Linear groups of degree at most 27 over residue rings modulo p^k

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1 Introduction

Let p be a prime and let A be a finite homocyclic p-group. This means that A is the direct product of n cyclic p-groups of the same order p^k for some positive integers n, k which determine A up to isomorphism. Homocyclic p-groups form one of the basic classes of finite groups. Understanding the automorphism group $\mathrm{Aut}A$ of A is important for many applications. It is well-known that $\mathrm{Aut}A$ is isomorphic to $GL(n, R_k)$ where $R_k = \mathbf{Z}/p^k\mathbf{Z}$ is the residue ring of the ring $\mathbf{Z}$ modulo p^k. Set $H_k = GL(n, R_k)$. For $i < k$ we set $N_i = \{h \in H_k : \ h - \mathrm{Id} \equiv 0 (\mathrm{mod}\, p^i)\}$. The groups N_i are known as congruence-subgroups. The group $GL(n, R_1)$ is usually denoted by $GL(n, p)$ or $GL(n, F_p)$ where $F_p = \mathbf{Z}/p\mathbf{Z}$ is the field with p elements. The study of subgroups of $GL(n, p)$ is a part of general theory of linear groups over finite fields which is a well-established area of contemporary research. In this area significant progress has been achieved during recent years, especially, in understanding maximal subgroups of $GL(n, p)$ (see, for example, [14]). There are two approaches which are effectively used: the modular representation theory of finite groups and the theory of representations of algebraic groups and finite groups of Lie type. One aspect of the theory of linear groups over fields is the classification of groups of small degrees, which is important both for particular applications and for induction purposes.

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[1] Supported in part by RFFI grant 99-01-00462.

Unlike the case $k = 1$, for $k > 1$ very little is known about subgroups of $H_k = GL(n, R_k)$ other than normal ones. Of course, H_k is an extension of the congruence subgroup N_1 by $GL(n, p)$ so each subgroup X of H_k has a normal p-subgroup whose quotient group is isomorphic to a subgroup of $GL(n, p)$. For some purposes this is sufficient. However, this in fact says very little about subgroups of H_k as the extension *does not split* over N_1, i.e., H_k does not have a subgroup isomorphic to $GL(n, p)$. This means that the knowledge of subgroups of $GL(n, p)$ does not lead to the knowledge of subgroups of H_k for $k > 1$. Therefore, it is important to determine subgroups of H_k that do not meet N_1 non-trivially. Thus, at the first stage it is reasonable to start with a more specific problem of determining liftable subgroups of $GL(n, p)$ which are defined as follows.

Definition 1.1 *Let $k > 1$. A subgroup G of $GL(n, p)$ is called liftable to H_k if there exists a subgroup Y of H_k such that $Y \cap N_1 = 1$ and the image of Y under the projection $H_k \to GL(n, p)$ is G.*

In this paper we determine liftable absolutely irreducible quasi-simple subgroups of $GL(n, p)$ for $p > 3$ and $n \leq 27$. We make essential use of the classification of absolutely irreducible finite quasisimple subgroups of $GL(n, p)$ for $n \leq 27$ obtained by the first author in [15], [16], [17], [18]. Besides, we consider the similar problem for subgroups of $GL(n, p^m)$ with $m > 1$ and $n \leq 27$.

Let M be the vector space of all $(n \times n)$-matrices over F_p. It is well-known that if $N_i \neq 0$ then N_i/N_{i+1} is isomorphic to M (a vector space isomorphism). The conjugation action of H_k on N_i turns N_i/N_{i+1} to an F_pH_k-module. As N_1 acts on this module trivially, N_i/N_{i+1} can be viewed as $F_pGL(n, p)$-module, which is in fact isomorphic to the $F_pGL(n, p)$-module M under the conjugation action of $GL(n, p)$ on M. It follows that there is a natural F_pG-module isomorphism $N_i/N_{i+1} \cong M$ where G is a subgroup of $GL(n, p)$. Therefore, if $H^2(G, M) = 0$ then G is liftable to H_k for all k. However, the general cohomology theory does not help much, except for the case of p'-groups. The experience from linear group theory suggests that analysis should start with irreducible groups G.

The irreducibility of G guarantees that $O_p(G) = 1$ which seems to be important assumption for the reducible case as well. The problem of lifting is close to that of determining the subgroups Y of H_k such that $O_p(G) = 1$ and the projection of Y in $GL(n,p)$ is irreducible. However, the latter additionally requires determining the conjugacy classes of such subgroups.

The fact that $SL(n,p)$ does not lift to H_k for $k > 1$ was proved in [22, Ch.IV] for $p > 3$ and in [20, Theorem 7] in the form below:

Theorem 1.2 *$SL(n,p)$ is not liftable to H_k for $k > 1$, unless $(n,q) = (3,2)$, $(2,2)$ or $(2,3)$.*

Under more general setting, there is Griess' condition for a subgroup X of a Chevalley group $H = H(p)$ to have no lift to H_2, see [10, page 303]. Namely, X is not liftable if $p > 3$ and for an element $g \in X$ of order p the Jordan form of g on the adjoint module of $H(p)$ has no block of size p. For the case $H = GL(n,p)$ Griess' condition is equivalent to the claim that a subgroup $X \subseteq GL(n,p)$ is not liftable if X contains an element whose Jordan form on the natural module of $GL(n,p)$ has no block of size $> \frac{p-1}{2}$. If X is a group of Lie type defined over a field of characteristic p, some progress has been made in a recent work of Tiep and Zalesski [25]. Let X be a quasi-simple Chevalley group of type other than A_1, G_2 or F_4 over a field of characteristic $p > 3$, and let ϕ be a non-trivial irreducible representation of X into $GL(n,p)$. It is shown in [25] that the group $\phi(X)$ is liftable to H_2 only if ϕ belongs to a block of defect 0, see Theorem 4.3 below. This is based on the following necessary condition for lifting which will be heavily used in this paper too.

Theorem 1.3 *Let $C \subset GL(n,p)$ be a subgroup of order p. Suppose that C is liftable to H_k for some $k > 1$. Then the Jordan form of C contains no block of size $2, \dots, p-2$.*

This result follows from the classification of representations of the cyclic group of order p over R_k obtained in [7]. The condition above is the starting point of that classification (see an exposition of this in [6, p.638-639]).

If G is liftable to $GL(n, \mathbf{Z}_p)$ where $\mathbf{Z}_p$ is the ring of p-adic integers then G is liftable to H_k for all k. The question of which quasi-simple subgroups $G \subseteq GL(n,p)$ for $n \leq 27$ are liftable to $GL(n, \mathbf{Z}_p)$ can be solved by inspection of the tables in [15], [16],[18]. In fact, we need to compute the character field for the groups listed in these works. Our first result is

Theorem 1.4 *Let G be a absolutely irreducible quasi-simple subgroup of $GL(n,p)$, where $n \leq 27$ and $p > 3$ is a prime divisor of $|G|$. Suppose that G is not isomorphic to a group of Lie type over a field of characteristic p. Then Table 1 gives all triples (n, G, p) such that G is liftable to $GL(n, \mathbf{Z}_p)$ and to $GL(n, \mathbf{Z}/p^k\mathbf{Z})$ for all $k > 1$.*

Is there any analogue of the problem under discussion for subgroups of $GL(n,q)$ for $q = p^m$ with $m > 1$? Let ε denote a primitive $(q-1)$-th root of 1. Set $\mathbf{Q}_p^q = \mathbf{Q}_p(\varepsilon)$ where $\mathbf{Q}_p$ is the field of p-adic numbers. Let $\mathbf{Z}_p^q$ be the ring integers of $\mathbf{Q}_p^q$ and let I be the maximal ideal in $\mathbf{Z}_p^q$. Then $F_q \simeq \mathbf{Z}_p^q/I$. We set $H_k^q = GL(n, \mathbf{Z}_p^q/I^k)$. A subgroup X of $GL(n,q)$ is called liftable to H_k^q if there exists a subgroup Y of H_k^q isomorphic to X such that $Y \equiv X (\mathrm{mod}\, I^k)$. Theorem 1.3 remains true under this more general setting.

For $q > p$ we prove the following analogue of Theorem 1.4:

Theorem 1.5 *Let $q = p^m > p > 3$ and let G be a absolutely irreducible quasi-simple subgroup of $GL(n,q)$, where F_q is the character field of G and p is a prime divisor of $|G|$. Suppose that G is not isomorphic to a group of Lie type over a field of characteristic p. Then Table 2 gives all quadruples (n, G, p, q) such that $n \leq 27$, and G lifts to $GL(n, \mathbf{Z}_p^q)$ and to H_k^q for all $k > 1$.*

Theorem 1.6 *Let $n \leq 27$, $q = p^m > 3$ and $G \subset GL(n,q)$ be absolutely irreducible quasi-simple subgroup isomorphic to no group of Lie type of charactersitic p. Suppose that G is not liftable to $GL(n, \mathbf{Z}_p^q)$. Then either G is not liftable to H_k^q for $k > 1$ or $q = p = 5$, $n = 23$ and $G \cong A_{25}$. In the exceptional case G lifts to H_2 but does not lift to any H_k with $k > 2$.*

For groups G of Lie type of characteristic p we prove the following result.

Theorem 1.7 *Let G be a quasisimple subgroup of Lie type defined over a field of characteristic $p > 3$ and let $\phi : G \to GL(n,p)$ be absolutely irreducible representation. Suppose that $1 < n < 28$. Then G is not liftable to H_k for $k > 1$, unless, possibly, one of the following holds:*

(1) *$G = SL(2,p)$ with $3 < p < 24$ and $\deg \phi = p - 1$ or p;*

(2) *$G \cong SL(2,25)$ and $\deg \phi \in \{16, 25\}$.*

The proof of Theorem 1.7 is based on the results obtained in [25]. Let $G \subset GL(n,q)$ be an absolutely irreducible subgroup. Recall that a linear group $X \subset GL(n,\mathbf{C})$ is conjugate to a subgroup of $GL(n,\mathbf{Z}_p^q)$ if and only if it is conjugate to a subgroup of $GL(n,\mathbf{Q}_p^q)$. For the proof of Theorems 1.4, 1.5 and 1.6 we distinguish the following cases.

Case A. $q = p$ and G lifts to $GL(n,\mathbf{Q}_p)$.

These groups G are listed in Table 1. It is trivial that G from Table 1 lifts to $GL(n,H_k)$ for each $k > 1$.

Case B. $G \subset GL(n,q)$ where $q = p^a > p$ and G lifts to $GL(n,\mathbf{Q}_p^q)$.

These groups G are listed in Table 2. The groups from Table 2 lift to $GL(n,H_k^q)$ for each $k > 1$.

Case C. $G \subset GL(n,q)$ with $q = p^a$ where $a \geq 1$ and G lifts to $GL(n,\mathbf{C})$ and does not lift to $GL(n,\mathbf{Q}_p^q)$.

Case D. G does not lift to $GL(n,\mathbf{C})$.

Applying the classification [15], [16], [17], [18] to cases A, B, C, and D separately, we obtain Tables 1, 2, 3, and 4, respectively. In §3 we show that in cases C and D the group G does not lift to H_k^q for all $k > 1$ except when $(n,G,p,k) = (23,A_{25},5,2)$ in case D (Theorem 3.3). In addition, in §3 we show that cases A and B correspond to Theorems 1.4 and 1.5, respectively. In §4 we prove Theorem 1.7.

Notation. All groups in this paper are finite. For a given representation ϕ of a group X and for an element x of X, let $\mathrm{Jord}(\phi(x))$ (or simply $\mathrm{Jord}(x)$) denote the Jordan form of $\phi(x)$. The Jordan block of size m with eigenvalue 1 is denoted by J_m. We set $kJ_m = \mathrm{diag}(J_m,\dots,J_m)$ (k times). For a prime p, a complex character χ of a group X is called p-unramified, if $\mathbf{Q}(\chi)$ is contained in the cyclotomic field $\mathbf{Q}_G = \mathbf{Q}(\varepsilon)$ where ε is a primitive $|G|_{p'}$-root of 1, and p-ramified otherwise. Observe that p-unramified characters are also called p-rational. If F is a field and χ is a function with values in $\bar{F}$ then $F(\chi)$ denotes the minimal subfield of $\bar{F}$ containing all values of χ. By S_n and A_n we denote the symmetric and alternating groups of degree n, respectively. For these groups we use Young diagram notation for conjugacy classes (say, $g \simeq (3^3,1^2)$ means that $g \in S_{11}$ has 3 orbits of size 3 and 2 orbits of size 1 on a set of 11 elements).

For simple groups and representation theory we use the standard notations (see, for example, [1], [5], [6], [8]). The conjugacy classes of elements of order n in a group X from [1] are denoted by $(nA)_X$, $(nB)_X$, $(nC)_X, \ldots$. The union of the conjugacy classes $(nA)_X$, $(nB)_X, \ldots$ is denoted by $(nAB\ldots)_X$.

Table 1 has to be read as follows: Column 1 lists the degree n, Column 2 lists quasi-simple groups G, Column 3 lists prime divisors p of $|G|$, Column 4 lists the number $qe(n, G, p)$ of conjugacy classes of liftable absolutely irreducible subgroups G in $GL(n, p)$, Column 5 lists $\mathbf{Q}(\chi)$ for each conjugacy class of G. The columns 1-5 of Table 2 are analogous to the respective columns of Table 1. Column 6 of Table 2 lists the value of q for each conjugacy class of G.

Columns 1-3 of Tables 3 and 4 are analogous to the corresponding columns of Table 1. Column 4 of Tables 3 and 4 lists the structure of $N_G(G_p)$. Column 5 of Table 3 lists $qe(n, G, p)$. Column 6 of Table 3 lists $\mathbf{Q}(\chi)$ which shows that χ is p-ramified. The last columns of Tables 3 and 4 list $\mathrm{Jord}(g)$ for an element g of order p of G.

2 Preliminary results

In this section p is a prime integer.

Lemma 2.1 *Let $2 < p \leq n \in \mathbf{N}$. Let M_n be the standard permutation F_pA_n-module (of dimension n), and let M_n^0 denote the unique nontrivial composition factor of M_n. If g is a single p-cycle in A_n then*

$$\mathrm{Jord}(g|M_n^0)) = \begin{cases} J_{p-2} & \text{if } n = p \\ J_p + (n-1-p)J_1 & \text{if } p \text{ does not divide } n \\ J_p + (n-2-p)J_1 & \text{otherwise.} \end{cases}$$

Proof. Let T_n denote the trivial F_pA_n-submodule. Suppose that p does not divide n. Then $M_n = M_n^0 \oplus T_n$ and hence $\mathrm{Jord}(g|M_n) = \mathrm{Jord}(g|M_n^0) + J_1$. Since $\mathrm{Jord}(g|M_n) = J_p + (n-p)J_1$, the lemma is true in this case. Suppose that $p < n$ and p divides n. Then $M_n|A_{n-1} \simeq M_{n-1} \oplus T_{n-1} = M_{n-1}^0 \oplus T_{n-1} \oplus T_{n-1}$. Since $n - 1 \not\equiv 0(p)$ and $g \in A_{n-1}$, we have $\mathrm{Jord}(g|M_{n-1}^0) = J_p + (n-2-p)J_1$, hence the lemma is true. The case $n = p$ is obvious.

Lemma 2.2 *Let $G = S_n$ or A_n, where $n = kp > p$ and p is odd and coprime to k. Let M_n and M_n^0 be as in Lemma 2.1. If $g \in G$ consists of k cycles of length p then* $\mathrm{Jord}(g|M_n^0) = J_{p-2} + (k-1)J_p$.

Proof. Let $a \in C_G(g)$ consists of p cycles of length k, and $A = \langle a \rangle$. It is clear that M_n is completely reducible as an F_pA-module. In fact, $M_n|A$ is isomorphic to a direct sum of p copies of the regular module. Set $\bar{M}_n = M_n \otimes \bar{F}_p$. Then $\bar{M}_n|A$ is the direct sum of one-dimensional submodules. It follows that each of the k eigenvalues of a has multiplicity p. Let $\lambda \in \bar{\mathbf{F}}_p$ be a primitive k-th root of 1. Then $\bar{M}_n = M_1 \oplus \cdots \oplus M_k$, where M_i is the eigenspace of an eigenvalue λ^i of a and $i = 1, \ldots, k$. Since $ag = ga$, every M_i is a $\langle g \rangle$-module. The composition series of M_n has three factors, two of them being trivial. Hence the multiplicities of all eigenvalues of $a|M_n^0$ are all equal to p, except for 1 which multiplicity is equal to $p-2$. Therefore M_n^0 is the direct sum of $k-1$ isomophic copies of a $\langle g \rangle$-submodule of the dimension p and one submodule of dimension $p-2$. The number of the Jordan blocks of an element acting on a module does not increase in a quotient module. Hence g acts on each of these module as a single Jordan block, so the lemma follows.

We shall use some results on the Green correspondence (see [6], § 20A) and on blocks with cyclic defect group (see [8], Chapter 7). To apply these results observe that if G is quasisimple and a Sylow p-subgroup G_p of G is cyclic then G_p is a TI-subgroup. (This means that $x^{-1}G_px \cap G_p$ is either G_p itself or trivial.)

Theorem 2.3 ([6], Theorem (20.6), Corollary (20.8)). *Let G be a finite group, and let R be a field of characteristic p or the ring of integers of a finite extension of $\mathbf{Q}_p$. Let V be a indecomposable RG-module with vertex D and $N = N_G(D)$. Then there exists an indecomposable RN-module W with vertex D such that $W^G = V \oplus X$ and $V|_N = W \oplus Y$, where X is a RG-module and Y is a RN-module. Moreover, X is (G,P)-projective for all p-subgroups P of the form $D \cap gDg^{-1}$ with $g \notin N$ (so that $P \neq D$) and Y is (N,L)-projective for all subgroups L of the form $N \cap gDg^{-1}$ with $g \notin N$.*

Corollary 2.4 *Under assumption and notation of Theorem* 2.3 *suppose additionally that a Sylow p-subgroup S of G is a cyclic TI-subgroup. Then $D = S$ and $V|_N = W \oplus Y$, where W is an indecomposable RN-module and Y is a projective RN-module. Besides,* $\dim V \equiv \dim W (\mathrm{mod}\, |S|)$.

Proof. We can assume that $D \leq S \leq N$. Let B be the p-block containing V and let P be the defect group of B. Since V is non-projective, B is not of defect 0. Then $P \neq 1$ (see §87 in [5]). Since P is an intersection of two Sylow p-subgroups of G (see [8, Theorem (III.8.14)]) and S is a TI-subgroup in G, we have $P = S$. By [8, Theorem (VII.15.1)], S is the vertex of the module V. Therefore, $D = S$ and $N = N_G(S)$. Now, it is obvious that $N \cap gSg^{-1} = 1$ for $g \notin N$. So $L = 1$ and Y is $(N, 1)$-projective. Therefore, Y is projective (see [5, page 449]). Hence $Y|S$ is free (see [8, (VII.15.1)]).

Lemma 2.5 *Under the assumptions of Corollary* 2.4*, suppose that R is a field of characteristic p. Then W has a unique composition series. If $l(W)$ is its length then either $0 < l(W) \leq e$, or $|S| - e \leq l(W) < |S|$, where e is some divisor of $p-1$. In particular, if N/S is abelian then* $\dim W < |S|$.

Proof. See [8, Theorems VII.2.4, VII.2.6, VII.2.7]. If N/S is abelian then the composition factors of W are one-dimensional and hence $l(W) = \dim W < p^a$.

Corollary 2.6 *Let G be a quasi-simple group with a cyclic Sylow p-subgroup $S = \langle s \rangle$ and let V be an irreducible $\bar{F}_pG$-module. Suppose that $N_G(S)/S$ is abelian. Then the Jordan form of s consists of blocks of size $|s|$ (except when* $\dim V < |s|$*) and a single block of size* $\dim V (\mathrm{mod}\, |s|)$ *(except when* $\dim V \equiv 0 (\mathrm{mod}\, |s|)$.

Proof. A Sylow p-subgroup of G is a TI-subgroup, see Blau [4]. So the claim follows from Lemma 2.5.

Lemma 2.7 ([8, Corollary IV.9.4] *Let F be a finite extension of $\mathbf{Q}_p$ and let $G \subset GL(n, F)$ be a finite group. If* $G(\mathrm{mod}\, p)$ *is irreducible then G is realizable over the field $\mathbf{Q}_p(\chi)$.*

Lemma 2.8 *Let P be a finite extension of $\mathbf{Q}_p$ and let $\phi: \ G \to GL(n, P)$ be a irreducible representation with character χ. Set $\bar{\chi} = \chi(\mathrm{mod}\, p)$. Then $[F_p(\bar{\chi}) : F_p]$ divides $[\mathbf{Q}_p(\chi) : \mathbf{Q}_p]$.*

Proof. Let $L = \mathbf{Q}_p(\chi)$. Let $\mathbf{Z}_L$ be the ring of integers of L, I the maximal ideal in $\mathbf{Z}_L$ and $R = \mathbf{Z}_L/I$. Then $[L : \mathbf{Q}_p]$ is divisible by $[R : F_p]$. As $\chi(G) \subset \mathbf{Z}_L$, $F_p(\bar{\chi})$ is a subfield of R. So $[L : \mathbf{Q}_p]$ is divisible by $[F_p(\bar{\chi}) : F_p]$, and we are done.

Lemma 2.9 ([5], Chapter III, §19, Exercise 5). *Let p be odd and let n be an integer coprime to p. Then n is a square in $\mathbf{Q}_p$ if and only if n is a quadratic residue modulo p.*

3 Proofs of the main results

In this section G is a quasi-simple group of order divisible by a prime $p > 3$. We assume that G is not isomorphic to a group of Lie type defined over a field of characteristic p. We fix a Sylow p-subgroup G_p of G. Throughout this section ϕ is a faithful absolutely irreducible representation of G of degree $n \leq 27$ over field F_q, where F_q is exactly the field generated by the values of the character of ϕ. It follows that G is not realizable over any proper subfield of F_q. Let V be the F_qG-module associated with ϕ and $\beta = \beta_\phi$ be its Brauer character.

Tables 1, 2 are obtained from [15], [16], [17], [18] by using Lemmas 2.7, 2.8, 2.9 and information given in [1], [2].

Lemma 3.1 *Let $q = p^m$ and let $\phi : X \to GL(n, q)$ be a faithful representation of a group X. Suppose that $O_p(X) = 1$. If ϕ lifts to $GL(n, \mathbf{Z}_p^q)$ then X lifts to H_k^q for all $k \geq 1$. If X does not lift to H_l^q for some $l > 1$ then X does not lift to H_k^q for all $k > l$.*

Proof. This is obvious.

Theorem 1.4 follows from Lemma 3.1 and Table 1.

Lemma 3.2 *If the quadruple (n, G, p, q) is from Table 2 then $q = p^m$ where $m = [\mathbf{Q}_p(\chi) : \mathbf{Q}_p]$.*

Proof. The lemma follows from Lemma 2.8.

Theorem 1.5 follows from Table 2 and Lemmas 3.1 and 3.2.

Using the classification of quasi-simple subgroups of $GL(n, \bar{F}_p)$ for $n \leq 27$ (where G is not a group of Lie type of charactersitic p) given in [15], [16], [17], [18] and the tables of ordinary and Brauer characters [1], [2], we obtain all columns, except the last one, of Tables 3 and 4. Justification of the values in the last columns of Tables 3 and 4 will be given by Lemmas 3.4 - 3.15 and 3.18.

By Theorem 1.3, the last columns of Tables 3 and 4 imply the following.

Theorem 3.3 (1) *If the triple* (n, G, p) *is from Tables* 4 *then* G *is not liftable to* $GL(n, \mathbf{Z}/p^k\mathbf{Z})$ *for l* $k > 1$ *except when* $(n, G, p) = (23, A_{25}, 5)$.

(2) *If* (n, G, p, q) *is from Table* 3 *then* G *is not liftable to* H_k^q *for* $k > 1$.

Below the notation J_2, J_3 for the Janko groups should not be confused with that for the Jordan blocks of size 2 or 3.

Lemma 3.4 *If the triple* (n, G, p) *is from Table* 3 *or* 4 *then one of the following holds:*

(i) $|G_p| = p$ *and* $N_G(G_p)/G_p$ *is abelian;*

(ii) $(n, G, p) = (m - 2, A_m, p)$*, where* $m = kp$ *and* $k < p < m$*;*

(iii) (n, G, p) *is one of the triples* $(6, 2{\cdot}J_2, 5)$, $(14, J_2, 5)$, $(17, L_2(16), 5)$, $(18, L_2(19), 5)$, $(18, 3{\cdot}J_3, 5)$, *or* $(21, J_2, 5)$ *from Table* 3;

(iv) (n, G, p) *is one of the triples* $(8, 2{\cdot}A_{10}, 5)$, $(13, A_8, 5)$, $(21, HiS, 5)$ *or* $(21, McL, 5)$ *from Table* 4.

(v) $(n, G, p) = (23, A_{25}, 5)$ *from Table* 4.

Proof. This is obvious.

Lemma 3.5 *In cases* (i) *and* (ii) *of Lemma* 3.4, *the last columns of Tables* 3 *and* 4 *are correct.*

Proof. The lemma follows from Lemma 2.5 for (i) and from Lemmas 2.1 and 2.2 for (ii).

Now, we consider cases (iii) and (iv) of Lemma 3.4.

Lemma 3.6 *If* $(n, G, p) = (6, 2{\cdot}J_2, 5)$ *in Table* 3 *and* g *is an element of order* 5 *from* $(5AB)_G$ *then* $\mathrm{Jord}(g) = 2J_3$.

Proof. By [1], $\bar{G} = G/Z(G)$ has a subgroup $\bar{M} \simeq 3{\cdot}PGL_2(9)$. Let M be the preimage of $\bar{M}$ in G. By [2], $6{\cdot}PGL_2(9)$ has no faithful 5-modular representations of degree < 12, hence $M' \simeq 3{\cdot}A_6$. Also by [2], $\beta|_{M'} = \beta_1 + \beta_2$, where β_1 and β_2 are irreducible constituents of degree 3 which are permuted by an element of $M \setminus M'$. Therefore, $\phi|_{M'}$ is the direct sum of two faithful irreducible representations of degree 3. Since $|M'|_5 = 5$ and $N_{M'}(\langle g\rangle)/\langle g\rangle \simeq 6$, by Lemma 2.5 $\mathrm{Jord}(g) = 2J_3$.

Lemma 3.7 *If* $(n, G, p) = (14, J_2, 5)$ *in Table* 3 *and* g *is an element of order* 5 *from* $(5AB)_G$ *then* $\mathrm{Jord}(g) = \mathrm{diag}(3J_3, J_5)$.

Proof. By [1], G has a subgroup $M \simeq 3{\cdot}PGL_2(9)$ which contains an element g from $(5AB)_G$. By [2], it is easy to see that $\beta|_M = \beta_1 + \beta_2 + \beta_3$, where β_1, β_2 and β_3 are irreducible constituents in $\beta|'_M$ of degree 3, 3 and 8, respectively, an element of $M \setminus M'$ fixes β_3 and permutes β_1 and β_2, $Z(M')$ is contained in the kernel of β_3 and is not contained in the kernels of β_1 and β_2. Hence, for $\langle x \rangle = Z(M')$, we have $V = C_V(x) \oplus [V, \langle x \rangle]$ and $[V, \langle x \rangle] = V_1 \oplus V_2$ where $C_V(x)$, V_1 and V_2 are irreducible M'-modules of dimensions 8, 3 and 3, respectively. By Lemma 2.5, $\mathrm{Jord}(g) = \mathrm{diag}(3J_3, J_5)$.

Lemma 3.8 *If* $(n, G, p) = (17, L_2(16), 5)$ *in Table* 3 *and* g *is an element of order* 5 *in* G *then* $\mathrm{Jord}(g) = \mathrm{diag}(J_2, 3J_5)$.

Proof. For $N = N_G(\langle g \rangle)$, we have $N/\langle g \rangle \simeq S_3$. By Corollary 2.4 and Lemma 2.5, $V|_N = Y + W$, where Y and W are projective and indecomposable N-modules, respectively, with $\dim W \leq 4 \dim U$ and $\dim U \leq 2$ for a composition factor U of the module W. Hence $\dim W \leq 8$ and $\dim Y \geq 17 - 8 = 9$, i. e. $\dim Y \geq 10$. It is clear that $\dim W \neq 7$, therefore $\dim Y = 15$ and $\dim W = 2$.

Lemma 3.9 *If* $(n, G, p) = (18, L_2(19), 5)$ *in Table* 3 *and* g *is an element of order* 5 *in* G *then* $\mathrm{Jord}(g) = \mathrm{diag}(J_3, 3J_5)$.

Proof. G has a maximal subgroup $M \simeq D_{18}$ of index 20 and acts 2-transitively on the conjugacy class M^G as a subgroup of A_{20} in its natural permutation action. By [19], the corresponding representation ϕ of G is the restriction of the 5-modular irreducible representation of degree 18 for A_{20}. Since an element g of order 5 in G acts on M^G fix-point-freely, g is the product of 4 disjoint 5-cycles in A_{20}. Therefore, by Lemma 1.4, $\mathrm{Jord}(g) = \mathrm{diag}(J_3, 3J_5)$.

Lemma 3.10 *If* $(n, G, p) = (18, 3{\cdot}J_3, 5)$ *in Table* 3 *and* g *is an element of order* 5 *in* G. *Then* $\mathrm{Jord}(g) = \mathrm{diag}(J_3, 3J_5)$.

Proof. Set $N = N_G(\langle g \rangle)$. Then we have $N/\langle g \rangle \simeq 3 \times S_3$. By Corollary 2.4 and Lemma 2.5, $V|_N = Y + W$, where Y and W are projective and indecomposable N-modules, respectively, $\dim W \leq 4 \dim U$ and $\dim U \leq 2$ for each composition factor U of W. Hence $\dim W \leq 8$ and $\dim Y \geq 18-8 = 10$. If $\dim W < 8$ then $\dim Y = 15$, hence $\dim W = 3$ and the lemma follows. Suppose that $\dim W = 8$. Then the length of the composition series of W is equal to 4 and all composition factors of W have dimension 2. By [1], N is contained in a maximal subgroup $M \simeq 3 \times L_2(16) : 2$ of G. By [2], the module $V|_M$ has a single one-dimensional composition factor. But then the module W has an one-dimensional composition factor, which is a contradiction.

Lemma 3.11 *If* $(n, G, p) = (21, J_2, 5)$ *in Table* 3 *and* g *is an element of order* 5 *from* $(5CD)_G$ *then* $\mathrm{Jord}(g) = \mathrm{diag}(2J_3, 3J_5)$.

Proof. By [1], G has a subgroup $M \simeq 2_-^{1+2} : A_5$ which contains an element g from $(5CD)_G$. Set $\langle x \rangle = Z(M)$ and $Q = O_2(M)$. It is easy to see from the character table of M (see p. 40 in [13]) and from [2] that $\beta|_M = \beta_1 + \beta_2 + \beta_3$, where β_1, β_2 and β_3 are irreducible constituents of $\beta|_M$ of degree 3, 8, 10 and with kernels Q, 1, $\langle x \rangle$, respectively. Thus $V = C_V(Q) \oplus [V, \langle x \rangle] \oplus [C_V(x), Q]$, where $C_V(Q)$, $[C_V(x), Q]$ and $[V, \langle x \rangle]$ are irreducible M-modules with the Brauer characters β_1, β_2, β_3, respectively. Since $N_M(\langle g \rangle) \simeq 2 \times D_{10}$, by Lemma 2.5 we have $\mathrm{Jord}(g) = \mathrm{diag}(2J_3, 3J_5)$.

Lemma 3.12 *If* $(n, G, p) = (8, 2^{\cdot}A_{10}, 5)$ *in Table* 4 *then* G *contains an element* g *of order* 5 *with* $\mathrm{Jord}(g) = \mathrm{diag}(J_3, J_5)$.

Proof. $\bar{G} = G/Z(G)$ has a subgroup $\bar{M} \simeq M_{10} \simeq A_6.2_3$. Let M be the preimage of $\bar{M}$ in G. By [1], $\bar{M}'$ contains an involution $\bar{t}$ from the conjugacy class $(2B)_{\bar{G}}$. If $M' \simeq SL_2(9)$ then M' has a single involution. But the coset $\bar{t}$ in G contains an involution (see [1]), which is a contradiction. Hence $M' \simeq A_6$. By [2], the 5-modular representation $\phi|_{M'}$ is irreducible. Let g be an element of order 5 in M'. Since $|M'|_5 = 5$ and $N_{M'}(\langle g \rangle)/\langle g \rangle \simeq 2$, by Lemma 1.7 $\mathrm{Jord}(g) = \mathrm{diag}(J_3, J_5)$.

Lemma 3.13 *If* $(n, G, p) = (13, A_8, 5)$ *in Table* 4 *and* $g \in G$ *is an element of order* 5 *then* $\mathrm{Jord}(g) = \mathrm{diag}(J_3, 2J_5)$.

Proof. G has a maximal subgroup $M \simeq 2^3 : L_3(2)$ of index 15 and acts 2-transitively on the conjugacy class M^G as a subgroup of A_{15} in its natural permutation action. By [19], the corresponding representation ϕ for

G is the restriction of the 5-modular irreducible representation of degree 13 for A_{15}. Since an element g of order 5 in G acts on M^G fix-point-freely, g is the product of 3 disjoint 5-cycles in A_{15}. Therefore, by Lemma 2.2, $\mathrm{Jord}(g) = \mathrm{diag}(J_3, 2J_5)$.

Lemma 3.14 *If* $(n, G, p) = (21, HiS, 5)$ *in Table* 4 *and* g *be an element of order* 5 *from* $(5A)_G$ *then* $\mathrm{Jord}(g)$ *contains an Jordan block* J_3*.*

Proof. By [1], G has a subgroup $M \simeq 4 \circ SL_2(5)$ which contains an element g from $(5A)_G$. By [2], it is easy to see that $\beta|_{M'} = 2\beta_1 + 2\beta_2 + \beta_3 + 2\beta_4$, where β_1, β_2, β_3 and β_4 are irreducible constituents in $\beta|_{M'}$ of degree 1, 3, 5 and 4, respectively, $Z(M')$ is contained in the kernels of β_1, β_2, β_3 and does not contained in the kernel of β_4. Set $\langle x \rangle = Z(M)$. We have $V = C_V(x^2) \oplus [V, \langle x^2 \rangle]$, and $C_V(x^2)$ is a M'-module with the Brauer character $2\beta_1 + 2\beta_2 + \beta_3$. By [1], $x \in (4A)_G$ and hence $\beta(x) = -7$. Now it easy to see that $C_V(x^2) = C_V(x) \oplus [V, \langle x \rangle]$, where $C_V(x)$ is irreducible M-module of dimension 3. By Lemma 2.1, this module corresponds to Jordan block J_3 in $\mathrm{Jord}(g)$.

Lemma 3.15 *If* $(n, G, p) = (21, McL, 5)$ *in Table* 4 *and* g *is an element of order* 5 *from* $(5A)_G$ *then* $\mathrm{Jord}(g)$ *contains an Jordan block* J_3*.*

Proof. By [1], the group G has a subgroup $M \simeq 2^{\cdot}A_8$ which contains an element g from $(5A)_G$. By [2], it is easy to see that $\beta|_M = \beta_1 + \beta_2$, where β_1 and β_2 are irreducible constituents in $\beta|_M$ of degree 13 and 8, respectively, $Z(M)$ is contained in the kernel of β_1 and does not contained in the kernel of β_2. Set $\langle x \rangle = Z(M)$ and $\bar{M} = M/\langle x \rangle$. We have $V = C_V(x) \oplus [V, \langle x \rangle]$, and $C_V(x)$ is a $\bar{M}$-module with the Brauer character β_1. The triple $(13, \bar{M}, 5)$ is from Table 4, hence by Lemma 3.13, $\mathrm{Jord}(g)$ contains a Jordan block J_3.

Lemmas 3.4-3.15 imply that the last columns of Tables 3 and 4 are correct. Hence for the proof of Theorems 3.3 and 1.6 it remains to consider the triple $(23, A_{25}, 5)$ from Table 4.

Lemma 3.16 *Let* $p \geq 2$ *and* $p^m \geq 4$*. Let* $G \cong A_{p^m}$ *be the alternating group realized in* $GL(p^m - 2, p)$*. Then* G *lifts to* $GL(p^m - 2, H_m)$*.*

Proof. Set $R = \mathbf{Z}/p^m\mathbf{Z}$. Denote by M the free R-module with basis $b_1, \dots, b_{p^m}$. We turn M into an RG-module by forcing G to act on $b_1, \dots, b_{p^m}$ by permutations. Set $K = R\langle b_1 + \dots + b_{p^m}\rangle$ and $L = R\langle b_2 - b_1, b_3 - b_1, \dots, b_{p^m} - b_1\rangle$. Clearly, K and L are RG-submodules of M. Observe first that $K \subset L$. Indeed, $b_1 + \dots + b_{p^m} = p^m b_1 + \sum_{i=2}^{p^m}(b_i - b_1) \in L$ as $p^m b_1 = 0$ in M. Consider the R-submodule M_1 generated by $b_i - b_1$ with $2 \leq i \leq p^m - 1$. Clearly, M_1 is a free R-module (if $\sum_{i=2}^{p^m-1} r_i(b_i - b_1) = 0$ for some $r_i \in R$ then $\sum_{i=2}^{p^m-1} r_i b_i - \sum_{i=2}^{p^m-1} r_i b_1 = 0$ which implies that $r_2 = \dots = r_{p^m-1} = 0$ as $b_1, \dots, b_{p^m}$ is a free basis in M). Besides, $M_1 \cap K = 0$ by the same reason (if $\sum_{i=2}^{p^m-1} r_i(b_i - b_1) + r(b_1 + \dots + b_{p^m}) = 0$ for some $r, r_i \in R$ then $\sum_{i=2}^{p^m-1}(r_i + r)b_i - (\sum_{i=2}^{p^m-1} r_i)b_1 + rb_1 = 0$; therefore, $r_i = -r$ for $i = 2, \dots, p^m - 1$ so $-(\sum_{i=2}^{p^m-1} r_i)b_1 + rb_1 = r(p^m - 1)b_1 = 0$ which is not true). Therefore, the R-module L/K is isomorphic to M_1 so L/K is a free R-module. Clearly, L/K is an RG-module, so we are done.

Lemma 3.17 *Let $p \geq 3$, $n = p^m - 2 > 1$ and $C = \langle J_n\rangle$ be a cyclic group of order p^m. Then C is not liftable to H_r for $r > m$.*

Proof. This is a particular case of a result of Thevenaz [24]. More precisely, n must be the degree of a monic polynomial $f \in (\mathbf{Z}/p^r\mathbf{Z})(x)$ that divides $x^{p^m} - 1$, see [24, Lemma 2.1]. Besides, $\deg f$ coincides with the degree of a monic divisor $t(x) \in \mathbf{Z}(x)$ of $x^{p^m} - 1$, by [24, Theorem 2.6]. It is known that $t(x)$ is a product of distinct cyclotomic polynomials $\Phi_d(x)$ for d dividing p^m (so $d = p^j$ with $j \in \{0, 1, \dots, m\}$). We have $\Phi_{p^j}(x) = (x^{p^j} - 1)/(x^{p^{j-1}} - 1)$. One can observe that for $p > 2$ the degree of $t(x)$ cannot be $p^m - 2$, see [24, Remark after Theorem 2.6].

Lemma 3.18 *Let $p \geq 3$, $n = p^m - 2 > 1$. Let $G \cong A_{p^m}$ and $G \subset GL(p^m - 2, p)$. Then G lifts to H_m and does not lift to H_{m+1}. In particular, if $G \cong A_{25}$ and $G \subset GL(23, 5)$ then G lifts to H_2 and does not lift to H_k for $k > 2$.*

Proof. This follows immediately from Lemma 3.17 as G contains J_{23}.

4 Groups of Lie type in defining characteristic

Lemma 4.1 *Let J_i stand for the Jordan block of size i. The Jordan form of $J_l \otimes J_m$ (where $1 \leq l \leq m \leq p$ is described as follows.*

(i) If $l+m \leq p$ then $\mathrm{Jord}(J_l \otimes J_m) = \mathrm{diag}(J_{m+l-1}, J_{m+l-3}, \ldots, J_{m-l+1})$;

(ii) If $l+m > p$ and $m < p$ then

$$\mathrm{Jord}(J_l \otimes J_m) = \mathrm{diag}(J_p, \ldots, J_p, J_{2p-m-l-1}, J_{2p-l-m-3}, \ldots, J_{m-l+1}),$$

where J_p is repeated $m+l-p$ times;

(iii) If $l+m > p$ and $m = p$ then $\mathrm{Jord}(J_l \otimes J_p) = \mathrm{diag}(J_p, \ldots, J_p)$, *where J_p is repeated l times.*

Proof. See [8, Theorem VIII.2.7].

Remark. The case (iii) is trivial as the module in (iii) is projective.

Corollary 4.2 *Let $1 < m_1 \leq m_2 \leq \ldots \leq m_k < p$. Then* $\mathrm{Jord}(J_{m_1} \otimes \ldots \otimes J_{m_k})$ *contains no block of size $2, \ldots p-2$ if and only if $m_1 = \ldots = m_k = p-1$.*

Proof. Let first $k = 2$. Then $m - l + 1 \in \{1, p-1\}$ implies $m = l$. Next, use induction on k.

Let $G = G(q)$ be the universal quasi-simple Chevalley group defined over a field of characteristic $p > 3$. Let $\mathcal{G}$ be the algebraic group associated with G and r be its rank. Let $\omega_1, \ldots, \omega_r$ denote the fundamental weights of $\mathcal{G}$. Let ϕ be an irreducible representation of G over $\bar{F}_p$. In general ϕ is tensor decomposable: $\phi = \phi_1 \otimes Fr \cdot \phi_2 \ldots \otimes Fr^{l-1} \cdot \phi_l$ for some $l \in \mathbf{N}$ where $\phi_1, \ldots, \phi_l$ are infinitesimally irreducible representations of G, and Fr denotes the Frobenius automorphism of $\bar{F}_p$. We say that ϕ is the *modular Steinberg representation* if ϕ is the restriction to G of an irreducible representation of $\mathcal{G}$ with highest weight $p^i(p-1)(\omega_1 + \ldots + \omega_r)$ for some integer $i \geq 0$.

Theorem 4.3 [25] *Let G be a quasi-simple finite group of Lie type defined over a field of characteristic $p > 3$. Let $\phi: G \to GL(n,p)$ be an absolutely irreducible representation of G for some $n > 1$. Then G is not liftable to H_k for $k > 1$, unless one of the following holds:*

(1) ϕ is of defect 0, equivalently, $n \equiv 0 (\mathrm{mod}\, |G|_p)$ where $|G|_p$ is the p-part of $|G|$);

(2) *G is of type $A_1(p^m)$ and $n = (p-1)^m$;*

(3) *G is of type $G_2(q)$ or $F_4(q)$ and ϕ is tensor decomposable.*

(4) *G is of type $G_2(5)$ and ϕ is tensor indecomposable of dimension* 8456 *or* 1715.

Proof of Theorem 1.7. We shall use Theorem 4.3. Suppose that case (1) holds. As the minimum value of N is 3 for G of rank > 1, we have $\dim \phi \geq 5^3 > 28$. So the rank of G is 1. Then either $n = p$ so $3 < p < 24$ or $p = 5$ and $n = 25$. Case (3) is also ruled out as the minimal dimension of a non-trivial representation of G is equal to 7 for G of type G_2 and 26 for type F_4 (for $p > 3$), see [9]. Therefore, ϕ is tensor indecomposable. Case (4) of Theorem 4.3 is obviously not compatible with the assumption. Consider case (2). As $(p-1)^m < 28$, we have either $m = 1$, $p \leq 23$ or $m = 2, p = 5$. This completes the proof.

Acknowledgements

The work was partially supported by the visiting fellowship research grant GR/L91733 from the Engineering and Physical Sciences Research Council (UK). The first author thanks the University of East Anglia for the hospitality during his visit to Norwich in March 1998.

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Table 1: Case A

Subgroups of $GL(n,p)$ liftable to $GL(n,\mathbf{Q}_p)$

n	G	p	$qe(n,G,p)$	$\mathbf{Q}(\chi)$
4	$L_2(9)$	5	1	$\mathbf{Q}$
5	A_5	5	1	$\mathbf{Q}$
	A_6	5	1	$\mathbf{Q}$
	$L_2(11)$	5	1	$\mathbf{Q}(\sqrt{-11})$
6	$L_3(2)$	7	1	$\mathbf{Q}$
	$SL_2(7)$	7	2	$\mathbf{Q}(\sqrt{2})$
	$SL_2(11)$	5	2	$\mathbf{Q}(\sqrt{-11})$
	$U_3(3)$	7	1	$\mathbf{Q}$
	$6{\cdot}L_3(4)$	7	1	$\mathbf{Q}(\sqrt{-3})$
	$PSp_4(3)$	5	1	$\mathbf{Q}$
	$6_1{\cdot}U_4(3)$	7	1	$\mathbf{Q}(\sqrt{-3})$
	A_7	5	1	$\mathbf{Q}$
	$3{\cdot}A_7$	7	1	$\mathbf{Q}(\sqrt{-3})$
	$6{\cdot}A_7$	7	1	$\mathbf{Q}(\sqrt{-3},\sqrt{2})$
7	$L_3(2)$	7	1	$\mathbf{Q}$
	$L_2(8)$	7	1	$\mathbf{Q}$
	A_8	5, 7	1	$\mathbf{Q}$
	$U_3(3)$	7	1	$\mathbf{Q}$
	$PSp_6(2)$	5, 7	1	$\mathbf{Q}$
8	$L_2(8)$	7	1	$\mathbf{Q}$
	$2{\cdot}A_8$	5, 7	1	$\mathbf{Q}$
	$2{\cdot}PSp_6(2)$	5, 7	1	$\mathbf{Q}$
	$2{\cdot}P\Omega_8^+(2)$	5, 7	1	$\mathbf{Q}$
	A_9	5, 7	1	$\mathbf{Q}$
	$2{\cdot}A_9$	5, 7	1	$\mathbf{Q}$
9	$L_2(19)$	5	1	$\mathbf{Q}(\sqrt{-19})$
	A_{10}	7	1	$\mathbf{Q}$
10	A_6	5	1	$\mathbf{Q}$
	$L_2(11)$	5	2	$\mathbf{Q}$
	$SL_2(11)$	5	1	$\mathbf{Q}$
	$SL_2(19)$	5	1	$\mathbf{Q}(\sqrt{-19})$

	$U_5(2)$	5, 11	1	$\mathbf{Q}$
	A_{11}	5, 7	1	$\mathbf{Q}$
	M_{11}	5	1	$\mathbf{Q}$
		11	2	$\mathbf{Q}(\sqrt{-2})$
	$2{\cdot}M_{12}$	11	2	$\mathbf{Q}(\sqrt{-2})$
	$2{\cdot}M_{22}$	11	2	$\mathbf{Q}(\sqrt{-7})$
11	$L_2(11)$	5	1	$\mathbf{Q}$
	A_{12}	5, 7, 11	1	$\mathbf{Q}$
	M_{11}	5, 11	1	$\mathbf{Q}$
	M_{12}	5, 11	1	$\mathbf{Q}$
12	$SL_2(25)$	13	1	$\mathbf{Q}$
	$L_3(3)$	13	1	$\mathbf{Q}$
	$U_3(4)$	5, 13	1	$\mathbf{Q}$
	$2.G_2(4)$	5, 7, 13	1	$\mathbf{Q}$
	A_{13}	5, 7, 11	1	$\mathbf{Q}$
	$2{\cdot}M_{12}$	5, 11	1	$\mathbf{Q}$
	$6{\cdot}Suz$	7, 13	1	$\mathbf{Q}(\sqrt{-3})$
13	$L_2(25)$	13	1	$\mathbf{Q}$
	$L_3(3)$	13	1	$\mathbf{Q}$
	$PSp_6(3)$	7, 13	1	$\mathbf{Q}(\sqrt{-3})$
	A_{14}	5, 11, 13	1	$\mathbf{Q}$
14	$L_2(13)$	7	1	$\mathbf{Q}$
	$SL_2(13)$	7	1	$\mathbf{Q}$
	$L_2(27)$	7, 13	1	$\mathbf{Q}(\sqrt{-3})$
	$SL_2(27)$	7, 13	1	$\mathbf{Q}(\sqrt{-3})$
	$SL_2(29)$	5, 7	1	$\mathbf{Q}(\sqrt{29})$
	$U_3(3)$	7	1	$\mathbf{Q}$
	$Sz(8)$	13	2	$\mathbf{Q}(\sqrt{-1})$
	$G_2(3)$	7, 13	1	$\mathbf{Q}$
	$Sp_6(3)$	7, 13	1	$\mathbf{Q}(\sqrt{-3})$
	A_7	7	1	$\mathbf{Q}$
	A_8	7	1	$\mathbf{Q}$
	$2{\cdot}A_7$	7	1	$\mathbf{Q}$
	A_{15}	7, 11, 13	1	$\mathbf{Q}$
	$2{\cdot}J_2$	5, 7	1	$\mathbf{Q}$
15	A_7	5	1	$\mathbf{Q}$

	$3{\cdot}A_7$	7	1	$\mathbf{Q}(\sqrt{-3})$
	A_{16}	5, 7, 11, 13	1	$\mathbf{Q}$
	$L_2(29)$	5, 7	1	$\mathbf{Q}(\sqrt{29})$
	$L_2(31)$	5	1	$\mathbf{Q}(\sqrt{-31})$
	$SL_3(4)$	7	1	$\mathbf{Q}(\sqrt{-3})$
	$PSp_4(3)$	5	2	$\mathbf{Q}$
	$PSp_6(2)$	5, 7	1	$\mathbf{Q}$
	$3_1{\cdot}U_4(3)$	7	1	$\mathbf{Q}(\sqrt{-3})$
16	$L_2(16)$	5	1	$\mathbf{Q}$
	$SL_2(31)$	5	1	$\mathbf{Q}(\sqrt{-31})$
	$2{\cdot}A_{10}$	7	1	$\mathbf{Q}$
	$2{\cdot}A_{11}$	5	1	$\mathbf{Q}(\sqrt{-11})$
	A_{17}	5, 7, 11, 13	1	$\mathbf{Q}$
	M_{11}	5	2	$\mathbf{Q}(\sqrt{-11})$
	M_{12}	5	1	$\mathbf{Q}(\sqrt{-11})$
17	$L_2(16)$	5, 17	1	$\mathbf{Q}$
	A_{18}	5, 7, 11, 13, 17	1	$\mathbf{Q}$
18	$SL_2(19)$	5	1	$\mathbf{Q}$
	$Sp_4(4)$	5, 17	1	$\mathbf{Q}$
	A_{19}	5, 7, 11, 13, 17	1	$\mathbf{Q}$
	$3{\cdot}J_3$	19	4	$\mathbf{Q}(\sqrt{-3}, \sqrt{5})$
19	A_{20}	7, 11, 13, 17, 19	1	$\mathbf{Q}$
20	$L_2(19)$	5	1	$\mathbf{Q}$
	$SL_2(19)$	5	1	$\mathbf{Q}$
	$SL_2(41)$	5	1	$\mathbf{Q}(\sqrt{41})$
	$L_3(4)$	5	1	$\mathbf{Q}$
	$4_2{\cdot}L_3(4)$	5	1	$\mathbf{Q}(\sqrt{-1})$
	$PSp_4(3)$	5	1	$\mathbf{Q}$
	$Sp_4(3)$	5	1	$\mathbf{Q}$
	$2{\cdot}A_7$	5	2	$\mathbf{Q}$
		7	1	$\mathbf{Q}$
	A_{21}	5, 11, 13, 17, 19	1	$\mathbf{Q}$
	A_8	5	1	$\mathbf{Q}$
	$U_3(5)$	7	1	$\mathbf{Q}$
	$2{\cdot}U_4(3)$	5, 7	1	$\mathbf{Q}$
	$4{\cdot}U_4(3)$	5	1	$\mathbf{Q}(\sqrt{-1})$

21	$L_2(41)$	5	1	$\mathbf{Q}(\sqrt{41})$
	$L_2(43)$	11	1	$\mathbf{Q}(\sqrt{-43})$
	$SL_3(4)$	7	1	$\mathbf{Q}(\sqrt{-3})$
	$U_3(3)$	7	1	$\mathbf{Q}$
	$U_3(5)$	7	1	$\mathbf{Q}$
	$SU_3(5)$	7	2	$\mathbf{Q}(\sqrt{-3})$
	$PSp_6(2)$	5,7	2	$\mathbf{Q}$
	$U_4(3)$	5,7	1	$\mathbf{Q}$
	$3_1{\cdot}U_4(3)$	7	1	$\mathbf{Q}(\sqrt{-3})$
	$3{\cdot}U_6(2)$	7	1	$\mathbf{Q}(\sqrt{-3})$
	A_7	7	1	$\mathbf{Q}$
	A_8	7	1	$\mathbf{Q}$
	A_{22}	5,7,13,17,19	1	$\mathbf{Q}$
	M_{22}	5,7	1	$\mathbf{Q}$
	$3{\cdot}M_{22}$	7	1	$\mathbf{Q}(\sqrt{-3})$
22	$L_2(23)$	11	3	$\mathbf{Q}$
			2	$\mathbf{Q}(\sqrt{3})$
	$SL_2(43)$	11	1	$\mathbf{Q}(\sqrt{-43})$
	$U_6(2)$	5,7,11	1	$\mathbf{Q}$
	A_{23}	5,7,11,13,17,19	1	$\mathbf{Q}$
	M_{23}	5,7,11	1	$\mathbf{Q}$
	HiS	7,11	1	$\mathbf{Q}$
	McL	7,11	1	$\mathbf{Q}$
23	$L_2(23)$	11	1	$\mathbf{Q}$
	A_{24}	5,7,11,13,17,19,23	1	$\mathbf{Q}$
	M_{24}	5,7,11,23	1	$\mathbf{Q}$
	Co_3	5,7,11,23	1	$\mathbf{Q}$
	Co_2	5,7,11,23	1	$\mathbf{Q}$
24	$L_2(49)$	5	1	$\mathbf{Q}$
	A_{25}	7,11,13,17,19,23	1	$\mathbf{Q}$
	$2{\cdot}Co_1$	5,7,11,13,23	1	$\mathbf{Q}$
25	$L_2(49)$	5	1	$\mathbf{Q}(\sqrt{3})$
	A_{26}	5,7,11,17,19,23	1	$\mathbf{Q}$
26	$L_2(25)$	13	3	$\mathbf{Q}$
			1	$\mathbf{Q}(\sqrt{3})$
	$L_2(27)$	13	2	$\mathbf{Q}(y_7)$

	$SL_2(27)$	7,13	1	$\mathbf{Q}$
	$SL_2(53)$	13	1	$\mathbf{Q}(\sqrt{53})$
	$L_3(3)$	13	1	$\mathbf{Q}$
	$L_4(3)$	5,13	1	$\mathbf{Q}$
	${}^3D_4(2)$	7,13	1	$\mathbf{Q}$
	A_{27}	5,7,11,13,17,19,23	1	$\mathbf{Q}$
27	$L_2(27)$	13	1	$\mathbf{Q}$
	$L_2(53)$	13	1	$\mathbf{Q}(\sqrt{53})$
	$PSp_6(2)$	5,7	1	$\mathbf{Q}$
	$3^{\cdot}G_2(3)$	7,13	2	$\mathbf{Q}(\sqrt{-3})$
	$3^{\cdot}P\Omega_7(3)$	7,13	1	$\mathbf{Q}(\sqrt{-3})$
	A_9	5	1	$\mathbf{Q}$
	A_{28}	5,11,13,17,19,23	1	$\mathbf{Q}$

Table 2: Case B

Subgroups of $GL(n,q)$ liftable to $GL(n,\mathbf{Q}_p^q)$ where $q \neq p$

n	G	p	$qe(n,G,p)$	$\mathbf{Q}(\chi)$	q
4	$Sp_4(3)$	5	1	$\mathbf{Q}(\sqrt{-3})$	p^2
	$2{\cdot}A_7$	5	1	$\mathbf{Q}(\sqrt{-7})$	p^2
5	$PSp_4(3)$	5	1	$\mathbf{Q}(\sqrt{-3})$	p^2
6	$3{\cdot}A_6$	5	1	$\mathbf{Q}(\sqrt{-3})$	p^2
	$6{\cdot}A_6$	5	1	$\mathbf{Q}(\sqrt{-3}, \sqrt{2})$	p^2
	$SL_2(13)$	7	1	$\mathbf{Q}(\sqrt{13})$	p^2
	$6{\cdot}L_3(4)$	5	1	$\mathbf{Q}(\sqrt{-3})$	p^2
	$6_1U_4(3)$	5	1	$\mathbf{Q}(\sqrt{-3})$	p^2
	$3{\cdot}A_7$	5	1	$\mathbf{Q}(\sqrt{-3})$	p^2
	$6{\cdot}A_7$	5	2	$\mathbf{Q}(\sqrt{-3}, \sqrt{2})$	p^2
	$2{\cdot}J_2$	7	1	$\mathbf{Q}(\sqrt{5})$	p^2
7	$L_2(8)$	7	1	$\mathbf{Q}(y_9)$	p^3
	$L_2(13)$	7	1	$\mathbf{Q}(\sqrt{13})$	p^2
	$U_3(3)$	7	1	$\mathbf{Q}(\sqrt{-1})$	p^2
8	$4_1L_3(4)$	7	1	$\mathbf{Q}(\sqrt{5})$	p^2
10	$2{\cdot}A_6$	5	1	$\mathbf{Q}(\sqrt{2})$	p^2
	A_7	5	1	$\mathbf{Q}(\sqrt{-7})$	p^2
	$SL_2(11)$	5	2	$\mathbf{Q}(\sqrt{3})$	p^2
	$2{\cdot}L_3(4)$	5	1	$\mathbf{Q}(\sqrt{-7})$	p^2
	$PSp_4(3)$	5	1	$\mathbf{Q}(\sqrt{-3})$	p^2
	M_{11}	5	2	$\mathbf{Q}(\sqrt{-2})$	p^2
	$2{\cdot}M_{12}$	5	2	$\mathbf{Q}(\sqrt{-2})$	p^2
	$2{\cdot}M_{22}$	5	2	$\mathbf{Q}(\sqrt{-7})$	p^2
12	$SL_2(23)$	11	1	$\mathbf{Q}(\sqrt{-23})$	p^2
	$Sp_4(5)$	13	1	$\mathbf{Q}(\sqrt{5})$	p^2
	$6{\cdot}Suz$	5, 11	1	$\mathbf{Q}(\sqrt{-3})$	p^2
13	$L_2(27)$	5, 11	1	$\mathbf{Q}(\sqrt{-3})$	p^2
	$U_3(4)$	13	1	$\mathbf{Q}(z_5)$	p^4
	$PSp_4(5)$	13	1	$\mathbf{Q}(\sqrt{5})$	p^2
	$PSp_6(3)$	13	1	$\mathbf{Q}(\sqrt{-3})$	p^2
14	$SL_2(13)$	7	2	$\mathbf{Q}(\sqrt{3})$	p^2

	$Sz(8)$	7	2	$\mathbf{Q}(\sqrt{-1})$	p^2
	$Sp_6(3)$	5	1	$\mathbf{Q}(\sqrt{-3})$	p^2
	$2{\cdot}A_7$	5	1	$\mathbf{Q}(\sqrt{2})$	p^2
	J_2	7	2	$\mathbf{Q}(\sqrt{5})$	p^2
15	$3{\cdot}A_6$	5	1	$\mathbf{Q}(\sqrt{-3})$	p^2
	$3{\cdot}A_7$	5	1	$\mathbf{Q}(\sqrt{-3})$	p^2
	$L_2(16)$	5	2	$\mathbf{Q}(y_{17})$	p^8
	$SL_3(4)$	5	1	$\mathbf{Q}(\sqrt{-3})$	p^2
	$3_1{\cdot}U_4(3)$	5	1	$\mathbf{Q}(\sqrt{-3})$	p^2
16	$2{\cdot}A_{11}$	7	1	$\mathbf{Q}(\sqrt{-11})$	p^2
17	$L_2(16)$	17	1	$\mathbf{Q}(\sqrt{5})$	p^2
			1	$\mathbf{Q}(y_{15})$	p^4
18	$SL_2(37)$	19	1	$\mathbf{Q}(\sqrt{37})$	p^2
	$3{\cdot}J_3$	17	4	$\mathbf{Q}(\sqrt{5}, \sqrt{-3})$	p^2
19	$L_2(37)$	19	1	$\mathbf{Q}(\sqrt{37})$	p^2
20	$L_2(19)$	5	3	$\mathbf{Q}(y_9)$	p^3
	$SL_2(19)$	5	3	$\mathbf{Q}(y9)$	p^3
	$SL_2(41)$	7	1	$\mathbf{Q}(\sqrt{41})$	p^2
	$4_2{\cdot}L_3(4)$	7	1	$\mathbf{Q}(\sqrt{-1})$	p^2
	$Sp_4(3)$	5	2	$\mathbf{Q}(\sqrt{-3})$	p^2
	$4{\cdot}U_4(3)$	7	1	$\mathbf{Q}(\sqrt{-1})$	p^2
21	$L_2(41)$	7	1	$\mathbf{Q}(\sqrt{41})$	p^2
	$L_2(43)$	7	1	$\mathbf{Q}(\sqrt{-43})$	p^2
	$SL_3(4)$	5	1	$\mathbf{Q}(\sqrt{-3})$	p^2
	$U_3(3)$	7	1	$\mathbf{Q}(\sqrt{-1})$	p^2
	$3_1{\cdot}U_4(3)$	5	1	$\mathbf{Q}(\sqrt{-3})$	p^2
	$3{\cdot}U_6(2)$	5,11	1	$\mathbf{Q}(\sqrt{-3})$	p^2
	$3{\cdot}A_7$	5	1	$\mathbf{Q}(\sqrt{-3})$	p^2
	A_8	7	1	$\mathbf{Q}(\sqrt{-15})$	p^2
	A_9	7	1	$\mathbf{Q}(\sqrt{-15})$	p^2
	$3{\cdot}M_{22}$	5,11	1	$\mathbf{Q}(\sqrt{-3})$	p^2
	J_2	7	1	$\mathbf{Q}(\sqrt{5})$	p^2
22	$SL_2(23)$	11	2	$\mathbf{Q}(\sqrt{2})$	p^2
			4	$\mathbf{Q}(\sqrt{2}, \sqrt{3}, y_{24})$	p^2
	$SL_2(43)$	7	1	$\mathbf{Q}(\sqrt{-43})$	p^2
23	$L_2(47)$	23	1	$\mathbf{Q}(\sqrt{-47})$	p^2

24	$SL_2(47)$	23	1	$\mathbf{Q}(\sqrt{-47})$	p^2
	$6{\cdot}A_7$	5	1	$\mathbf{Q}(\sqrt{-3}, \sqrt{-7})$	p^2
	$12_1L_3(4)$	5	2	$\mathbf{Q}(\sqrt{-1}, \sqrt{-3}, \sqrt{-7})$	p^2
	$Sp_4(7)$	5	1	$\mathbf{Q}(\sqrt{-7})$	p^2
25	$PSp_4(7)$	5	1	$\mathbf{Q}(\sqrt{-7})$	p^2
26	$SL_2(25)$	13	1	$\mathbf{Q}(\sqrt{2})$,	p^2
			2	$\mathbf{Q}(\sqrt{2}, \sqrt{3}, y_{24})$	
	$L_3(3)$	13	1	$\mathbf{Q}(\sqrt{-2})$	p^2
	${}^2F_4(2)'$	5,13	1	$\mathbf{Q}(\sqrt{-2})$	p^2
27	$3{\cdot}P\Omega_7(3)$	5	1	$\mathbf{Q}(\sqrt{-3})$	p^2
	${}^2F_4(2)'$	5,13	2	$\mathbf{Q}(\sqrt{-1}, \sqrt{-2})$	p^2

Table 3

Subgroups of $GL(n,q)$ liftable to $GL(n,\mathbf{C})$ and non-liftable to $GL(n,\mathbf{Q}_p^q)$

n	G	p	$N_G(G_p)$	$qe(n,G,p)$	$\mathbf{Q}(\chi)$	Jord(g)
2	$SL_2(5)$	5	5:4	1	$\mathbf{Q}(\sqrt{5})$	J_2
3	A_5	5	5:2	1	$\mathbf{Q}(\sqrt{5})$	J_3
	$L_3(2)$	7	7:3	1	$\mathbf{Q}(\sqrt{-7})$	J_3
	$3{\cdot}A_6$	5	5:6	1	$\mathbf{Q}(\sqrt{-3},\sqrt{5})$	J_3
4	$SL_2(7)$	7	7:6	1	$\mathbf{Q}(\sqrt{-7})$	J_4
	$2{\cdot}A_7$	7	7:6	1	$\mathbf{Q}(\sqrt{-7})$	J_4
6	$2{\cdot}J_2$	5	$(5\times 5):(2.D_{12})$	1	$\mathbf{Q}(\sqrt{5})$	$2J_3$ for $g\in(5A)_G$
8	A_6	5	5:2	1	$\mathbf{Q}(\sqrt{5})$	J_3+J_5
	$4_1L_3(4)$	5	$5:(2\times 4)$	1	$\mathbf{Q}(\sqrt{-1},\sqrt{5})$	J_3+J_5
10	A_7	7	7:3	1	$\mathbf{Q}(\sqrt{-7})$	J_3+J_7
	$2{\cdot}L_3(4)$	7	7:6	1	$\mathbf{Q}(\sqrt{-7})$	J_3+J_7
	$2{\cdot}M_{22}$	7	7:6	1	$\mathbf{Q}(\sqrt{-7})$	J_3+J_7
12	$L_2(13)$	7	$7:(2\times 2)$	1	$\mathbf{Q}(y7)$	J_5+J_7
14	J_2	5	$(5\times 5):D_{12}$	1	$\mathbf{Q}(\sqrt{5})$	$3J_3+J_5$
15	$L_2(16)$	17	17:2	2	$\mathbf{Q}(y17)$	J_{15}
16	$L_3(3)$	13	13:3	2	$\mathbf{Q}(d13)$	J_3+J_{13}
	$2{\cdot}A_{11}$	11	11:10	1	$\mathbf{Q}(\sqrt{-11})$	J_5+J_{11}
	M_{11}	11	11:5	1	$\mathbf{Q}(\sqrt{-11})$	J_5+J_{11}
	M_{12}	11	11:5	1	$\mathbf{Q}(\sqrt{-11})$	J_5+J_{11}
17	$L_2(16)$	5	$5:S_3$	2	$\mathbf{Q}(\sqrt{5})$	J_2+3J_5
18	$L_2(19)$	5	D_{20}	1	$\mathbf{Q}(\sqrt{5})$	J_3+3J_5
	$3{\cdot}J_3$	5	$5:(3\times S_3)$	1	$\mathbf{Q}(\sqrt{5},\sqrt{-3})$	J_3+3J_5
21	A_8	5	$5:\hat{S}_3\simeq 15:4$	1	$\mathbf{Q}(\sqrt{-15})$	$3J_2+3J_5$
	A_9	5	$5:\hat{S}_3\simeq 15:4$	1	$\mathbf{Q}(\sqrt{-15})$	$3J_2+3J_5$
	J_2	5	$(5\times 5):D_{12}$	1	$\mathbf{Q}(\sqrt{-15})$	$2J_3+3J_5$ for $g\in(5CD)_G$
24	$L_2(25)$	13	D_{52}	1	$\mathbf{Q}(y13)$	$J_{11}+J_{13}$
	$6{\cdot}A_7$	7	$6\times(7:3)$	1	$\mathbf{Q}(\sqrt{-3},\sqrt{-7})$	J_3+3J_7
	$2{\cdot}A_8$	7	$7:6$	1	$\mathbf{Q}(\sqrt{-7})$	J_3+3J_7
	$12_1L_3(4)$	7	$12\times(7:3)$	1	$\mathbf{Q}(\sqrt{-1},\sqrt{-3},\sqrt{-7})$	J_3+3J_7

26	$L_2(27)$	7	D_{28}	1	$\mathbf{Q}(\sqrt{-7})$	$J_5 + 3J_7$

Table 4

Subgroups of $GL(n, q)$ not liftable to $GL(n, \mathbf{C})$

n	G	p	$N_G(G_p)$	Jord(g)
2	$L_3(2)$	7	7:3	J_2
3	$3{\cdot}A_7$	5	$3 \times (5:4)$	J_3
5	$L_3(2)$	7	7:3	J_5
	A_7	7	7:3	J_5
7	J_1	11	11:10	J_7
8	$2{\cdot}Sz(8)$	5	$2 \times (5:4)$	$J_3 + J_5$
	A_7	5	5:4	$J_3 + J_5$
	A_{10}	5	$(5 \times 5) : [2^4]$	$J_3 + J_5$ for $g \simeq (5,5)$
	$2{\cdot}A_{10}$	5	$(5 \times 5) : [2^5]$	$J_3 + J_5$ for $g \in (5B)_G$
9	$3{\cdot}A_7$	7	$3 \times (7:3)$	$J_2 + J_7$
	A_{11}	11	11:5	J_9
	M_{11}	11	11:5	J_9
11	$L_3(3)$	13	13:3	J_{11}
	A_{13}	13	13:6	J_{11}
12	$12\dot{}_2L_3(4)$	7	$12 \times (7:3)$	$J_5 + J_7$
	$6{\cdot}A_7$	5	5:24	$J_2 + 2J_5$
	A_{14}	7	$(7 \times 7) : [36]$	$J_5 + J_7$ for $g \simeq (7,7)$
13	A_8	5	$5 : \hat{S}_3 \simeq (5 \times 3) : 4$	$J_3 + 2J_5$
	A_{15}	5		$J_3 + 2J_5$ for $g \simeq (5,5,5)$
14	J_1	11	11:10	$J_3 + J_{11}$
15	A_{17}	17	17:8	J_{15}
16	$2{\cdot}A_8$	7	7:6	$J_2 + 2J_7$
	$2{\cdot}Sz(8)$	13	$13 : (2 \times 4)$	$J_3 + J_{13}$
	$2{\cdot}A_7$	7	7:6	$J_2 + 2J_7$
	$4{\cdot}M_{22}$	7	7:12	$J_2 + 2J_7$
17	A_{19}	19	19:9	J_{17}
18	$3{\cdot}A_7$	5	5:12	$J_3 + 3J_5$
	A_{20}	5		$J_3 + 3J_5$ for $g \simeq (5,5,5,5)$

19	$L_3(4)$	7	7:3	$J_5 + 2J_7$
	A_8	7	7:3	$J_5 + 2J_7$
	A_{21}	7		$J_5 + 2J_7$ for $g \simeq (7,7,7)$
20	A_{22}	11		$J_9 + J_{11}$ for $g \simeq (11,11)$
	M_{22}	11	11:5	$J_9 + J_{11}$
21	A_{23}	23	23:11	J_{21}
	M_{23}	23	23:11	J_{21}
	HiS	5	$(5_+^{1+2} : (8 : 2)$	$J_3 + \ldots$ for $g \in (5A)_G$
	McL	5	$(5_+^{1+2} : (3 : 8)$	$J_3 + \ldots$ for $g \in (5A)_G$
23	$PSp_4(3)$	5	5:4	$J_3 + 4J_5$
	A_{25}	5		J_{23} for $g \simeq (25)$
24	$2^{\cdot}Sz(8)$	13	13:4	$J_{11} + J_{13}$
	A_{26}	13		$J_{11} + J_{13}$ for $g \simeq (13,13)$
	$12^{\cdot}M_{22}$	11	5:60	$J_2 + 2J_{11}$
	$U_3(3)$	7	7:3	$J_5 + 3J_7$
	$PSp_6(2)$	7	7:6	$J_5 + 3J_7$
27	A_{29}	29	29:14	J_{27}
	J_1	11	11:10	$J_5 + 2J_{11}$