Pcf theory and cardinal invariants of the reals

Lajos Soukup

Alfréd Rényi Institute of Mathematics
Hungarian Academy of Sciences

A snapshot of logic
some recent results in mathematical logic

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- Use the spectrum of some cardinal invariant to code!

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Theorem (Fuchino, $\sim$)

There is a c.c.c poset P such that in V^P for each $X \subset \omega$ there is an ***ordinal*** α *such that* $X = \{n \in \omega : \aleph_{\alpha+n} \in \text{spectrum}(\mathfrak{a})\}$.

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Characterize spectrum $(\mathfrak{x})$ **for different cardinal invariants!**

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- **Problem**: Full characterization of $CF(Sym(\omega))$

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Theorem

Assume that $\mathcal{I} \subset \mathcal{P}(I)$ is a σ-complete ideal, $Y \in \mathcal{I}^+$, and $A \subset \mathrm{ADD}(\mathcal{I}, Y)$ is countable. Then pcf$(A) \subset \mathrm{ADD}(\mathcal{I}, Y)$.

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- For $a \in A$ let $\mathfrak{F}_a = \{F^a_\alpha : \alpha < a\} \subset \mathcal{I}$ increasing $\bigcup \mathfrak{F}_a = Y$.
- Let $\kappa \in pcf(A)$. Fix an **ultrafilter** $\mathcal{U}$ on A such that $\mathrm{cf}(\prod A/\mathcal{U}) = \kappa$
- Let $\{g_\alpha : \alpha < \kappa\} \subset \prod A$ be $\leq_{\mathcal{U}}$-increasing, $\leq_{\mathcal{U}}$-cofinal sequence.
- For $g \in \prod A$ let $U(g) = \{x \in I : \{a \in A : x \in F^a_{g(a)}\} \in \mathcal{U}\}$.
- $\langle U(g_\alpha) : \alpha < \kappa \rangle$ **witnesses that** $\kappa \in \mathrm{ADD}(\mathcal{I}, Y)$

  (1) $U(g) \in \mathcal{I}$ for each $g \in \prod A$

  (2) If $g_1 \leq_{\mathcal{I}} g_2$ then $U(g_1) \subset U(g_2)$.

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Does $A \subset \mathrm{ADD}(\mathcal{I})$ imply $pcf(A) \subset \mathrm{ADD}(\mathcal{I})$?
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Theorem

If $A \subset \mathrm{ADD}(\mathcal{B})$ is progressive and $|A| < \mathfrak{h}$, then $\mathrm{pcf}(A) \subset \mathrm{ADD}(\mathcal{B})$.

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If $\aleph_{\omega+1} < \max pcf(\{\aleph_n : 1 \leq n < \omega\})$ then there is an infinite $Y \subset \{\aleph_n : 1 \leq n < \omega\}$ such that $\mathrm{ADD}(\mathcal{I}) = Y \cup \{\aleph_{\omega+2}\}$ in some c.c.c generic extension V^P.

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$\mathcal{I}$ has the **Hechler property** iff given any σ-directed poset Q there is a c.c.c poset P such that
$V^P \models$ a cofinal subset $\{I_q : q \in Q\}$ of $\langle \mathcal{I}, \subset \rangle$ is **isomorphic to Q**.

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- Φ is a natural, cofinal, order preserving embedding.

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- Key observation:
  If $B = pcf(B)$ is a **progressive set of regular cardinals**, $\lambda \notin B$, then for each $\{f_i : i < \lambda\} \subset \prod B$ there is $g \in \prod B$ such that $|\{i : f_i \leq g\}| = \lambda$.

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Theorem (Farah)

If $\kappa > \omega_1$ is regular then $V^{\mathcal{H}_E} \models$ ***there is a κ-chain in*** *$\langle \omega^\omega, \leq^* \rangle$ iff one of the followings happens*

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Assume GCH in the ground model. Given any nonempty set K of uncountable regular cardinals there is a c.c.c poset $\mathcal{H}_{E_K}$ such that

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