Passman's problem on adjoint representations

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Abstract We study the representations of finite simple groups over the complex numbers obtained from the action of a group on itself by conjugation. For finite simple groups G the main question under discussion is whether each irreducible representation of G is a constituent of this one. The answer was known to be positive for alternating groups. We prove that the same is true for all sporadic simple groups. However, the main result is to provide the negative answer for a series of the unitary simple groups $PSU(n,q)$ where n is coprime to $2(q+1)$.

1. Introduction

In 1992 Passman [**5**] discussed the problem on determining the kernel of the adjoint representation of a group algebra. By dealing with this problem in general, he also made a considerable contribution to the finite group case where the problem had a certain history already.

Let G be a group, F a field and FG the group algebra of G over F. The conjugation representation of G is the one arising from the action of G on FG by conjugation. It extends to a representation $\Delta : FG \to FG$ which is called the **adjoint representation** of FG. The term comes from Hopf algebra theory where the representation in question is the adjoint representation of Hopf algebra FG.

PROBLEM 1.1. (D.S. Passman) Determine the kernel of Δ.

For group algebras of finite groups over the complex numbers the problem was first touched on by Roth [**6**] who conjectured a very straightforward answer, namely, that the kernel coincides with the kernel of the natural homomorphism $FG \to F(G/Z(G))$. However, it was immediately observed by J.S. Frame in his review to Roth's paper and then by Formanek [**2**] that this is invalid. Nevertheless the conjecture does reflect a certain reasonable philosophy.

The adjoint representation is quite large as its dimension coincides with that of the group algebra. Therefore, its kernel is expected to be very small. In his review J.S. Frame just warns that the conjecture cannot be true without exceptions. Exceptions do exist, and in the above work Passman has discovered new principles for constructing examples. However, the nature of the exceptions remains rather obscure. We think that the problem is of greater interest for simple groups as the study of the adjoint action may yield additional tools for constructing their representations. For symmetric and alternating groups the problem above was studied earlier, see [**7**] and [**5**].

In this paper we provide an infinite family of finite simple groups for whose complex group algebras Roth's conjecture is false. Observe that Frame's example with $G = PSU(3,3^2)$ is a member of this family.

THEOREM 1.2. Let $G = SU(n,q^2)$ where n is odd and coprime to $q+1$. Then $\ker\Delta \neq 0$.

In contrast with Theorem 1.2 $\ker\Delta = 0$ for all other classical simple groups. A proof of this result will appear elsewhere.

THEOREM 1.3. If G is a simple sporadic group then $\ker\Delta = 0$.

Let Ad_G be the module of the adjoint representation. Obviously, it is a permutation module and it is the direct sum of the permutation modules associated with the action of G on its conjugacy classes. Let $g \in G$ and let g^G be the conjugacy class of g. Let $C = C_G(g)$. Then the permutation module associated with the action of G on g^G is isomorphic to the induced module 1_C^G where 1_C stands for the

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trivial FC-module. Therefore, Problem 1.1 has an obvious translation to the language of representation theory. For finite groups and F of characteristic 0, Problem 1.1 is equivalent to

Problem 1.1a. Determine the irreducible representations of G that do not occur as constituents of the induced representation $1_{C_G(g)}^G$ for any $g \in G$.

Using Frobenius' reciprocity theorem one can further restate it as follows:

Problem 1.1b. Determine the irreducible representations ϕ of G such that $\phi|_{C_G(g)}$ does not contain the trivial representation of $C_G(g)$ for every $g \in G$.

In examining this problem we prove the following much stronger version of Theorem 1.2:

THEOREM 1.4. Let $G = SU(n,q)$ where n is odd and coprime to $q+1$. Let ϕ be an irreducible representation of G of dimension $(q^n - q)/(q+1)$. Then for every $g \in G$ the group $C_G(g)$ contains an abelian subgroup T (which depends on g) such that 1_T is not a constituent of $\phi|_T$.

If G is finite and F is an algebraically closed field of characteristic 0 then FG is the direct sum of simple components $R_i = M(n_i, F)$ where $M(n,F)$ is the algebra of $(n \times n)$-matrices over F. If $M(n_i, F) \subseteq \ker \Delta$ then the representation of G associated with the homomorphism $G \to GL(n_i, F)$ does not occur as a constituent of Ad_G. This implies that $\ker \Delta = 0$ if and only if every irreducible representation τ of G is a constituent of $\sigma \otimes \sigma^*$ where σ is some irreducible representation of G and $*$ stands to denote the dual of σ.

LEMMA 1.1. *Let G be a finite group and let ω be the character of an irreducible representation of G occuring in* $\ker \Delta$. *Then* $(\omega, \chi\overline{\chi}) = (\omega\chi, \chi) = 0$ *for every irreducible character χ of G.*

Solomon [**11**] observes that the multiplicity of an irreducible character ω of G in Ad_G is equal to the sum of the entries of the ω-row in the character table of G. Therefore, if ω is as in Lemma 1.1 then the entries of the ω-row sum to 0.

Our experience obtained from the study of Passman's problem leads us to a quite unexpected conjecture. To state it, let us call a conjugacy class K of G **global** if every irreducible representation of G is a constituent of $1_{C_G(x)}^G$ for $x \in K$.

CONJECTURE 1.5. Suppose that G is a finite simple group. Then $\Delta = 0$ if and only if G has a global conjugacy class.

In this paper we prove the conjecture for alternating groups and for sporadic simple groups:

THEOREM 1.6. Let G be an alternating group A_n for $n > 4$ or a simple sporadic group. Then G contains a global conjugacy class.

The conjecture is also true for classical groups, a proof of which will be published elsewhere. For alternating groups our proof of Theorem 1.6 is a modification of Passman's argument [**5**] used in proving that $\Delta = 0$. For sporadic groups the proof of Theorem 1.6 is based on the representation theory of groups with cyclic Sylow p-subgroups. The key observation is the following:

THEOREM 1.7. Let G be a finite group with cyclic Sylow p-subgroup P. Suppose that $C_G(g) = P$ for $g \in P$ of order p. Let ϕ be an irreducible representation of G over the complex numbers. Then either $\dim \phi < |P|$ or ϕ is a constituent of the induced representation 1_P^G. In particular, if G has no non-trivial representations of dimension less than $|P|$ then the conjugacy class of g is global.

Every sporadic simple group G has a self-centralizing Sylow p-subgroup of order p for some p. If p is the minimal prime with this property then p is always less than the dimension of any non-trivial representation of G. So for sporadic groups Theorem 1.6 follows from Theorem 1.7.

Notation. The greatest common divisor of two integers a, b is denoted by (a,b). If G is a group, $Z(G)$ is its center and G' the commutator subgroup of G. The finite field of q elements is denoted by F_q. Our notations for classical groups are standard, in particular, $GL(V)$ denotes the group of all linear transformation of a vector space V. The algebra of $(n \times n)$-matrices over a field F is denoted by $M(n,F)$ and $GL(n,q)$ stands for the group of all non-singular matrices. We usually identify $GL(V)$ with $GL(n,q)$ if $\dim V = n$. The unitary group over F_{q^2} is denoted by $U(n,q)$ and the special unitary group by $SU(n,q)$. Our notation for sporadic simple groups agrees with the Atlas of Finite Groups [**1**]. The symmetric and alternating groups on n symbols are denoted by S_n and A_n, respectively. If ϕ is a representation or a character of a group G and $H \subset G$ is a subgroup, the symbol $\phi|_H$ denotes the restriction of ϕ to H.

A similar notation is used for modules and for characters of G. If τ is a representation of H (module, character) then τ^G stands for the induced representation (induced module, induced character) of G. The trivial one dimensional representation (G-module, character) of G is denoted by 1_G.

2. The method

From now on F denotes an algebraically closed field of characteristic 0 and all groups are finite. In order to show that $\Delta \neq 0$ it suffices to find an irreducible representation ϕ of G such that $1_{C_G(x)}$ is not a constituent of $\phi|_{C_G(x)}$ for every $x \in G$. Conversely for showing that $\Delta = 0$ it is sufficient to detect a global conjugacy class, that is, to find $x \in G$ such that $1_{C_G(x)}$ is a constituent of $\phi|_{C_G(x)}$ for every irreducible representation ϕ of G.

Let V be a unitary vector space, that is, a vector space endowed with a non-degenerate sesquilinear Hermitian form. An element e in $GL(V)$ is said to be **indecomposable** if V is not the direct sum of two e-stable non-zero subspaces of V.

Definition 2.1. Let V be the natural module for $U(n, q)$. We call a subgroup T of $U(n, q)$ a **star subgroup** if there is an odd $k \leq n$ such that T stabilizes a non-degenerate k-dimensional subspace W of V, acts trivially on $W^\perp$ and $T|_W = C_{U(W)}(e)$ where e is an indecomposable element of $U(W)$.

The centralizers of irreducible elements in classical groups are called Singer subgroups. These are special cases of star subgroups for $n = k = t$.

Proposition 2.2. Let $G = U(n, q)$. Then the following holds.

(1) Every star subgroup T of G is abelian.

(2) Star subgroups of G are parametrized up to conjugacy by the pairs of odd integers (k, t) where t divides k and $k \leq n$. If $T = T(k, t)$ is parametrized by (k, t) then $|T| = (q^t + 1)q^{n-t}$.

(3) Suppose that n is odd. Then for every $g \in G$ the group $C_G(g)$ contains a star subgroup. In addition, if $H = G/Z(G)$ is the projective unitary group then for every $h \in H$ the group $C_H(h)$ contains the projection in H of a star subgroup of G.

Representatives of the conjugacy classes of star subgroups can be described explicitly as follows. Let $l = k/t$. Start from group $X = U(l, q^t)$. Let J_l be the Jordan block of size l with eigenvalue 1. Replacing X by a conjugate we can assume that $J_l \in X$. Let $T_0 = C_X(J_l)$. The centralizer C of J_l in the matrix ring $M(l, q^{2t})$ is spanned over $F_{q^{2t}}$ by the matrices $\mathrm{Id}, a, a^2, \ldots, a^{l-1}$ where $a = J_l - \mathrm{Id}$. Then $T_0 = C \cap X$ can be described explicitly, see Section 5 below for details. Then $T(k, t)$ is obtained from T_0 through the natural embeddings $X \to U(k, q) \to U(n, q)$. Observe that the images T_0 in $U(k, q)$ when t runs over divisors of k yield all centralizers of indecomposable elements in $U(k, q)$ up to conjugation.

A key to our proof of Theorem 1.4 is the analysis of Weil representations of unitary groups. The main reference is Gerardin [**3**, 4.9.21]. Some relevant results on Weil representations can be found in [**8**] and [**13**].

Proposition 2.3. [**3**, 4.9.21] Let $G = U(n, q) = U(V)$. There is a representation of G whose character χ is given by

$$\chi(g) = (-1)^n(-q)^{d(g)}$$

where $d(g) = n - \dim(g - \mathrm{Id})V$.

In particular, $\chi(1) = q^n$. We refer to χ as the **Weil character**. The representation with character χ is called the **Weil representation**. The advantage of using star subgroups T in $C_G(g)$ for $g \in G$ is that one can easily count the number of the elements $x \in T$ with a given value $\chi(x)$. This allows us to perform explicitly the computation of the inner product $(\chi|_T, 1_T)$ for the Weil character χ. This yields the following result.

Theorem 2.4. Let T be a star subgroup of $G = U(n, q)$ and let χ be the Weil character of G. Then $(\chi|_T, 1_T) = 0$.

Let $G = U(n, q)$ and $Z = Z(G)$ so $|Z| = q + 1$. Let W be the module of the Weil representation. Then $W|_Z = \oplus W_\zeta$ where ζ runs over the irreducible characters of Z and $W_\zeta = \{w \in W : zw = \zeta(z)w$ for all $z \in Z\}$. The module W_ζ with $\zeta = 1_Z$ is denoted by W_0.

Proposition 2.5. [**8**] Let $n > 2$. Then W_ζ are irreducible G-modules. Moreover, every W_ζ remains irreducible under restriction to $G' \cong SU(n, q)$. In addition, $\dim W_\zeta = \frac{q^n-(-1)^n}{q+1}$ for $\zeta \neq 1_Z$ and $\frac{q^n+(-1)^n q}{q+1}$ otherwise.

If n is odd then $\dim W_0 = \frac{q^n - q}{q+1}$ and this is the minimum degree of a non-trivial irreducible representation of G'. Moreover, W_0 is a unique irreducible G'-module of this degree. Observe that W_0 can be viewed as a $PU(n,q)$- and $PSU(n,q)$-module as Z acts trivially on W_0. Combining Theorem 2.4 with Proposition 2.2(3) we now get

THEOREM 2.6. Let n be odd and let θ be the representation of $H = PU(n,q)$ realized in W_0. Then $1_{C_H(h)}$ is not a constituent of $\theta|_{C_H(h)}$ for all $h \in H$. Therefore, $\ker \Delta \neq 0$.

Theorem 1.2 is a particular case of Theorem 2.6. Indeed, if $(n, q+1) = 1$ then $PSU(n,q) \cong PU(n,q)$.

3. Groups with cyclic Sylow p-subgroup

LEMMA 3.1. *Let G be a finite group with a cyclic Sylow p-subgroup P. Suppose that $C_G(x)$ is abelian for elements $x \in G$ of order p. Let ϕ be an irreducible representation of G such that $\phi|_P$ contains no subrepresentation isomorphic to the regular representation of P. Then* $\dim \phi < |C_G(P)/Z(G)|$.

PROOF. This is essentially Proposition 2.8 in [**15**]. It assumes P to be a TI-subgroup of G which means that $P \cap gPg^{-1} = 1$ for every $g \in G \setminus N_G(P)$. If $P \cap gPg^{-1} \neq 1$ then there is $x \in (P \cap gPg^{-1})$ of order p hence both P and gPg^{-1} are Sylow p-subgroups of $C_G(x)$. As $C_G(x)$ is abelian, they coincide hence $g \in N_G(P)$. □

Proof of Theorem 1.7. Suppose first that ϕ is of p-defect 0. Then $\phi|_P$ is well known to be a multiple of the regular $\mathbf{C}\,P$-module. Hence 1_P is a constituent of $\phi|_P$ and the result follows by Frobenius's reciprocity theorem.

Suppose that ϕ is not of p-defect 0. By Lemma 3.1, either $\dim \phi < |P|$ or $\phi|_P$ contains a subrepresentation isomorphic to the regular representation of P. In particular, 1_P is a constituent of $\phi|_P$. By Frobenius' reciprocity, ϕ is a constituent of 1_P^G.

We illustrate the use of Theorem 1.7 with a few examples. Let first $G = A_n$. If P is a Sylow p-subgroup of G then P is cyclic if and only if $p \leq n \leq 2p-1$, and if P is cyclic then $P = C_G(P)$ if and only if $p \leq n \leq p+2$.

LEMMA 3.2. *Let $G = A_n$ with $n > 4$ and let P be a Sylow p-subgroup of G.*

(1) *Suppose that $n = p+1$ or $p+2$. If $1 \neq x \in P$ then the conjugacy class x^G is global.*

(2) *Suppose that $n = p$. Let ρ be an irreducible representation of G. Then $\rho|_P$ contains 1_P unless* $\dim \rho = n-1$.

PROOF. (1) Observe that G has no non-trivial irreducible representation of dimension $d < p$. By Theorem 1.7, x^G is a global conjugacy class. (2) The representations of dimension $d < p$ are of dimension 1 or $p-1$ except for $p = 5$ where there is an irreducible representation of dimension 3. One can check that the representation of dimension $p-1$ is the only irreducible representation of G whose restriction to $C_G(P)$ does not contain 1_P. □

LEMMA 3.3. *Every sporadic simple group G has a self-centralizing Sylow p-subgroup P of order p for some p. If p is the minimal prime with this property then $p \leq$* $\dim \phi$ *for every non-trivial representation ϕ of G.*

Proof. We have the following table where the 1st row lists the sporadic simple groups G, the 2nd row gives the minimum prime p such that a Sylow p-subgroup of G is cyclic of order p and the 3rd row gives the minimum degree $d(G)$ of a non-trivial ordinary representation of G.

TABLE 1

G	M_{11}	M_{12}	M_{22}	M_{23}	M_{24}	J_1	J_2	J_3	J_4	HS	McL	He	Ru
p	5	11	5	11	11	7	7	17	29	7	11	17	29
$d(G)$	10	11	21	22	23	56	14	85	1333	22	22	51	378

G	Suz	$O'N$	Co_1	Co_2	Co_3	Fi_{22}	Fi_{23}	Fi'_{24}	HN	Ly	Th	BM	M
p	11	11	23	11	23	13	17	17	19	31	19	31	41
$d(G)$	143	10944	276	23	23	78	782	8671	133	2480	248	4371	196883

In each case the minimal prime does not exceed $d(G)$. Therefore, Theorem 1.3 and Theorem 1.6 follow from Theorem 1.7.

Example. For some choices of n, q the group $SU(n,q)$ has a self-centralizing Sylow p-subgroup. If n is odd, this happens exactly when $\frac{q^n+1}{q+1} = q^{n-1} - q^{n-2} + \cdots - q + 1$ is a prime power. Groups $G = SU(5,2)$ for $p = 11$, $SU(7,2)$ for $p = 43$, $SU(3,3)$ for $p = 7$, $SU(3,7)$ for $p = 43$ and many others have this property. By Theorem 1.7, the induced module 1_P^G contains all irreducible representations ϕ of G provided $\dim \phi \geq |P|$. On the other hand, the minimum dimension of a non-trivial representation of G is known to be $\frac{q^n+1}{q+1} - 1 = |P| - 1$. Moreover, this representation is known to be unique. Therefore, for the groups $SU(n,q)$ with this property there is at most one irreducible representation ϕ such that ϕ is not a constituent of 1_P^G. This shows that $\dim \ker \Delta = (\frac{q^n+1}{q+1} - 1)^2$.

Groups G that have faithful representations of degree less than $|P|$ are known, see Zhang [**16**] and [**17**]. This allows us to determine all simple groups for which Theorem 1.7 provides a global conjugacy class of elements of order p. The following lemma follows from Zhang [**16**], [**17**].

Lemma 3.4. *Let $G \subset GL(n, \mathbf{C})$ be an irreducible subgroup with cyclic self-centralizing Sylow p-subgroup P. Suppose that $n < |P|$. Then one of the following holds:*

(1) *P is normal in G;*
(2) *$G = PSL(2,p)$ with n odd and $(p-1)/2 \leq n \leq (p+1)/2$;*
(3) *$G = PSL(2,p)$ or $PGL(2,p)$ where $p \equiv 3 \mod 4$ and $n = p - 1$;*
(4) *$SL(2,2^a) \subseteq G \subseteq Aut(SL(2,2^a))$, $|P| = 2^a + 1$ and $n = 2^a$ or $n = 2^a - 1$;*
(5) *$PSL(d,q) \subseteq G \subseteq Aut(PSL(d,q))$ where d is odd, $|P| = (q^d - 1)/(q - 1)$ and $n = |P| - 1$;*
(6) *$PSU(d,q) \subseteq G \subseteq Aut(PSU(d,q))$ where d is odd, $|P| = (q^d + 1)/(q + 1)$ and $n = |P| - 1$;*
(7) *$A_p \subseteq G \subseteq S_p$ and $n = p - 1$;*
(8) *M_{11} for $|P| = 11$, M_{23} for $|P| = 23$. In these cases $n = |P| - 1$.*

Proof. Suppose that P is not normal in G. We use [**16**] if $|P| = p$ and [**17**] if $|P| > p$. Our assumptions are stronger than those in [**16**], [**17**] so we have to drop a few groups which occur in [**16**]. First of all, we exclude the groups whose center is non-trivial, as P is self-centralizing here. We also have to exclude the case of $G = PSp(2k,q)$ with $|P| = (q^k + 1)/2$ for q odd in [**17**]. Indeed, the group $PSp(2k,q)$ has no irreducible representation of even degree $(q^k - 1)/2$. Hence $(q^k - 1)/2$ is odd so $|P| = (q^k + 1)/2$ is even which is impossible as P is of odd order. For sporadic groups the cases where $G = M_{12}$, M_{22} with $|P| = 11$, J_3 for $p = 19$ and Ru for $p = 29$ have to be dropped as their representations of dimension less than $|P|$ are projective.

Theorem 3.1. Let G be a finite simple group with cyclic self-centralizing Sylow p-subgroup P. Let $1 \neq x \in P$. Then the conjugacy class x^G is global unless one of the following holds:
(1) $G = A_p$;
(2) $G = PSL(2,p)$;
(3) $SL(2,2^a)$ and $|P| = 2^a + 1$;
(4) $PSL(d,q)$ where d is odd and $|P| = (q^d - 1)/(q - 1)$;
(5) $PSU(d,q)$ where d is odd and $|P| = (q^d + 1)/(q + 1)$;
(6) M_{11} for $|P| = 11$ and M_{23} for $|P| = 23$.

Proof. We only have to inspect the cases (2) - (6) of Lemma 3.4. This is rather straightforward except when $G = SL(2,2^a)$ and $n = 2^a - 1$. The latter case can be sorted out by using the character table for $SL(2,2^a)$.

4. Alternating Groups

Lemma 4.1. *For every natural number $n > 15$ there exists a prime p such that $\frac{n}{2} < p \leq n - 5$.*

Proof. Use induction on n. For $n = 16, 17$ choose $p = 11$. Let $n > 17$. By Passman [**5**, p. 869], there are at least three primes p such that $\frac{n}{2} < r \leq n$. If the lemma is false then $n - 4, n - 2, n$ have to be primes if n is odd, otherwise, $n - 1, n - 3, n - 5$ have to be primes. This is impossible as for any integer m at least one of $m, m - 2, m - 4$ is divisible by 3.

Lemma 4.2. *If $n > 15$ then there exists a sequence $n = n_0 > n_1 > \cdots > n_k$ such that $5 \leq n_k \leq 15$ and $n_i < p_i = n_{i-1} - n_i$ for $i = 1, \ldots, k$ where $p_1, \ldots, p_k$ are distinct primes.*

Proof. Choose p_1 according to Lemma 4.1 and set $n_1 = n - p_1$. Then $5 \leq n_1 < p_1$. If $n_1 \leq 15$, we are done. Otherwise, we can use induction on n to produce a sequence $n_1 > \cdots > n_k$ with the properties required. As $p_1 > n_1$, it is distinct from p_i with $i > 1$.

THEOREM 4.1. Let $G = A_n$ for $n > 4$. Then there is an element $x \in G$ such that $H = C_G(x) = \langle x \rangle$ and 1_H is a constituent of $\phi|_H$ for every irreducible representation ϕ of G.

Proof. Suppose first that $5 \leq n \leq 15$. Choose x with $|x|$ according to the following table:

n	5	6	7	8	9	10	11	12	13	14	15
$\lvert x\rvert$	3	5	5	7	7	21	21	11	11	13	13

If $n \neq 10, 11$ then $|x|$ is a prime and in this case the lemma follows from Lemma 3.2(1). Let $n = 10$ and use the notation of [**1**] for the characters of $G = A_{10}$. Observe that g^7 and g^3 are rational in G while g is not. Let χ be an irreducible character of G. Then the character values of all elements of order 21 are the same unless $\chi(1) = 384$. Therefore, with this exception, one can compute the multiplicity of the eigenvalue 1 of $\phi(g)$ as $\frac{1}{21}(\chi(1) + 2\chi(g^7) + 6\chi(g^3) + 12\chi(g))$. It is easy to conclude by using the character table of G that this is always non-zero. Let $\chi(1) = 384$. In this case the multiplicity of eigenvalue 1 of $\phi(g)$ is equal to 16.

For $n = 11$ the argument is similar to the case $n = 10$.

Let $n > 15$. Choose a sequence $n = n_0 > n_1 > \cdots > n_k$ as defined by Lemma 4.2 and set $m = n - n_k$. Let $x_1 \in A_m$ be of order $p_1 \cdots p_k$ and let $x_2 \in A_{n-m}$ be of order $|x_2|$ chosen from the above table. Then $|x_1|$ and $|x_2|$ are coprime as $p_k > n - m$. It is easy to observe that G contains an element x of order $|x_1| \cdot |x_2|$ and $H := C_G(x)$ coincides with $\langle x \rangle$. We use induction on k starting with $k = 0$ which we interpret as $n \leq 15$. So for $k = 0$ the theorem has already been shown. Express $x = yz$ where $|y| = p_1$ and $yz = zy$. Then there are subgroups G_1, G_2 of G such that $[G_1, G_2] = 1$, $G_1 \cong A_{p_1}$, $G_2 \cong A_{n-p_1}$, $y \in G_1$ and $z \in G_2$. It is easy to observe that $H = C_G(x) = C_{G_1}(y)C_{G_2}(z)$.

Let ϕ be an irreducible representation of G and let τ be an irreducible constituent of $\phi|_{G_1G_2}$. Then $\tau = \tau_1 \otimes \tau_2$ where τ_i is an irreducible representation of G_i for $i = 1, 2$. By induction, $\tau_2|_{C_{G_2}(z)}$ contains a constituent isomorphic to $1_{C_{G_2}(z)}$. If $1_{C_{G_1}(y)}$ is a constituent of $\tau_1|_{C_{G_1}(y)}$ then we are done. Otherwise, by Lemma 3.2(2) for every choice of τ we have that $\dim \tau_1 = p_1 - 1$. It follows that all irreducible constituents of $\phi|_{G_1}$ are of dimension $p_1 - 1$. This is impossible. Indeed, it suffices to show that there is no irreducible representation σ of A_{p_1+1} such that all irreducible constituents of $\sigma|_{A_{p_1}}$ are of dimension $p_1 - 1$. This can be easily deduced from the branching rule describing the irreducible constituents of $\sigma|_{A_{p_1}}$.

5. Indecomposable elements and their centralizers

In this section we describe the centralizers in $G = U(n, q)$ of indecomposable elements $g \in U(n, q)$. Let V be the natural $F_{q^2}G$-module.

LEMMA 5.1. *Let A be a commutative algebra over F_{q^2} with 1, and $a \in A$. Suppose that elements $1, a, a^2, ..., a^k$ form a basis of A and $a^{k+1} = 0$. Set $A_i = \langle a^i, ..., a^k \rangle$. Let σ be a ring automorphism of A such that $\sigma^2 = \mathrm{Id}$ and $\sigma|_{1 \cdot F_{q^2}} \neq \mathrm{Id}$. Set $B = \{x \in A : \sigma(x)x = 1\}$ and $B_i = \{b \in B : b - 1 \in A_i\}$ for $i > 0$. Then $|B_i| = q^{k-i+1}$ and $|B| = (q+1)q^k$.*

Proof. Obviously, A_1 is the nilpotent radical of A so $\sigma(A_1) = A_1$ hence $\sigma(A_i) = A_i$ for every i. Observe that $\sigma|_{A_k}$ is not scalar as $\sigma(\alpha y) = \sigma(\alpha)\sigma(y)$ for $\alpha \in F_{q^2}$ and $y \in A$.

Set $A'_k = \{x \in A_k : \sigma(x) = x\}$. Then A'_k is an F_q-subspace. As A_k is of dimension 1 over F_{q^2}, it is of dimension 2 over F_q. Hence $\dim_{F_q} A'_k = 1$ so $|A'_k| = q$. Let $\lambda : A_k \to A_k$ be the mapping defined for $a \in A_k$ by $\lambda(a) = \sigma(a) + a$. Then $\lambda(A_k)$ is contained in A'_k. As $\sigma|_{A_k}$ is not scalar, $\lambda \neq 0$ and $\ker \lambda \neq 0$. Since λ is linear over F_q, the image of λ is of dimension 1. Therefore, $\lambda(A_k) = A'_k$.

Let $x \in A_k$. Obviously, $1 + x \in B_k$ if and only if $\sigma(x) = -x$, that is, $x \in A'_k$. Therefore, $|B_k| = q$.

We prove the lemma by induction on k. The case $k = 1$ has just been settled. Let $\pi : A \to A/A_k$ be the natural projection and $\tilde{A} = \pi(A) = A/A_k$. Let $\tilde{B} = \{x \in \tilde{A} : \sigma(x)x = 1\}$. By induction, $|\tilde{B}| = q^{k-1}$. Let $\tilde{b} \in \tilde{B}$ and let $c \in A$ be such that $\pi(c) = b$. Then $c\sigma(c) = 1 + y$ for some $y \in A_k$. As A is commutative, $\sigma(1+y) = 1+y$ hence $y \in A'_k$. By the above, there is $z \in A_k$ such that $\lambda(z) = z + \sigma(z) = -y$. Observe that $z\sigma(c) = z$ and $c\sigma(z) = \sigma(z)$ as $c - 1, \sigma(c) - 1 \in A_1$ and $A_1A_k = 0$. Then we compute $(c+z)\sigma(c+z) = 1 + y + z\sigma(c) + c\sigma(z) = 1 + y + z + \sigma(z) = 1$ as $z\sigma(z) = 0$. It follows that $c + z \in B$. Therefore, $\pi(B) = \tilde{B}$ so $|\pi(B)| = q^{k-1}$. Observe that $\pi|_B$ is a group homomorphism whose kernel is $\pi^{-1}(1) = \{b \in B : b - 1 \in A_k\} = B_k$. Therefore, $|B| = q^k$ as claimed.

LEMMA 5.2. *Let $X \in GL(n, q^2)$. Then X is conjugate to an element in $U(n, q)$ if and only if x is conjugate to $\sigma(X^t)^{-1}$, where σ is the automorphism of $M(n, q)$ defining $U(n, q)$. Furthermore, two elements in $U(n, q)$ are conjugate in $U(n, q)$ if and only if they are conjugate in $GL(n, q)$.*

Proof. [**14**, p. 34]

LEMMA 5.3. *Let J_n denote the Jordan block of size n with identity diagonal. Then*

(1) $U(n,q)$ contains a conjugate of J_n.

(2) Let $u \in U(n,q)$ be a unipotent element. Then there exists an orthogonal decomposition $V = V_1 \oplus \cdots \oplus V_k$ such that u restricted to V_i is indecomposable for all $i = 1,...,k$. In other words the Jordan normal form of u on V_i is J_{n_i} where $n_i = \dim V_i$.

Proof. It is well known that J_n is conjugate to J_n^{-1} in $GL(n,q^2)$. So (1) follows from Lemma 5.2. Since $U(r,q)$ is naturally embedded into $U(n,q)$ for $r < n$, it follows from (1) that $U(n,q)$ contains a matrix u' such that $u'V_i = V_i$ and $\mathrm{Jord}(u'|_{V_i}) = J_{n_i}$. By Lemma 5.2, u and u' are conjugate in $U(r,q)$ and (2) follows.

LEMMA 5.4. *Let $G = U(k,q)$ and $g \in G$ an indecomposable element. Express $g = su$ where $(|s|,q) = 1$ and u is a p-element. Then:*

(1) the F_{q^2}-span L of $\langle s\rangle$ is a field;

(2) $C_G(s) \cong U(l,L)$ where $l = \dim_L V$ is the composition length of s on V;

(3) u is indecomposable element of $U(l,L)$ (so u is conjugate in $U(l,L)$ to the Jordan block J_l).

Proof. (1) By Maschke's theorem, L is a semisimple F_{q^2}-algebra. If L is not a field then L contains a non-trivial idempotent e, say, and then $V = eV \oplus (1-e)V$ is a decomposition of V as a direct sum of proper g-modules, which is a contradiction.

(2) It is well known that $R := C_{M(k,F_{q^2})}(L) \cong M(l,L)$. Let σ be an involutive anti-automorphism of $\mathrm{End}\, V$ such that $U(k,q) = \{x \in GL(n,q^2) : \sigma(x)x = \mathrm{Id}\}$. As σ acts non-trivially on scalar matrices, the restriction $\sigma|_L$ is a non-trivial automorphism of L. Hence $H =: U(k,q) \cap R$ is the unitary subgroup of R. As $\dim V = l \cdot \dim L$, we have that $|L| = q^{2n/l}$. So $U(l,L) \cong U(l,q^{k/l})$, as desired.

(3) As V is a vector space over L, it can be viewed as an LR-module which is obviously isomorphic to the natural R-module. Hence it is also isomorphic to the natural module for $U(l,L)$. Clearly, V is indecomposable as an $L\langle g\rangle$-module. As s acts scalarly on the latter module, it is indecomposable as an $L\langle u\rangle$-module. Therefore, u is similar to the Jordan block J_l, see Corollary 5.3.

By Lemma 5.3 $U(n,q)$ contains a unipotent matrix whose Jordan normal form consists of the single Jordan block J_n. Since we can always replace $U(n,q)$ with a conjugate, we can assume without loss of generality that $J_n \in U(n,q)$.

LEMMA 5.5. *Let $G = U(n,q)$ and let $u \in G$ be a unipotent indecomposable element. Let V be the natural module for G. Then $C_G(u) = Z(G)\cdot B_1$ is an abelian group of order $(q+1)q^{n-1}$. In addition, there is a descending chain of subgroups $B_1, \dots, B_n = \mathrm{Id}$ of B_1 such that $B_i \setminus B_{i+1} = \{b \in B : \dim C_V(b) = i\}$.*

Proof. By Lemma 5.3, we can assume that $u = J_n$. It is well known that $U(n,q)$ can be described as $U(n,q) = \{m \in M(n,q^2) : \sigma(m)m = \mathrm{Id}\}$ where σ is a suitable anti-automorphism of $M(n,q^2)$ such that $\sigma^2 = 1$ and $\sigma|F_{q^2} \neq \mathrm{Id}$. Therefore, $\sigma(J_n) = J_n^{-1}$. Let $a = J_n - \mathrm{Id}$. Then a^i is a matrix with 1 at the positions $(1,i), (2,i+1), \dots, (n-i,n)$ and zero entries elsewhere. The matrices $\mathrm{Id}, a, a^2, \dots, a^{n-1}$ generate a commutative algebra over F_{q^2}, which we denote by A. Then $\sigma(A) = A$. We set $A_i = \langle a^i, \dots, a^{n-1}\rangle$ so A_i is an ideal of A. Let $e_1, \dots, e_n$ be the canonical basis of V. Define V_i to be the subspace spanned by $e_1, \dots, e_i$. It is easy to observe that $A_iV_i = 0$ and $V_i = \{v \in V : A_iv = 0\}$.

Set $B = C_G(u)$. It is well known that A coincides with the centralizer of J_n in the full matrix algebra $M(n,q^2)$. Therefore, $B = A \cap G = \{x \in A : \sigma(x)x = \mathrm{Id}\}$. As A is commutative, B is abelian. Define $B_i = \{x \in B : x - \mathrm{Id} \in A_i\}$. By Lemma 5.1, $|B| = (q+1)q^{n-1}$ and $|B_i| = q^{n-i}$. As $|Z(G)| = q+1$ and $Z(G) \subset B$, one observes that $B = Z(G)\cdot B_1$. In addition, $V_i = \{v \in V : B_iv = v\}$ which follows from the similar assertion above for A_i.

THEOREM 5.1. Let $g \in G = U(k,q)$ be an indecomposable element. Then

(1) $C_G(g)$ is an abelian group of order $(q^t+1)q^{k-t}$ for some odd divisor t of k;

(2) for $l = k/t$ there is a descending chain of subgroups $B_1, \dots, B_l = \mathrm{Id}$ of $C_G(g)$ such that $B_i \setminus B_{i+1} = \{b \in B_1 : \dim C_V(b) = it\}$ for $1 \le i < l$;

(3) $C_G(g) = KB_1$ where $|K| = q^t+1$ and every elements of K fixes no non-zero vector on V.

Proof. Express $g = su$ where s is semisimple and u is unipotent. As $|s|$ and $|u|$ are coprime, $C_G(g) = C_G(s) \cap C_G(u) = C_{C_G(s)}(u)$. By Lemma 5.4, $C_G(s) \cong U(l,L)$ where l is the composition length of s on V and L is a subfield of $M(n,q^2)$ containing the identity. In addition, u is indecomposable as an element of $U(l,L)$. By Lemma 5.5 applied to $u \in U(l,L)$, we have that $C_{C_G(s)}(u) = Z(U(l,L)) \cdot B_1$ where B_1 is described in Lemma 5.5 with the replacement of q by q^t and n by $l = n/t$. Therefore, B_1 is abelian of order $q^{t(l-1)} = q^{k-t}$. Let B_i be the subgroups introduced in Lemma 5.5. As V can be viewed as the natural module for $U(l,L)$, by Lemma 5.5 $|C_V(b)| = q^{it}$ for any $b \in B_i \setminus B_{i+1}$ with $1 \le i < l$.

6. Centralizers of elements of $U(n,q)$

In this section $G = U(n,q)$ and V is the natural $F_{q^2}G$-module.

Lemma 6.1. *Let $u \in H = U(n,q)$ be a unipotent element. Suppose that u is decomposable. Then $C_H(u) \setminus Z(H)$ contains an element x of order $q+1$. Furthermore, if $(n, q+1) = 1$ then x can be chosen in $SU(n,q)$.*

Proof: It follows from Lemma 5.3 that $U(n,q)$ contains a matrix y such that $y|_{V_1}$ is scalar of order $q+1$ and y is the identity on $V_2 \oplus \cdots \oplus V_n$. If $\det(y) = \lambda$ and $(n, q+1) = 1$, then there exists a scalar matrix z in $U(n,q)$ such that $\det(z) = \lambda^{-1}$. Then $x = yz$ as desired.

The following lemma is well known, a proof can be found in [**9**, Lemma 3.3].

Lemma 6.2. *Let $X \subset G$ be a completely reducible subgroup and let W be some homogeneous component of X on V. Then W is either non-degenerate or totally isotropic. In the second case there is another totally isotropic homogeneous component W' of V such that $\dim W = \dim W'$ and $W + W'$ is non-degenerate.*

Lemma 6.3. *Let $g \in G = U(n,q)$ where n is odd. Then there exists a non-degenerate g-submodule W of V such that $\dim W$ is odd and $g|_W$ is indecomposable.*

Proof: Choose W to be a non-degenerate g-submodule of V of minimal odd dimension. As n is odd, W exists. Express $g = su$ where u is unipotent and s is semisimple. Let $k = \dim(W)$ and let $s' = s|_W$. If s' is not homogeneous then by Lemma 6.2 W contains a non-degenerate homogeneous s'-component W' of odd dimension. But then $gW' = W'$, which contradicts the minimality of W. Therefore s' is homogeneous. Then $\langle s' \rangle_{F_{q^2}} = L$ is a field, and if $L \cong F_{q^{2t}}$ then t divides k and $C_{U(W)}(s') \cong U(k/t, q^t)$. Let $u' = u|_W$. View W as a vector space over L. Then W is indecomposable as g-module if and only if W is indecomposable as $L\langle u \rangle$-module. By Lemma 5.3, W is a direct sum of indecomposable $L\langle u \rangle$-modules. As $\dim_L W$ is odd, one of them is of odd dimension. This contradicts the minimality of W, unless it coincides with W.

An element $s \in GL(V)$ is called **homogeneous** if V is the sum of irreducible s-submodules isomorphic to each other.

Lemma 6.4. *Let $s \in G$ be a homogeneous element. Then $V = V_1 \oplus \cdots \oplus V_l$ where $V_1, \dots, V_l$ are non-degenerate irreducible s-submodules of the same dimension which is an odd integer.*

Proof. Recall that the enveloping algebra L of s is a field, hence all irreducible s-submodules have the same dimension t, say. In addition, for any $0 \neq v \in V$ the subspace $W = Lv$ is an irreducible s-module. Choose v to be an anisotropic vector. Then W is not totally isotropic. As W is an irreducible s-module, $W \cap W^{\perp} = 0$ hence W is non-degenerate. Therefore, $s|_W$ is contained in $U(k,q)$, and it is irreducible. Hence t is odd, see Huppert [**4**]. The lemma now follows by induction as $s|_{W^{\perp}}$ is homogeneous and $L|_{W^{\perp}}$ is the enveloping algebra of $s|_{W^{\perp}}$.

Proof of Proposition 2.2 (1) Let T be a star subgroup of G. Then there exists a non-degenerate subspace W of V and $e \in G$ such that $eW = W$, $e|_{W^{\perp}} = \mathrm{Id}$ and $e|_W$ is indecomposable. This determines a group $T = \{x \in C_G(e) : xW = W, x|_{W^{\perp}} = \mathrm{Id}\}$. Therefore, $T \cong T|_W = C_{U(W)}(e|_W)$ where $U(W)$ is the group of isometries of W. By Lemma 5.1, T is abelian.

(2) Let T_j for $j = 1, 2$ be two star subgroups with the same data (t, k) where t divides k. Then for $j = 1, 2$ there exists a non-degenerate subspace W_j of dimension k and element $e_j \in G$ such that $e_j W_j = W_j$, the restriction $e_j|_{W_j}$ is indecomposable and $T_j = \{x \in C_G(e_j) : xW_j = W_j \text{ and } x|_{W_j^{\perp}} = \mathrm{Id}\}$. So $T_j|_{W_j} = C_{U(W_j)}(e_j|_{W_j})$. By Witt's theorem we can assume that $W_1 = W_2$. Therefore, it suffices to prove the claim for the case where $W_1 = W_2 = V$. Then $n = \dim V = k$. Express $e_j = s_j u_j$ where s_j is semisimple and u_j is unipotent. Let l_j be the composition length of s_j on V. Then $l_j = k/t$ so $l_1 = l_2$. Set $l = l_1$. By Lemma 6.4 and Witt's theorem we can assume that $V = V_1 \oplus \cdots \oplus V_l$ where $V_1, \dots, V_l$ are non-degenerate irreducible s- and s'-submodules of the same dimension l. Let E_j be the enveloping algebra of s_j. By Lemma 5.4, E_j is a field which dimension over F_{q^2} equals $k/l = t$. Therefore, E_1 is isomorphic to E_2. Since $s_j|_{V_i}$ is irreducible, $C_{U(V_i)}(s_j)$ is of order $q^t + 1$ (Huppert [**4**]). Subgroups of order $q^t + 1$ are conjugate in $U(W_j)$, hence we can assume that $C_{U(V_i)}(s_1) = C_{U(V_i)}(s_2)$. Clearly, $E_j|_{V_i}$ is the enveloping algebra of $s_j|_{V_j}$. As $s_j|_{V_i}$ is irreducible, it follows from Schur's lemma that $C_{U(V_i)}(s_j)$ is contained in $E_j|_{V_i}$. Hence $C_{U(V_i)}(s_j)$ spans $E_j|_{V_i}$. Therefore, $E_1|_{V_i} = E_2|_{V_i}$, and hence $E_1 = E_2$. Let $R_j = C_{\mathrm{End}\, V}(s_j)$. Then $R_j = C_{\mathrm{End}\, V}(E_j)$ so $R_1 = R_2 \cong M(l, q^{2t})$. Let σ be the anti-automorphism defining G, that is, $G = \{x \in GL(k, q^2) : \sigma(x)x = \mathrm{Id}\}$. As $C_G(s_j) = R_j \cap G$, by Lemma 5.4 $C_G(s_1) = C_G(s_2) \cong U(l, q^t)$.

Furthermore, u_1, u_2 are indecomposable elements in $U(l, q^t)$. Therefore they are conjugate in $U(l, q^t)$ to a Jordan block J_l, hence they are conjugate in $C_G(s_1) = C_G(s_2)$. So we can assume that $u_1 = u_2$. As $C_G(s_j) = C_G(s_j) \cap C_G(u_j)$, (2) follows.

(3) Express $V = V_1 \oplus \cdots \oplus V_m$ where $V_1, \dots, V_m$ are non-degenerate orthogonally indecomposable g-modules. As n is odd, by Lemma 6.3, at least one of them is of odd dimension k, say, and we can assume that $\dim V_1 = k$. Set $g_1 = g|_{V_1}$ so g_1 is indecomposable. Let $H = C_G(V_1^{\perp})$ be a subgroup consisting on all elements of G acting trivially on $V_1^{\perp}$. Then $H \cong U(k, q)$. Let $T_1 = C_{U(k,q)}(g_1)$ and $T = C_H(g)$. Then T is contained in $C_G(g)$. As $T|_{V_1} = T_1$, it follows that T is a star subgroup of G, as required.

7. Weil representations

LEMMA 7.1. *Let $T = T(k, t)$ be a star subgroup of $G = U(n, q)$ and $g \in T$. Let $m = n - k$, $l = k/t$ and let $B_1, \dots, B_{l-1}$ be the unipotent subgroups of T introduced in Lemma* 5.5. *Let χ be the Weil character of G. Then*

$$\chi(g) = \begin{cases} (-1)^n(-q)^m & \text{if } g \notin B, \\ (-1)^n(-q)^{m+it} & \text{if } g \in (B_i \setminus B_{i+1}),\ 1 < i < l, \\ q^n & \text{if } g = \mathrm{Id}. \end{cases}$$

Proof. By Theorem 5.1 we have

$$d(g) = \begin{cases} m & \text{if } g \notin B, \\ m + it & \text{if } g \in (B_i \setminus B_{i+1}),\ 1 < i < l-1, \\ n & \text{if } g = \mathrm{Id}. \end{cases}$$

So the lemma follows from Proposition 2.3

Proof of Theorem 2.4. Expand $(\chi|_T, 1_T)$ as follows:

$$(\chi|_T, 1_T) = \frac{1}{|T|}\Big(\chi(1) + \sum_{g \in T \setminus B} \chi(g) + \sum_{i=1}^{l-1} \sum_{g \in (B_i \setminus B_{i+1})} \chi(g)\Big).$$

By Proposition 2.3, $\chi(1) = q^n$ and $\chi(g) = (-1)^n(-q)^m = -q^m$ for $g \in T \setminus B$ as $m - n = k$ is odd. In addition, $|T| - |B| = (q^t + 1)|B| - |B| = q^t|B| = q^t q^{t(l-1)} = q^{tl}$. Hence

$$\sum_{g \in T \setminus B} \chi(g) = q^{tl}(-q^m) = -q^n.$$

Next compute $\sum_{g \in (B_i \setminus B_{i+1})} \chi(g)$. The number of summands here is $|B_i| - |B_{i+1}| = q^{t(l-i)} - q^{t(l-i-1)}$. Therefore, as t is odd,

$$\sum_{g \in (B_i \setminus B_{i+1})} \chi(g) = (-1)^n(q^{t(l-i)} - q^{t(l-i-1)})(-q)^{m+it} = -(q^{n-ti} - q^{n-t(i+1)})(-q)^{it} = (-1)^{i+1}(q^n - q^{n-t}),$$

and hence

$$\sum_{i=1}^{l-1} \sum_{g \in (B_i \setminus B_{i+1})} \chi(g) = (q^n - q^{n-t}) \sum_{i=1}^{l-1} (-1)^{i+1} = 0.$$

The last equality follows as $l - 1$ is even (since $k = tl$ is odd).

Proof of Theorem 2.6 Let $G = U(n, q)$ and let $g \in G$ be an element whose projection in H coincides with h. Then the projection of $C_G(g)$ in H is contained in $C_H(h)$. View θ as a representation of G trivial on $Z(G)$. Then it suffices to prove that $1_{C_G(g)}$ is not a constituent of $\theta_{C_G(g)}$. By Proposition 2.2(3), $C_G(g)$ contains a star subgroup. So the result follows from Theorem 2.4 as θ is a constituent of the Weil representation of G.

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