

Minimal polynomials of elements of order p in p -modular projective representations of alternating groups

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Abstract. Let F be an algebraically closed field of characteristic $p > 0$ and G be a quasi-simple group with $G/Z(G) \cong A_n$. We describe the minimal polynomials of elements of order p in irreducible representations of G over F . If $p = 2$ we determine the minimal polynomials of elements of order 4 in 2-modular irreducible representations of A_n , S_n , $3 \cdot A_6$, $3 \cdot S_6$, $3 \cdot A_7$, and $3 \cdot S_7$.

1 Introduction

Throughout the paper, F is an algebraically closed field of characteristic $p > 0$ and all representations are F -representations unless otherwise stated. Let A_n and S_n denote the alternating and symmetric groups on n letters. We always assume that $n \geq 5$. Let G be a quasi-simple group with $G/Z(G) \cong A_n$, and

$$\pi : G \rightarrow A_n$$

be the natural projection. Thus G is one of the following groups: A_n , $\tilde{A}_n := 2 \cdot A_n$, $k \cdot A_6$, or $k \cdot A_7$ for $k = 3, 6$.

Our goal is to determine the minimal polynomials of the elements $g \in G$ of order p in the irreducible representations of G . Minimal polynomials of such elements are always of the form $(x - 1)^d$ for some $d \leq p$, and we determine all configurations where $d < p$.

Theorem 1.1 *Let G be a quasi-simple group with $G/Z(G) \cong A_n$, $g \in G \setminus Z(G)$ be an element of order p , and ϕ be a faithful irreducible representation of G over F . Then the degree d of the minimal polynomial of $\phi(g)$ is less than p if and only if one of the following happens:*

- (i) $\pi(g)$ is a product of two 3-cycles, $G = \tilde{A}_6$, $p = 3$, and ϕ is a basic spin representation of dimension 2.

(ii) $\pi(g)$ is a p -cycle and one of the following holds:

- (a) $G = A_p$, and ϕ is the ‘natural’ representation of dimension $p - 2$;
- (b) $G = \tilde{A}_n$, $p = 3$ or 5 , and ϕ is a basic spin representation;
- (c) $G = 3 \cdot A_7$ or $6 \cdot A_7$, $p = 7$, and $\dim \phi = 6$;
- (d) $G = \tilde{A}_7$, $p = 7$, and $\dim \phi = 4$;
- (e) $G = \tilde{A}_5$, $p = 5$, and $\dim \phi = 4$;
- (f) $G = 3 \cdot A_6$ or $3 \cdot A_7$, $p = 5$, and $\dim \phi = 3$.

Moreover, $d = p - 1$ in the case (ii)(b) above, and $d = \dim \phi$ in the remaining exceptional cases.

In particular, we see that there are two ‘reasons’ for the minimal polynomial of an element of order p to have degree less than p in an irreducible representation of G . One is trivial—the dimension of our representation might be less than p . The other is less obvious— $p = 3$ or 5 and the representation is a basic spin representation (these representations are known to be a source of many counterexamples and are pretty well-understood). We note that the degrees of the basic representations of \tilde{A}_n in prime characteristic may differ from those in zero characteristic, see Lemma 2.5 below.

In the proofs we only have to deal with the case $p > 3$, as the case $p = 3$ of Theorem 1.1 has recently been settled by Chermak [3].

Obviously, the case $p = 2$ is trivial for elements of order 2. However, a version of the question for $g \in G$ of order 4 is of essential interest. Of course, when $p = 2$ we do not need to deal with two-fold coverings. However, the case $G = S_n$ does not automatically reduce to A_n as g may not belong to A_n . So we consider S_n as well.

Theorem 1.2 *Let $p = 2$, $n \geq 5$, $G \in \{A_n, S_n, 3 \cdot A_6, 3 \cdot S_6, 3 \cdot A_7, 3 \cdot S_7\}$, $g \in G$ be an element of order 4, and ϕ be a faithful irreducible representation of G over F . Then the degree d of the minimal polynomial of $\phi(g)$ is less than 4 if and only if $d = 3$ and one of the following happens:*

- (a) g is of cycle type $(4, 2)$, and either $G \cong 3 \cdot A_6$, $\dim \phi = 3$ or $G \cong 3 \cdot S_6$, $\dim \phi = 6$;
- (b) $G \cong A_8 \cong SL(4, 2)$, g is of cycle type $(4, 4)$, and ϕ is either the natural representation of $SL(4, 2)$, or its dual, or its exterior square.
- (c) $G \cong S_8$, g is of cycle type $(4, 4)$, and $\dim \phi = 8$ or 6 .

2 Preliminaries

If M is a matrix, we denote by $\deg M$ the degree of the minimal polynomial of M and by $\text{Jord } M$ the Jordan normal form of M (defined up to the ordering of Jordan blocks). The Jordan block of size k with eigenvalue 1 is denoted by J_k . The symbol $\text{diag}(a_1, \dots, a_k)$ denotes the block-diagonal matrix with square matrices a_1, \dots, a_k along the diagonal.

If G is any group, we denote by 1_G the trivial FG -module (or the corresponding representation). If M is an FG -module (resp. $\phi : G \rightarrow GL(M)$ is a representation of G), and $H < G$ is a subgroup, then $M|_H$ (resp. $\phi|_H$) stands for the restriction of M (resp. ϕ) to H .

We record the following obvious fact.

Lemma 2.1 *Let G be a finite group and $g \in G$. If ρ and ϕ are representations of G such that ρ is a subfactor of ϕ then $\deg \rho(g) \leq \deg \phi(g)$.*

Let $m < n$. Throughout the paper we will often consider S_m as a subgroup of S_n , \tilde{A}_m as a subgroup of \tilde{A}_n , etc. Unless otherwise stated, the embeddings are assumed to be *natural*, i.e. the subgroup acts on the *first* m letters.

Now, let $G = A_n$ or S_n . We will refer to the non-trivial composition factor of the natural n -dimensional permutation FG -module as the *natural irreducible module* and denote it by E_n . Denote by ε_n the corresponding representation. We have $\dim \varepsilon_n = n - 2$ if $p|n$ and $\dim \varepsilon_n = n - 1$ otherwise.

Lemma 2.2 *Let $G = S_n$ or A_n and $g \in G$ be an element of order p . Then the degree d of the minimal polynomial of $\varepsilon_n(g)$ is p , unless $n = p$, in which case $d = p - 2$.*

Proof. An easy explicit calculation (see e.g. [8, Lemmas 2.1,2.2]). \square

Let $1_{\tilde{S}_n}^-$ be the sign module over FS_n , and set $E_n^- = E_n \otimes 1_{\tilde{S}_n}^-$. Define

$$\mathcal{E}_n := \{1_{S_n}, 1_{\tilde{S}_n}^-, E_n, E_n^-\}.$$

If λ is a p -regular partition, D^λ denotes the irreducible FS_n -module corresponding to λ , see [6]. The following is a useful inductive characterization of the FS_n -modules from \mathcal{E}_n .

Proposition 2.3 *Let $n \geq 6$ and D be an irreducible FS_n -module. Suppose that all composition factors of the restriction $D|_{S_{n-1}}$ belong to \mathcal{E}_{n-1} . Then $D \in \mathcal{E}_n$, unless $n = 6, p = 3$ and $D \in \{D^{(4,2)}, D^{(2^2, 1^2)}\}$, or $n = 6, p = 5$ and $D \in \{D^{(4,1^2)}, D^{(3,1^3)}\}$.*

Proof. By tensoring with $1_{\tilde{S}_n}^-$ if necessary, we may assume that $1_{S_{n-1}}$ or E_{n-1} occurs in the socle of $D|_{S_{n-1}}$. Then it follows from [7, Theorem 0.5] that either $D \in \mathcal{E}_n$ or $D \in \{D^{(n-2,2)}, D^{(n-2,1^2)}\}$. However, by [7, Theorem 0.4(ii)], $D^{(n-2,2)}|_{S_{n-1}}$ contains $D^{(n-3,2)}$ as a composition factor, and $D^{(n-3,2)} \notin \mathcal{E}_{n-1}$ unless $n = 6$ and $p = 3$. Similarly, $D^{(n-2,1^2)}|_{S_{n-1}}$ contains $D^{(n-3,1^2)}$ as a composition factor and $D^{(n-3,1^2)} \notin \mathcal{E}_{n-1}$ unless $n = 6$ and $p = 5$. \square

Corollary 2.4 *Let $n \geq 7$, and V be an irreducible FA_n -module such that all composition factors of the restriction $V|_{A_{n-1}}$ belong to $\{1_{A_{n-1}}, E_{n-1}\}$. Then $V \in \{1_{A_n}, E_n\}$.*

Proof. Follows from Clifford theory and Proposition 2.3 \square

Let \tilde{S}_n denote a (non-trivial) two-fold central cover of S_n . Of course, \tilde{A}_n is a subgroup in \tilde{S}_n of index 2. The group \tilde{S}_n has (one or two) remarkable complex representations called *basic (spin) representations*. These can be characterized as its faithful complex representations of minimal degree and constructed using Clifford algebras. A basic spin representation can also be defined as an irreducible representation of \tilde{S}_n whose character is labelled by the partition (n) in the Schur's parametrization of irreducible characters. The degree of a basic representation of \tilde{S}_n is $2^{(n-1)/2}$ if n is odd, and $2^{(n-2)/2}$ if n is even. On restriction to \tilde{A}_n , basic representations remain irreducible if n is even and split as a direct sum of two non-equivalent irreducibles if n is odd. In both cases the corresponding complex representations of \tilde{A}_n are also called *basic*.

Finally, for both \tilde{S}_n and \tilde{A}_n , every irreducible constituent of Brauer reduction of a basic representation modulo p is called a (modular) *basic* representation. Dimensions of modular basic representations of \tilde{S}_n have been determined by Wales [11]. For $p > 2$, these are the same as for complex representations, unless p divides n , in which case they are twice as small. Moreover, in [11, Table III], Wales provides a complete information concerning tensoring basic modular representations with sign, from which the dimensions of basic modular representations of \tilde{A}_n also follow, at least for $p > 2$. If $p = 2$ one can use Benson [1]. To summarize, we have:

Lemma 2.5 *Let $d_n(p)$ be the dimension of a modular basic representation of \tilde{A}_n .*

- (i) *Let $p > 2$ and $p \nmid n$. Then $d_n(p) = 2^{(n-3)/2}$ if n is odd, and $2^{(n-2)/2}$ if n is even.*
- (ii) *Let $p > 2$ and $p|n$. Then $d_n(p) = 2^{(n-3)/2}$ if n is odd, and $2^{(n-4)/2}$ if n is even.*
- (iii) *Let $p = 2$. Then $d_n(2) = 2^{(n-3)/2}$ if n is odd, $2^{(n-2)/2}$ if $n \equiv 2 \pmod{4}$, and $2^{(n-4)/2}$ if $n \equiv 0 \pmod{4}$.*

We cite another result of Wales for future reference:

Proposition 2.6 *Let $n > 5$ and ϕ be a faithful irreducible representation of \tilde{A}_n . Then ϕ is basic if and only if all composition factors of $\phi|_{\tilde{A}_{n-1}}$ are basic.*

Proof. For \tilde{S}_n a similar result is contained in the proof of [11, Theorem 8.1]. Then Clifford theory implies the result for \tilde{A}_n . \square

Finally we record a lemma of G. Higman which is often used below.

Lemma 2.7 [2, Ch. IX, Theorem 1.10] *Let $G \subset GL(n, F)$ be a finite subgroup with abelian normal subgroup A of order coprime to p . Let $g \in G$ be an element of order p^k such that $g^{p^{k-1}} \notin C_G(A)$. Then $\deg g = p^k$.*

3 Main Results

The following result of the second author provides us with an induction base for future arguments:

Lemma 3.1 [12, Lemma 2.12] *Let $n < 2p$, G be a quasi-simple group with $G/Z(G) \cong A_n$, $g \in G$ be an element with $g^p \in Z(G)$. Suppose that ϕ is a faithful irreducible representation of G such that $\deg \phi(g) < p$. Then one of the following holds:*

- (i) $Z(G) = 1$, $n = p$, and $\phi = \varepsilon_n$ with $\dim \phi = p - 2$;
- (ii) $p = 3$, $G = \tilde{A}_5$, and $\dim \phi = 2$;
- (iii) either $p = 5$, $G \cong \tilde{A}_6$, or $p = 5, 7$, $G \cong \tilde{A}_7$, and in the both cases $\dim \phi = 4$;
- (iv) $p = 5$, $G = \tilde{A}_8$ or \tilde{A}_9 , and $\dim \phi = 8$;
- (v) $p = 5$, $G = \tilde{A}_5$, and $\dim \phi = 2$;
- (vi) $p = 5$, $G = \tilde{A}_5$, and $\dim \phi = 4$;
- (vii) $p = 5$, $G = 3 \cdot A_6$ or $3 \cdot A_7$, and $\dim \phi = 3$;
- (viii) $p = 7$, $G = 3 \cdot A_7$ or $6 \cdot A_7$, and $\dim \phi = 6$.

Moreover, in all the cases above, except (iv), the Jordan normal form of $\phi(g)$ has a single block, and in case (iv) it has two blocks of size 4.

Remark. The representations ϕ appearing in (ii)-(v) are basic.

Lemma 3.2 *Let $G = A_n$ or \tilde{A}_n , with $n \geq 2p > 6$, and $g \in G$ be an element of order p . If $p = 5$, suppose additionally that $\pi(g)$ is a 5-cycle. If $p = 7$ suppose additionally that either $G = A_n$ or $\pi(g)$ is a 7-cycle. If ϕ is a faithful irreducible representation of G with $\deg \phi(g) < p$, then $p = 5$, $G = \tilde{A}_n$, and ϕ is basic.*

Proof. We may assume that $\pi(g)$ is a product of cycles of the form:

$$\pi(g) = (1, 2, \dots, p)(p + 1, \dots, 2p) \cdots$$

Recall that for $m < n$, A_m is assumed to be embedded into A_n as acting on the first m letters, unless otherwise stated. Define a subgroup H of G by requiring that (1) $H \supset Z(G)$; (2) $\pi(H) \cong A_7$ if $p = 5$; $\pi(H) \cong A_8$ if $p = 7$ and $G = \tilde{A}_n$; $\pi(H) \cong A_p$ otherwise.

Set $X = \langle g, H \rangle$. Then we have $H \cong X/O_p(X)$ and $g = hg_1$, where $h = (1, 2, \dots, p) \in H$ and $g_1 \in O_p(X)$. Let τ be a non-trivial composition factor of $\phi|_X$. Then $\tau(O_p(X)) = \text{Id}$, so we can also consider τ as a representation of H . We have $\tau(g) = \tau(h)$. In view of Lemma 2.1, $\deg \tau(g) < p$.

If $Z(G) = \{1\}$ then $Z(H) = \{1\}$, and so $\tau = \varepsilon_n$, thanks to Lemma 3.1. By induction on n it follows from Corollary 2.4 that $\phi = \varepsilon_n$. The result now follows from Lemma 2.2.

Finally, let $|Z(G)| = 2$. By Lemma 3.1, $p = 5$ and τ is basic. So Proposition 2.6 implies that ϕ is basic. \square

Lemma 3.3 *Let $G = \tilde{A}_n$ or A_n , and $g \in G$ be an element of order $p > 3$ such that $\pi(g)$ has k non-trivial cycles. If $\deg \phi(g) < p$ for some faithful irreducible representation ϕ of G then $k < 3$.*

Proof. Suppose $k \geq 3$. We may assume that

$$\pi(g) = (1, 2, \dots, p)(p+1, p+2, \dots, 2p)(2p+1, 2p+2, \dots, 3p) \dots$$

Let A be the elementary abelian 3-subgroup of A_n of order 3^p generated by the commuting 3-cycles $(j, p+j, 2p+j)$ for $1 \leq j \leq p$. If $G = \tilde{A}_n$, let $B = \pi^{-1}(A)$. If $G = A_n$, take $B = A$. In both cases B is abelian of order prime to p , and $g \in N_G(B) \setminus C_G(B)$. Now we apply Lemma 2.7. \square

Lemma 3.4 *Let $G = \tilde{A}_n$ or A_n , and $g \in G$ be an element of order $p = 5$ or 7 . If $\deg \phi(g) < p$ for some faithful irreducible representation ϕ of G , then $\pi(g)$ is a p -cycle.*

Proof. In view of Lemma 3.3, we may assume that

$$\pi(g) = (1, 2, \dots, p)(p+1, p+2, \dots, 2p).$$

Set $h_{ij} = (i, i+p)(j, j+p) \in A_n$ for $1 \leq i < j \leq p$. The subgroup H generated by the h_{ij} is abelian of order 2^{p-1} . If $G = A_n$, we may apply Lemma 2.7, as $g \in N_G(H) \setminus C_G(H)$. Now, let $G = \tilde{A}_n$.

Assume first that $p = 7$. Observe that H can be considered as an $\mathbf{F}_2\langle\pi(g)\rangle$ -module via conjugation, and $\langle\pi(g)\rangle$ is a cyclic group of order p . Then the dimension of H over \mathbf{F}_2 is 6, hence $\langle\pi(g)\rangle$ has an irreducible constituent M on H of dimension 3. In other words, $\pi(g)$ normalizes M , and $[\pi(g), M] \neq 1$. Let $L = \pi^{-1}(M)$. Then $|L| = 16$, hence it is not extraspecial. Now it is easy to deduce, using conjugation with g , that L is abelian. As $g \in N_G(L) \setminus C_G(L)$, the result follows from Lemma 2.7.

Finally, let $p = 5$. Then g is contained in a group X isomorphic to the central product of two copies of \tilde{A}_5 . Let τ be an irreducible constituent of the restriction ϕ to X . Then $\tau = \tau_1 \otimes \tau_2$ where τ_1 and τ_2 are faithful representations of the respective copies of \tilde{A}_5 . In view of Lemma 3.1 and [5, Chapter VIII, Theorem 2.7], $\deg \tau(g) < 5$ only if $\dim \tau_1 = \dim \tau_2 = 2$. This means that every irreducible constituent of the restriction of ϕ to the naturally embedded \tilde{A}_5 is basic. By Proposition 2.6, ϕ is basic. Then $\deg \phi(g) = 5$ by [8, Lemma 3.12]. \square

Proof of Theorem 1.1. For $p = 3$, see Chermak [3], and for $n < 2p$, see Lemma 3.1. Let $p > 3$ and $n \geq 2p$. Then the ‘only-if’ part follows from Lemmas 3.2–3.4. For the ‘if’ part it remains to show that $d := \deg \phi(g) = 4$ for ϕ basic spin, $p = 5$, and $\pi(g)$ a 5-cycle. Restricting to a natural subgroup \tilde{A}_7 containing g and using Lemma 3.1, we see that $d \geq 4$. On the other hand, for complex representations of \tilde{A}_n a theorem similar to Theorem 1.1 has been proved in [13]. In particular, if $g \in \tilde{A}_n$ is a 5-cycle then $\deg \beta(g) = 4$ for complex basic spin representations β . As ϕ is a constituent of a reduction of β modulo 5, we have $d \leq 4$. \square

Now we prove Theorem 1.2. The result is contained in Lemmas 3.5–3.11.

Lemma 3.5 *Theorem 1.2 is true for $n = 5$.*

Proof. As A_5 has no elements of order 4 we may assume that $G = S_5$. Then G has two non-trivial irreducible representations, both of dimension 4, see [6, Tables]. One of them is ε_5 , for which $\varepsilon_5 \oplus 1_{S_5} = \pi$, where π is the natural permutation representation of dimension 5. Clearly, $\text{Jord } \pi(g) = \text{diag}(J_4, J_1)$, so $\text{Jord } \varepsilon_5(g) = J_4$. Another irreducible representation of G corresponds to the partition $(3, 2)$, and so it is reducible on A_5 , thanks to [1] or [9]. Therefore $\text{Jord } \phi(g^2) = \text{diag}(J_2, J_2)$ whence $\text{Jord } \phi(g) = J_4$. \square

Lemma 3.6 *Let $n \geq 5$, $G \in \{A_n, S_n, 3 \cdot A_6, 3 \cdot S_6, 3 \cdot A_7, 3 \cdot S_7\}$, and $g \in G$ be an element of order 4 fixing at least one point of the natural permutation set. Then $\deg \phi(g) = 4$ for any faithful irreducible representation ϕ of G .*

Proof. We may assume that g transitively permutes $1, 2, 3, 4$ and fixes 5. Let $H := \text{Alt}\{1, 2, 3, 4, 5\}$, and \hat{H} be the preimage of H in G . Set $X := \langle g, \hat{H} \rangle$. As H contains no element of order 4, the restriction homomorphism $h : X \rightarrow \text{Sym}\{1, 2, 3, 4, 5\} \cong S_5$ is surjective. Let $K = \ker h$. Clearly, K is central in X . As S_5 has no non-split central extension with center of order 3, we have $X \cong Z(G) \times Y$ for some subgroup Y with $g \in Y$. Let τ be a composition factor of $\phi|_Y$ with $\dim \tau > 1$. Then $\tau(Y) \cong S_5$. By Lemma 3.5, $\deg \tau(g) = 4$, hence $\deg \phi(g) = 4$ in view of Lemma 2.1. \square

Lemma 3.7 *Theorem 1.2 is true for $G = A_6$ and S_6 .*

Proof. For $g \in S_6 \setminus A_6$ this follows from Lemma 3.6. So we may assume that $G = A_6$. We use [9]. Irreducible FG -modules of dimension 8 are projective. So the Jordan form of g on each of these modules is $\text{diag}(J_4, J_4)$. Other non-trivial irreducible FG -modules are of dimension 4. As $A_6 \subseteq S_6 \cong Sp(4, 2)$, one of them is the natural $Sp(4, 2)$ -module V restricted to A_6 . As the Jordan form of a unipotent element of $Sp(4, 2)$ does not have a block of size 3, the theorem is true for the natural representation. The second FG -module of dimension 4 is obtained from V by twisting with the outer automorphism σ of $S_6 = Sp(4, 2)$. As A_6 has only one conjugacy class of elements of order 4, $\sigma(g)$ is conjugate to g in A_6 , so $\text{Jord } \sigma(g) = \text{Jord } g$. \square

Lemma 3.8 *Let $n \geq 6$, $G = A_n$ or S_n , and $g \in G$ be an element of order 4 having a 2-cycle in its cycle type. Then $\deg \phi(g) = 4$ for any faithful irreducible representation ϕ of G .*

Proof. Clearly g normalizes a subgroup $H \cong A_6$ fixing $n - 6$ points such that g has a 2- and 4-cycle on the remaining 6 points. Then $g = g_1g_2$ where $g_1 \in H$, $g_2 \in C_G(H)$. Set $X = \langle g, H \rangle = \langle g_2, H \rangle$, and let τ be a non-trivial composition factor of $\phi|_X$. As $X/O_2(X) \cong H$, Lemma 3.7 gives $\deg \tau(g) = 4$. So by Lemma 2.1, $\deg \phi(g) = 4$. \square

Lemma 3.9 *Theorem 1.2 is true for $n = 8$.*

Proof. In view of Lemmas 3.8 and 3.6, we may assume that the cycle type of g is $(4, 4)$ and $G = A_8 \cong SL(4, 2)$. Note that the group A_8 has 2 conjugacy classes of elements of order 4, corresponding to cycle types $(4, 2)$ and $(4, 4)$, and only the first one meets the subgroup A_6 . The group $SL(4, 2)$ has 2 conjugacy classes of elements of order 4, with Jordan forms J_4 and $\text{diag}(J_3, J_1)$, and the second one does not meet $Sp(4, 2)$. As $A_6 \cong Sp(4, 2)'$, we conclude that the class $(4, 4)$ corresponds to the class $\text{diag}(J_3, J_1)$. So g belongs to the intermediate subgroup $H \cong SL(3, 2)$.

Let τ be an irreducible representation of H . Then τ is a restriction of a rational representation of \bar{H} , the algebraic group of type A_2 . The irreducible representations of \bar{H} are labelled by their highest weights $a_1\omega_1 + a_2\omega_2$, where a_1, a_2 are non-negative integers and ω_1, ω_2 are the fundamental weights. It is well known that τ is a restriction of one of the four irreducible representations of \bar{H} labelled by $0, \omega_1, \omega_2$, or $\omega_1 + \omega_2$. The last one corresponds to the Steinberg module, whose restriction to H is projective, and so all Jordan blocks of g are of size 4. Two other representations are the natural and its dual. So the Jordan form of g on both of them is $\text{diag}(J_3, J_1)$. Finally, corresponding to the zero highest weight we have the trivial representation.

Now, let $\lambda = a_1\omega_1 + a_2\omega_2 + a_3\omega_3$ be the highest weight of ϕ . By a theorem of Smith [10] (also proved independently by R. Dipper), the restriction $\phi|_H$ contains a direct summand τ with highest weight $a_1\omega_1 + a_2\omega_2$. By the previous paragraph, we may assume that at least one of a_1, a_2 is zero. By duality, the same is true for a_2, a_3 . So we are left with the cases $\lambda \in \{\omega_1, \omega_2, \omega_3, \omega_1 + \omega_3\}$. The last one is the adjoint representation of G . Clearly, its restriction to H contains a composition factor isomorphic to the adjoint representation of H . As the last representation is projective, it is a direct summand. Hence this case is ruled out. The cases $\lambda = \omega_1, \omega_3$ are obvious. Finally, the module corresponding to $\lambda = \omega_2$ is the exterior square of the natural module. So its restriction to H is a direct sum of the natural and dual natural modules, hence the Jordan blocks of $\phi(g)$ are of size 3. \square

Lemma 3.10 *Let $G = A_n$ or S_n , and $g \in G$ be an element of order 4 containing at least three 4-cycles. Then $\deg \phi(g) = 4$ for any faithful irreducible representation ϕ of G .*

Proof. We may assume that

$$g = (1, 2, 3, 4)(5, 6, 7, 8)(9, 10, 11, 12) \dots$$

Set $h_j := (j, j + 4, j + 8)$ for $j = 1, 2, 3, 4$, and $H := \langle h_1, h_2, h_3, h_4 \rangle$. Then H is an abelian 3-group and $g \in N_G(H)$. Moreover, $g^2 \notin C_G(H)$, so the result follows from Lemma 2.7.

□

Lemma 3.11 *Theorem 1.2 is true for $G = 3 \cdot A_6$ and $3 \cdot A_7$.*

Proof. For $G = 3 \cdot A_7$ see Lemma 3.6. Let $G = 3 \cdot A_6$. Then $\dim \phi = 3$ or 9, see [9]. In the former case $\deg \phi(g) = 3$, as $\deg \phi(g) < 3$ implies $\phi(g)^2 = 1$, which is false. Let $\dim \phi = 9$. Observe that g^2 normalizes a cyclic group $\langle c \rangle$ of order 5. Set $X := \langle g^2, c \rangle$. As $g^2 c g^{-2} = c^{-1}$ and the multiplicity of every eigenvalue $\alpha \neq 1$ of $\phi(c)$ is 2 (see [9]), it follows that $\phi|_X$ has four composition factors of dimension 2 and one composition factor of dimension 1. Therefore $\text{Jord } g^2 = \text{diag}(J_2, J_2, J_2, J_2, J_1)$, whence $\text{Jord } g = \text{diag}(J_4, J_4, J_1)$.

□

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