

Ramsey Cardinals and the HNN Embedding Theorem

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The HNN Embedding Theorem

Theorem (Higman-Neumann-Neumann 1949)

Every countable group G can be embedded into a 2-generator group.

Sketch Proof.

- Let $(g_n \mid n \in \mathbb{N})$ be a sequence of generators with $g_0 = 1$.
- Let \mathbb{F} be the free group on $\{a, b\}$ and let $G * \mathbb{F}$ be the free product.
- Then $\{b^{-n}ab^n \mid n \in \mathbb{N}\}$ and $\{g_n a^{-n} b a^n \mid n \in \mathbb{N}\}$ freely generate free subgroups of $G * \mathbb{F}$.
- Hence we can construct the *HNN* extension

$$G \hookrightarrow K_G = \langle G * \mathbb{F}, t \mid t^{-1} b^{-n} a b^n t = g_n a^{-n} b a^n \rangle$$

- Since $g_n \in \langle a, b, t \rangle$ and $t^{-1} a t = b$, it follows that $K_G = \langle a, t \rangle$.



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A natural question

Observation

It is *reasonably clear* that the isomorphism type of the 2-generator group K_G usually depends upon both the generating set of G and the particular enumeration that is used.

Question

Does there exist a *more uniform* construction with the property that the isomorphism type of K_G only depends upon the isomorphism type of G ?

The answer ...

Notation

- \mathcal{G} denotes the Polish space of countably infinite groups.
- \mathcal{G}_{fg} denotes the Polish space of finitely generated groups.

Theorem

There does **not** exist a Borel map $G \mapsto K_G$ from \mathcal{G} to \mathcal{G}_{fg} such that for all $G, H \in \mathcal{G}$,

- $G \leftrightarrow K_G$; and
- if $G \cong H$, then $K_G \cong K_H$.

Definition

A function $\varphi : X \rightarrow Y$ is **Borel** if $\text{graph}(\varphi)$ is a Borel subset of $X \times Y$.

The Polish space of countably infinite groups

- Let \mathcal{G} be the set of groups with underlying set \mathbb{N} .
- We can identify each group

$$G \in \mathcal{G} \longleftrightarrow m_G \in 2^{\mathbb{N} \times \mathbb{N} \times \mathbb{N}}$$

with the graph of its multiplication operation.

- Then \mathcal{G} is a **G_δ subset** of the Cantor space $2^{\mathbb{N} \times \mathbb{N} \times \mathbb{N}}$; i.e. \mathcal{G} is a countable intersection of open subsets.
- Hence \mathcal{G} is a **Polish subspace** of the Cantor space $2^{\mathbb{N} \times \mathbb{N} \times \mathbb{N}}$.

The Polish space of f.g. groups

- A **marked group** (G, \bar{s}) consists of a f.g. group with a distinguished sequence $\bar{s} = (s_1, \dots, s_m)$ of generators.
- For each $m \geq 1$, let \mathcal{G}_m be the set of **isomorphism types** of marked groups $(G, (s_1, \dots, s_m))$ with m distinguished generators.
- Then there exists a canonical embedding $\mathcal{G}_m \hookrightarrow \mathcal{G}_{m+1}$ defined by

$$(G, (s_1, \dots, s_m)) \mapsto (G, (s_1, \dots, s_m, 1_G)).$$

- And $\mathcal{G}_{fg} = \bigcup \mathcal{G}_m$ is the **space of f.g. groups**.

The Polish space of f.g. groups

- Let $(G, \bar{s}) \in \mathcal{G}_m$ and let d_S be the corresponding word metric. For each $\ell \geq 1$, let

$$B_\ell(G, \bar{s}) = \{g \in G \mid d_S(g, 1_G) \leq \ell\}.$$

- The basic open neighborhoods of (G, \bar{s}) in \mathcal{G}_m are given by

$$U_{(G, \bar{s}), \ell} = \{(H, \bar{t}) \in \mathcal{G}_m \mid B_\ell(H, \bar{t}) \cong B_\ell(G, \bar{s})\}, \quad \ell \geq 1.$$

Example

For each $n \geq 1$, let $C_n = \langle g_n \rangle$ be cyclic of order n . Then:

$$\lim_{n \rightarrow \infty} (C_n, g_n) = (\mathbb{Z}, 1).$$

Why is the Theorem “obviously true”?

Definition

Let E, F be equivalence relations on the Polish spaces X, Y . Then the Borel map $\varphi : X \rightarrow Y$ is a **homomorphism** if

$$x E y \implies \varphi(x) F \varphi(y).$$

The Non-Embedding Theorem

If $\varphi : \langle \mathcal{G}, \cong_{\mathcal{G}} \rangle \rightarrow \langle \mathcal{G}_{fg}, \cong_{\mathcal{G}_{fg}} \rangle$ is **any** Borel homomorphism, then there exists a group $G \in \mathcal{G}$ such that $G \not\mapsto \varphi(G)$.

Heuristic Reason

Otherwise, since $\cong_{\mathcal{G}}$ is **much more complex** than $\cong_{\mathcal{G}_{fg}}$, the Borel homomorphism φ must have a “large kernel” and hence “too many” groups $G \in \mathcal{G}$ will be mapped to a fixed $K \in \mathcal{G}_{fg}$.

But Greg Cherlin isn't satisfied ...

Main Theorem (LC)

- Suppose that $G \mapsto K_G$ is *any* Borel map from \mathcal{G} to \mathcal{G}_{fg} such that $G \hookrightarrow K_G$ for all $G \in \mathcal{G}$.
- Then there exists an uncountable Borel family $\mathcal{F} \subseteq \mathcal{G}$ of pairwise isomorphic groups such that the groups $\{K_G \mid G \in \mathcal{F}\}$ are pairwise incomparable with respect to relative constructibility; i.e., if $G \neq H \in \mathcal{F}$, then $K_G \notin L[K_H]$ and $K_H \notin L[K_G]$.

Remarks

- (LC): There exists a Ramsey cardinal κ .
- In ZFC, we can find an uncountable Borel family \mathcal{F} such that the groups $\{K_G \mid G \in \mathcal{F}\}$ are pairwise incomparable with respect to embeddability.

Countable Quasi-orders

Definition

The relation \preceq on the Polish space X is a **countable quasi-order** if:

- (a) \preceq is reflexive and transitive.
- (b) For all $x \in X$, the set $\{y \in X \mid y \preceq x\}$ is countable.

Some countable Borel quasi-orders

- The embeddability relation on \mathcal{G}_{fg} .
- The Turing reducibility relation \leq_T on $2^{\mathbb{N}}$.

A countable Σ_2^1 quasi-order (LC)

The relative constructibility relation \leq_c on $2^{\mathbb{N}}$ defined by

$$x \leq_c y \iff x \in L[y].$$

Towards a proof of the Main Theorem ...

Question

What is known about the kernels of homomorphisms from *complete analytic* equivalence relations to *countable Borel* equivalence relations?

Answer (Kechris)

Not a lot!

Definition

- $\text{Inj}(\mathbb{N}, 2^{\mathbb{N}})$ is the Polish space of all *injective* maps $z : \mathbb{N} \rightarrow 2^{\mathbb{N}}$.
- E_{cntble} is the Borel equivalence relation on $\text{Inj}(\mathbb{N}, 2^{\mathbb{N}})$ defined by

$$z E_{\text{cntble}} z' \iff \{z(n) \mid n \in \mathbb{N}\} = \{z'(n) \mid n \in \mathbb{N}\}.$$

Main Lemma

Suppose that X is a Polish space and that $\theta : \text{Inj}(\mathbb{N}, 2^{\mathbb{N}}) \rightarrow X$ is *any* Borel map. Then at least one of the following must hold:

- (a) There exists $x \in X$ such that for all $r \in 2^{\mathbb{N}}$, there exists $z \in \text{Inj}(\mathbb{N}, 2^{\mathbb{N}})$ with $r \in \text{range}(z)$ such that $\theta(z) = x$.
- (b) For each countable Borel quasi-order \preceq on X , there exists a perfect subset $P \subseteq \text{Inj}(\mathbb{N}, 2^{\mathbb{N}})$ such that
 - (i) $y E_{\text{cntble}} z$ for all $y, z \in P$; and
 - (ii) $\theta(y), \theta(z)$ are incomparable with respect to \preceq for all $y \neq z \in P$.

Moreover, if (LC) holds, then the conclusion also holds with respect to the quasi-order \leq_c of relative constructibility.

The Proof of the Main Theorem

- Suppose that $\varphi : \mathcal{G} \rightarrow \mathcal{G}_{fg}$ is a Borel map such that $G \hookrightarrow \varphi(G)$ for all $G \in \mathcal{G}$.
- Let $\{H_r \mid r \in 2^{\mathbb{N}}\} \subseteq \mathcal{G}$ be a Borel family of pairwise nonisomorphic 2-generator groups. (B. H. Neumann 1937)
- Let $\psi : \text{Inj}(\mathbb{N}, 2^{\mathbb{N}}) \rightarrow \mathcal{G}$ be the injective Borel map defined by

$$\psi(z) = H_{z(0)} \times H_{z(1)} \times \cdots \times H_{z(n)} \times \cdots$$

and consider $\theta = \varphi \circ \psi : \text{Inj}(\mathbb{N}, 2^{\mathbb{N}}) \rightarrow \mathcal{G}_{fg}$.

- First suppose that there exists a group $G \in \mathcal{G}_{fg}$ such that for all $r \in 2^{\mathbb{N}}$, there exists $z \in \text{Inj}(\mathbb{N}, 2^{\mathbb{N}})$ such that $r \in \text{range}(z)$ and $\theta(z) = G$.
- Then H_r embeds into G for all $r \in 2^{\mathbb{N}}$, which is impossible since G has only countably many 2-generator subgroups!

The Proof of the Main Theorem

- Let \preceq be either the embeddability relation or the relative constructibility relation on \mathcal{G}_{fg} .
- Then there exists a perfect subset $P \subseteq \text{Inj}(\mathbb{N}, 2^{\mathbb{N}})$ such that
 - (i) $y E_{cntble} z$ for all $y, z \in P$; and
 - (ii) $\theta(y), \theta(z)$ are incomparable with respect to \preceq for all $y \neq z \in P$.
- Hence $\mathcal{F} = \psi(P) \subseteq \mathcal{G}$ is an uncountable Borel family of pairwise isomorphic groups such that the groups $\{\varphi(G) \mid G \in \mathcal{F}\}$ are pairwise incomparable with respect to \preceq .

Notation

From now on, we work within a fixed set-theoretic universe V .

Definition

- The relation R on the Polish space X is Σ_n^1 if $R(\bar{v})$ has the form

$$(\exists x_1 \in X_1)(\forall x_2 \in X_2) \cdots B(x_1, x_2, \dots, \bar{v}),$$

where X_1, \dots, X_n are Polish spaces and $B(\bar{x}, \bar{v})$ is a Borel relation.

- In this case, $R^{V^{\mathbb{P}}}$ denotes the relation obtained by applying the definition of R within the generic extension $V^{\mathbb{P}}$.
- R is **absolute** for $V^{\mathbb{P}}$ iff $R^{V^{\mathbb{P}}} \cap V = R$.

Shoenfield Absoluteness

Theorem (Shoenfield)

If $R \in V$ is a Σ_2^1 relation, then R is absolute for **every** generic extension $V^{\mathbb{P}}$.

An Application

If \preceq is a countable Borel quasi-order on the Polish space X , then $\preceq^{V^{\mathbb{P}}}$ is a countable Borel quasi-order on $X^{V^{\mathbb{P}}}$.

Proof.

Let $\text{Perf}(X)$ be the Polish space of nonempty perfect subsets of X . Then \preceq is countable if and only if

$$(\forall x \in X) (\forall P \in \text{Perf}(X)) (\exists y \in X) [y \in P \wedge y \not\preceq x].$$



Martin-Solovay Absoluteness

Theorem (Martin-Solovay)

Suppose that κ is a Ramsey cardinal. If $R \in V$ is a Σ_3^1 relation and $|\mathbb{P}| < \kappa$, then R is absolute for $V^{\mathbb{P}}$.

An Application (LC)

\leq_c is a countable Σ_2^1 quasi-order on $2^{\mathbb{N}}$.

Proof.

If \mathbb{P} is the poset of finite functions $p : \omega \rightarrow \omega_1$, then for all $x \in 2^{\mathbb{N}} \cap V$,

$$V^{\mathbb{P}} \models (\exists f \in (2^{\mathbb{N}})^{\mathbb{N}})(\forall z \in 2^{\mathbb{N}})[z \in L[x] \implies (\exists n) f(n) = z].$$

By Martin-Solovay, this $\Sigma_3^1(x)$ statement also holds in V . □

Definition (Kanovei après Hjorth)

Let E be a Borel equivalence relation on X and let \mathbb{P} be a forcing notion. Then a \mathbb{P} -name τ is a **virtual E -class** if:

- $\Vdash_{\mathbb{P}} \tau \in X^{V^{\mathbb{P}}}$
- $\Vdash_{\mathbb{P} \times \mathbb{P}} \tau_{\text{left}} E^{V^{\mathbb{P} \times \mathbb{P}}} \tau_{\text{right}}$

Here $\tau_{\text{left}}, \tau_{\text{right}}$ are the $(\mathbb{P} \times \mathbb{P})$ -names such that if $G \times H$ is $(\mathbb{P} \times \mathbb{P})$ -generic, then $\tau_{\text{left}}[G \times H] = \tau[G]$ and $\tau_{\text{right}}[G \times H] = \tau[H]$.

Example

- Let $E = E_{cntble}$ and let \mathbb{P} consist of all finite injective partial functions $p : \mathbb{N} \rightarrow 2^{\mathbb{N}}$.
- If G is \mathbb{P} -generic, then $g = \bigcup G$ is a bijection between \mathbb{N} and $2^{\mathbb{N}} \cap V$.
- Hence if τ is the canonical \mathbb{P} -name such that $\tau[G] = g$, then τ is a virtual E_{cntble} -class.

Main Lemma

Suppose that X is a Polish space and that $\theta : \text{Inj}(\mathbb{N}, 2^{\mathbb{N}}) \rightarrow X$ is any Borel map. Then at least one of the following must hold:

- (a) There exists $x \in X$ such that for all $r \in 2^{\mathbb{N}}$, there exists $z \in \text{Inj}(\mathbb{N}, 2^{\mathbb{N}})$ with $r \in \text{range}(z)$ such that $\theta(z) = x$.
- (b) For each countable Borel quasi-order \preceq on X , there exists a perfect subset $P \subseteq \text{Inj}(\mathbb{N}, 2^{\mathbb{N}})$ such that
 - (i) $y E_{\text{cntble}} z$ for all $y, z \in P$; and
 - (ii) $\theta(y), \theta(z)$ are incomparable with respect to \preceq for all $y \neq z \in P$.

Moreover, if (LC) holds, then the conclusion also holds with respect to the quasi-order \leq_c of relative constructibility.

Towards a proof of the Main Lemma ...

- Let $\theta : \text{Inj}(\mathbb{N}, 2^{\mathbb{N}}) \rightarrow X$ be any Borel map.
- Let \preceq be either a countable Borel quasi-order on X or else the relative constructibility relation \leq_c .

Notation

- $x \perp y \iff x, y$ are \preceq -incomparable.
 - $x \parallel y \iff x, y$ are \preceq -comparable.
- Let \mathbb{P} consist of all finite injective partial functions $p : \mathbb{N} \rightarrow 2^{\mathbb{N}}$ and let τ be the corresponding virtual E_{cntble} -class.

The Fundamental Dichotomy

Are $\theta(\tau_{\text{left}}), \theta(\tau_{\text{right}})$ comparable with respect to $\preceq^{V^{\mathbb{P} \times \mathbb{P}}}$?

Case 1: $(\exists p_0 \in \mathbb{P}) \langle p_0, p_0 \rangle \Vdash \theta(\tau_{\text{left}}) \parallel \theta(\tau_{\text{right}})$.

Claim

There exists $p_1 \leq p_0$ such that $\langle p_1, p_1 \rangle \Vdash \theta(\tau_{\text{left}}) = \theta(\tau_{\text{right}})$.

Proof.

- Suppose not and let \mathbb{Q} collapse $\mathcal{P}(\mathbb{P} \times \mathbb{P})$ to a countable set.
- Working in $V^{\mathbb{Q}}$, there exists a perfect subset $P \subseteq \text{Inj}(\mathbb{N}, 2^{\mathbb{N}})$ such that $\theta(P)$ is an uncountable Borel set of pairwise \preceq -comparable elements.
- Let $Z \subseteq \theta(P)$ be a perfect subset.
- By Kuratowski-Ulam, both $A = \{(x, y) \in Z \times Z \mid x \preceq y\}$ and $B = \{(x, y) \in Z \times Z \mid y \preceq x\}$ are meager subsets of $Z \times Z$.
- Since $Z \times Z = A \cup B$, this contradicts the Baire Category Theorem.



Case 1: $(\exists p_0 \in \mathbb{P}) \langle p_0, p_0 \rangle \Vdash \theta(\tau_{\text{left}}) \parallel \theta(\tau_{\text{right}})$.

Working in V and assuming that $X = [0, 1]$, we can inductively define conditions

$$p_1 \geq p_2 \geq p_3 \geq \cdots \geq p_n \geq \cdots$$

and closed intervals $I_n \subseteq [0, 1]$ with rational endpoints

$$I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq \cdots$$

such that the following conditions hold:

- $|I_n| = 2^{-(n-1)}$
- $p_n \Vdash \theta(\tau) \in I_n$.

Still working in V , let

$$\bigcap_{n \geq 1} I_n = \{x\}.$$

Case 1: $(\exists p_0 \in \mathbb{P}) \langle p_0, p_0 \rangle \Vdash \theta(\tau_{\text{left}}) \parallel \theta(\tau_{\text{right}})$.

Claim

$$p_1 \Vdash \theta(\tau) = x.$$

Proof.

- Otherwise, there exists $q \leq p_1$ and $n \geq 1$ such that $q \Vdash \theta(\tau) \notin I_n$.
- But then $\langle q, p_n \rangle \leq \langle p_1, p_1 \rangle$ satisfies

$$\langle q, p_n \rangle \Vdash \theta(\tau_{\text{left}}) \notin I_n \text{ and } \theta(\tau_{\text{right}}) \in I_n,$$

which is a contradiction.



Case 1: $(\exists p_0 \in \mathbb{P}) \langle p_0, p_0 \rangle \Vdash \theta(\tau_{\text{left}}) \parallel \theta(\tau_{\text{right}})$.

- Let $G \subseteq \mathbb{P}$ be V -generic with $p_1 \in G$.
- Then $V[G] \models \theta(\tau[G]) = x$.
- Hence for each $r \in 2^{\mathbb{N}} \cap V$,

$$V[G] \models (\exists z \in \text{Inj}(\mathbb{N}, 2^{\mathbb{N}})) (\exists n \in \mathbb{N}) [z(n) = r \text{ and } \theta(z) = x].$$

- By Shoenfield Absoluteness, this Σ_1^1 property of the reals $r, x \in 2^{\mathbb{N}} \cap V$ must also hold in V .
- Thus, in V , for all $r \in 2^{\mathbb{N}}$, there exists $z \in \text{Inj}(\mathbb{N}, 2^{\mathbb{N}})$ with $r \in \text{range}(z)$ such that $\theta(z) = x$.

Case 2: $(\forall p \in \mathbb{P}) \langle p, p \rangle \not\perp \theta(\tau_{\text{left}}) \parallel \theta(\tau_{\text{right}})$.

- Once again, let \mathbb{Q} collapse $\mathcal{P}(\mathbb{P} \times \mathbb{P})$ to a countable set.
- Then $V^{\mathbb{Q}}$ satisfies the following statement:

$$(\exists P \in \text{Perf}(\text{Inj}(\mathbb{N}, 2^{\mathbb{N}}))) (\forall x) (\forall y) \\ [(x, y \in P \wedge x \neq y) \implies (x E_{\text{cntble}} y \wedge \theta(x) \perp \theta(y))].$$

- Applying either Shoenfield or Martin-Solovay Absoluteness, this statement also holds in V .
- This completes the proof of the Main Lemma.

The word problem for finitely generated groups

Theorem (Folklore)

For each subset $A \subseteq \mathbb{N}$, there exists a finitely generated group G_A such that $\text{Word}(G_A) \equiv_T A$.

Theorem

- *Suppose that $A \mapsto G_A$ is a Borel map from $2^{\mathbb{N}}$ to \mathcal{G}_{fg} such that $\text{Word}(G_A) \equiv_T A$ for all $A \in 2^{\mathbb{N}}$.*
- *Then there exists a Turing degree \mathbf{d}_0 such that for all $\mathbf{d} \geq_T \mathbf{d}_0$, there exists an infinite subset $\{A_n \mid n \in \mathbb{N}\} \subseteq \mathbf{d}$ such that the groups $\{G_{A_n} \mid n \in \mathbb{N}\}$ are pairwise incomparable with respect to embeddability.*

Sketch Proof.

An easy consequence of Borel Determinacy. □