now we turn to the exceptional groups where we first consider the Suzuki and Ree groups.

Lemma 2.9 *Theorem* 1.1 *is true for the groups* $^2B_2(q^2)$, $^2G_2(q^2)$, $^2F_4(q^2)'$.

Proof. The character table of the universal covering group of $^2B_2(8)$ is contained in [4], as well as the one for $^2F_4(2)'$. For the remaining groups the list of character degrees can be found in [19], [22] and [9]. It is an easy exercise to go through the lists and check that no case arises. Note that $^2G_2(3)$ is not simple, with derived group $\mathrm{L}_2(8)$.

Let us next get rid of exceptional covering groups.

Lemma 2.10 *Theorem* 1.1 *is true for the groups* $3\cdot G_2(3)$, $2\cdot G_2(4)$, $2\cdot F_4(2)$, $2\cdot {}^2E_6(2)$ *and for the faithful characters of* $6\cdot {}^2E_6(2)$.

Proof. The character tables of all but the last group are contained in [4], so the assertion can be verified. All faithful characters of the sixfold cover of $^2E_6(2)$ have degree divisible by 6.

We can now turn to the generic situation. Here again we first choose some tori.

TABLE 2: Tori for exceptional groups

G	$\lvert T_1\rvert$	$\lvert T_2\rvert$	$\lvert T_3\rvert$
G_2	Φ_6	Φ_3	
${}^3D_4$	Φ_{12}		
F_4	Φ_{12}	Φ_8	
E_6	$\Phi_{12}\Phi_3$	Φ_9	$\Phi_8\Phi_2\Phi_1$
${}^2E_6$	Φ_{18}	$\Phi_{12}\Phi_6$	$\Phi_8\Phi_2\Phi_1$
E_7	$\Phi_{18}\Phi_2$	$\Phi_{14}\Phi_2$	$\Phi_{12}\Phi_3\Phi_1$
E_8	Φ_{30}	Φ_{24}	Φ_{20}

We are now able to reduce to the case of unipotent characters.

Lemma 2.11 *Let $G(q)$ be a simply connected exceptional group of type $G_2(q)$ ($q \geq 5$), ${}^3D_4(q)$, $F_4(q)$ ($q \neq 2$), $E_6(q)$, ${}^2E_6(q)$, $E_7(q)$ or $E_8(q)$, and $\chi \in \mathrm{Irr}(G)$ of prime power degree, $\chi(1) > 1$. Then χ is a unipotent character of G.*

Proof. In each case, for each of the tori T_i listed in Table 2, there exists a Zsigmondy prime l_i. From the explicit order polynomials (see for example [3]) it is easy to check that l_i divides just one of the factors $\Phi_m(q)$, and that the relevant cyclotomic polynomial $\Phi_m(x)$ occurs just once in $|G|(x)$. In particular by Lemma 2.3 the Sylow l_i-subgroups of $G(q)$ are cyclic. If $\chi(1)$ is a power of l_i then $\chi(1) \leq \Phi_m(q)$. This contradicts the lower bounds for character degrees in [12]. Hence $\chi(1)$ is prime to all the l_i. It was shown in [13], §4, using Lusztig's Jordan decomposition of characters, that this implies that either χ is unipotent or $G =^3 D_4(q)$ and $\chi(1) = (q^6 - 1)^2$. The latter degree is certainly not a prime power.

The following is easily checked from [3, §13].

Lemma 2.12 *Let G be a Chevalley group of exceptional Lie type. Let $1_G \neq \chi \in \mathrm{Irr}(G)$ be a unipotent character of G. Then $\chi(1)$ is a multiple of q (respectively of q^2 in the case of the Suzuki and Ree groups) except for the following cases:*

$G = G_2(q)$ *with* $q = 2$ (4 *characters*) *or* $q = 3$ (6 *characters*);
$G = F_4(q)$ *with* $q = 2$ (4 *characters*);
$G = {}^2B_2(q^2)$ *with* $q^2 = 2$ (2 *characters*);
$G = {}^2G_2(q^2)$ *with* $q^2 = 3$ (4 *characters*);
$G = {}^2F_4(q^2)$ *with* $q^2 = 2$ (10 *characters*).

All these characters are not of prime power degree except when $G = G_2(3)$ and $\chi(1) = 64$.

(Note that among the exceptions only $G_2(3)$ and $F_4(2)$ are simple groups.)

Proof of Theorem 1.1 for exceptional groups. Suppose that G is a group of exceptional Lie type. By Lemma 2.9 we can assume that G is not a Suzuki- or Ree-group, and Lemma 2.10 shows that we need not worry about the exceptional Schur multipliers. So G is one of groups in 2.11 and thus χ is a unipotent character. The degrees of unipotent characters are of the shape $q^i d$ where $i \geq 0$ and d is a product of at least two factors of shape $\Phi_m(q)$, see [3]. In particular, $(q, d) = 1$ so $i = 0$. This case is dealt with in Lemma 2.12.

3 The sporadic groups

The complex character tables of the covering groups of all sporadic simple groups are known and can be found in the Atlas [4]. Thus the verification of Theorem 1.1 is a routine check.

References

[1] K.S. Abdukhalikov, Automorphism groups of invariant lattices in the Steinberg module of groups of Lie type of odd characteristic, *Sbornik: Mathematics* **189** (1998), 1273–1294.

[2] C. Bessenrodt, A. Balog, J.B. Olsson, K. Ono, Prime power degree representations of the symmetric and alternating groups, preprint 2000.

[3] R. Carter, Finite Groups of Lie Type: Conjugacy Classes and Complex Characters. Wiley-Interscience, 1985.

[4] J.H. Conway, R.T. Curtis, S.P. Norton, R.A. Parker, and R.A. Wilson, Atlas of Finite Groups, Clarendon Press, Oxford, 1985.

[5] M.R. Darafshen, p-Steinberg characters of alternating and projective special linear groups, *J. Algebra* **181** (1996), 196–206.

[6] J. Dixon and A. E. Zalesskiĭ, Finite primitive linear groups of prime degree, *J. London Math. Soc. (2)* **57** (1998), 126–134.

[7] H. Enomoto, The characters of the finite symplectic group $\mathrm{Sp}_4(q)$, $q = 2^f$, *Osaka J. Math.* **9** (1972), 75–94.

[8] W. Feit, Extending Steinberg characters, In "Linear algebraic groups and their representations, Los Angeles, 1992", Contemp. Math. 153 (1993), 1–9.

[9] M. Geck, G. Hiss, F. Lübeck, G. Malle, and G. Pfeiffer, CHEVIE — A system for computing and processing generic character tables, *AAECC* **7** (1996), 175–210.

[10] D. Gorenstein and R. Lyons, The local structure of finite groups of characteristic 2 type, Memoirs of the Amer. Math. Soc. 42 (1983), no. 276, 731 pp.

[11] B. Huppert and N. Blackburn, Finite Groups II. Springer-Verlag, 1982

[12] V. Landázuri and G. Seitz, On the minimal degrees of the projective representations of the finite Chevalley groups, *J. Algebra* **32** (1974), 418–443.

[13] G. Malle, Almost irreducible tensor squares, *Comm. in Algebra* **27** (1999), 1033–1051.

[14] G. Malle, J. Saxl, and T. Weigel, Generation of classical groups, *Geom. Dedicata* **49** (1994), 85–116.

[15] J. Olsson, Remarks on symbols, hooks and degrees of unipotent characters, *J. Algebra* **42** (1986), 223–238.

[16] M. Roitman, On Zsigmondy primes, *Proc. Amer. Math. Soc.* **125** (1997), 1913–1919.

[17] W. Simpson and J. Frame, The character tables for $\mathrm{SL}(3,q)$, $\mathrm{SU}(3,q)$, $\mathrm{PSL}(3,q)$, $\mathrm{PSU}(3,q)$, *Can. J. Math.* **25** (1973), 486–494.

[18] B. Srinivasan, The characters of the finite symplectic group $\mathrm{Sp}(4,q)$, *Trans. Amer. Math. Soc.* **131** (1968), 488–525.

[19] M. Suzuki, On characterisations of linear groups, I,II, *Trans. Amer. Math. Soc.* **92** (1959), 191–204, 205–219.

[20] P.H. Tiep, p-Steinberg characters of finite simple groups, *J. Algebra* **187** (1997), 304 - 319.

[21] P.H. Tiep and A.E. Zalesskiĭ, Minimal characters of finite classical groups, *Comm. in Algebra* **24** (1996), 2093–2167.

[22] H. Ward, On Ree's series of simple groups, *Trans. Amer. Math. Soc.* **121** (1966), 62–89.

Authors' addresses:
Gunter Malle
FB Mathematik/Informatik
Universität Kassel
Heinrich-Plett-Straße 40
D 34132 Kassel
Germany
e-mail: malle@mathematik.uni-kassel.de

Alexandre E. Zalesskiĭ
School of Mathematics
University of East Anglia
Norwich NR4 7TJ, England
e-mail: A.Zalesskii@uea.ac.uk