Snapshots of Logic

University of East Anglia

London

June 16, 2010

Duals Vanishing from Black Boxes

Oren KOLMAN

Some conjectures involving non-elementary algebraic or model-theoretic properties, in one form, are independent of ordinary set theory (ZFC), but when appropriately reformulated, are provable using combinatorial principles (black boxes) that are theorems of ZFC.

The conjectures may have simple formats:

There exists a structure A possessing a property φ.

There exist arbitrarily large very different structures possessing φ.

The property φ is non-elementary.

The constructions of first-order logic are potentially insufficient.

Residues of infinite cardinal exponentiation (e.g., the Generalized Continuum Hypothesis) remain active, inducing phenomena of set-theoretic independence.

Confirmations of these conjectures frequently appears to rest on the validity of strong prediction principles.

Internal forcing axioms (Martin's Axiom, Proper Forcing Axiom, ...) yield refutations.

Black box principles deliver theorems of ZFC.

Large cardinal axioms may limit the size of conjecture witnesses.

However, although these newer combinatorial principles are provable in ordinary set theory, the structures, whose constructions they enable, are not necessarily absolute (in the set-theoretic sense).

This talk touches on some of these issues.

THE STATIONARY BLACK BOX

Suppose that $\mu^{\aleph_0} = \mu$ *and* $\chi > \lambda = \mu^+$. *Let* E *be a stationary subset of* λ *consisting of ordinals of cofinality* ω. *Let* $\mathbf{N}$ *be an expansion in a countable language of the structure* $(H(\chi), \in, \lambda, <,)$.

There exists a family of countable sets $\{(M_i, X_i) : i \in I\}$ *such that the following hold:*

(1) $M_i \leq \mathbf{N}$ *and* $X_i \subseteq \lambda$.

(2) *Let* $\delta(i) = \sup(M_i \cap \lambda)$. *If* $\delta(i) = \delta(j)$ *and* $i \neq j$, *then* $(M_i, X_i) \cong (M_j, X_j)$ *and* $M_i \cap M_j \cap \lambda$ *is a proper initial segment of* $M_i \cap \lambda$. *For each* $\delta < \lambda$, *there are at most* μ *many* i *with* $\delta(i) = \delta$.

(3) *For all* $X \subseteq \lambda$, *the set*

$$\{\delta \in E : \exists\, i \text{ with } \delta(i) = \delta \text{ and } (M_i, X_i) \equiv_{M_i \cap \lambda} (\mathbf{N}, X)\}$$

is a stationary subset of λ.

(4) *A technical uniformity condition.*

HISTORY

[Shelah 1974] – combinatorial principle to prove existence of large rigid systems in every infinite cardinality

[Shelah 1984] – separation of combinatorics from algebra

[Corner Göbel 1985] – unified treatment of Shelah's combinatorial principle under name of black box

[Eklof Mekler 1990; 2002] – proof of Shelah's stationary black box

[Göbel Trlifaj 2006] – systematic exposition of several black box principles: general black box, strong black box; comprehensive treatment of applications in endomorphism algebras

[Shelah 2007] – n+1 dimensional black boxes

[Göbel Shelah 2009] – new simplified black box

[Shelah 898] - BB(λ, μ, θ, κ)

THE TRIVIAL DUAL CONJECTURE

Let G be an abelian group.

The *dual* of G, denoted G^*, is $\mathrm{Hom}(G, \mathbb{Z})$ – the group of homomorphisms from G into the additive infinite cyclic group $\mathbb{Z}$ of integers.

An abelian group A is *free* if A is isomorphic to a direct sum of infinite cyclic groups:

$$A \cong \oplus_{\alpha < \lambda} \mathbb{Z} .$$

Let μ be an infinite cardinal.
An abelian group G is μ-*free* if every subgroup of cardinality less than μ is free.

THE TRIVIAL DUAL CONJECTURE FOR μ : *TDU*$_\mu$

There exists a μ-*free abelian group G with* $G^* = 0$.

THE TRIVIAL DUAL CONJECTURE FOR μ AT λ : $TDU_{\mu, \lambda}$

There exists a μ*-free abelian group G of cardinality* λ *with* $G^* = 0$.

Remarks

$TDU_\kappa \Rightarrow TDU_\mu$ for $\mu \leq \kappa$.

$TDU_{\mu, \kappa} \Rightarrow TDU_{\mu, \lambda}$ for $\kappa \leq \lambda$.

A candidate for TDU_μ can never be free:
$\mathrm{Hom}(\oplus_{\alpha < \lambda} \mathbb{Z}, \mathbb{Z}) \cong \Pi_{\alpha < \lambda} \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}^\lambda \neq 0$.

If μ-free implies free, then TDU_μ is false.

If μ-free implies λ^+-free, then $TDU_{\mu, \lambda}$ is false.

Corollary

1. *The conjecture* $TDU_{\aleph_\omega,\ \aleph_\omega}$ *is false.*
2. *The conjecture* $TDU_{\mu,\ \mu}$ *is false if* μ *is a singular cardinal, or* μ *is a weakly compact cardinal.*
3. *Suppose* μ *is an* $L_{\omega_1,\ \omega}$*-compact cardinal. Then* TDU_μ *is false.*

Proof. In each case μ-freeness implies too much freeness (a theorem of Hill, Shelah's Singular Compactness theorem).

A cardinal μ is $L_{\omega_1,\ \omega}$*-compact* if for every set I every μ-complete filter on I is contained in an $\aleph_1$-complete ultrafilter on I.

How about the simplest cases: $TDU_{\aleph_n,\ \aleph_n}$?

$TDU_{\aleph_0,\ \aleph_0}$ is easy:
The additive group $\mathbb{Q}$ of rationals is $\aleph_0$-free [$\mathbb{Q}$ is *torsion-free*: it has no non-zero elements of finite order], $\mathbb{Q}^* = 0$, and $\mathbb{Q}$ has cardinality $\aleph_0$.

Theorem [Eda 1989; Corner, Göbel]
The conjecture $TDU_{\aleph_1, \aleph_1}$ *is true, i.e., there exists an* $\aleph_1$*-free abelian group G of cardinality* $\aleph_1$ *such that* $G^* = 0$.

However, $TDU_{\aleph_2, \aleph_2}$ falls victim to set-theoretic ambush: it is consistent with ZFC that all $\aleph_2$-free groups of cardinality less than $2^{\aleph_0}$ are separable.

A group *A* is *separable* if every finite subset of *A* is contained in a free direct summand of *A*.

Equivalently, any pure cyclic subgroup of *A* is a free direct summand of *A*.

A subgroup *H* is *pure* in *G* if for every *n*, if every equation $nx = h \in H$ having a solution in G has a solution in *H*.

No TDU_{μ} candidate can ever be separable, since any free direct summands would make its dual non-trivial.

Theorem [Göbel Shelah 1994]
Assume Martin's Axiom. If G is $\aleph_2$-free and has cardinality less than $2^{\aleph_0}$, then G is separable.

Proof. Enough to show any pure cyclic subgroup $\langle h \rangle$ is a free direct summand of G. Let $\eta(h) = 1$. We shall use MA to extend η to $\Phi \in \mathrm{Hom}(G, \mathbb{Z})$.

Let $\boldsymbol{P}$ be the partial order $\{\varphi : \varphi \in \mathrm{Hom}(A_\varphi, \mathbb{Z})$, $\varphi(h) = 1$, A_φ is a pure, finitely generated subgroup of $G\}$ under extension of maps.

The poset $\boldsymbol{P}$ satisfies the hypotheses of MA.
Let F be a generic filter intersecting the dense set $D_g = \{\varphi \in \boldsymbol{P} : g \in A_\varphi\}$, for every $g \in G$. Then $\Phi = \cup F \in \mathrm{Hom}(G, \mathbb{Z})$, and $G / \ker(\Phi) \cong \mathbb{Z}$.

Corollary
Martin's Axiom implies $TDU_{\aleph_2, \lambda}$ is false for every $\lambda < 2^{\aleph_0}$.

Consider two possibilities for $TDU_{\mu,\lambda}$:

- Strengthen the set theory.
- Weaken the bound on λ.

THE STRONG TRIVIAL DUAL CONJECTURE FOR μ AT λ :

Strong $TDU_{\mu,\lambda}$
There exists a strongly μ*-free abelian group* G *of cardinality* λ *with* $G^* = 0$.

An abelian group G is *strongly* μ*-free* if every subset of G of cardinality less than μ is contained in a free subgroup H such that G/H is μ-free.

Equivalent, G has a μ-filtration $\langle G_\alpha : \alpha < \mu \rangle$ such that every G_α is free, and if α is a successor ordinal and $\alpha < \beta < \mu$, then G_β / G_α is free too.

A μ-*filtration* of G is an ascending continuous chain $\langle G_\alpha : \alpha < \mu \rangle$ such that $G = \cup_{\alpha < \mu} G_\alpha$, and each subgroup G_α has cardinality less than μ.

Theorem [Göbel Shelah 1996; rev. 2003]

Assume $\lambda = 2^{\mu} = \mu^{+}$. *If there exists a strongly* μ*-free abelian group of regular cardinality* μ *with trivial dual, then there exists a strongly* λ*-free abelian group of cardinality* λ *with trivial dual. That is:*

Strong $TDU_{\mu,\,\mu} \Rightarrow$ *Strong* $TDU_{\mu+,\,\mu+}$

Corollary [Göbel Shelah 1996; rev. 2003]

Assume $2^{\aleph_k} = \aleph_{k+1}$ *for* $k = 0, 1, \ldots n - 1$.

Then there exists a (strongly) $\aleph_n$*-free abelian group of cardinality* $\aleph_n$ *with trivial dual.*

It follows that for each $n > 1$, $TDU_{\aleph_n,\,\aleph_n}$ is independent of ZFC.

Next, instead of strengthening the set theory, let us consider weakening the bound on λ in $TDU\aleph_n, \lambda$.

Recall the beth function: $\beth_0 = \aleph_0$ and $\beth_{n+1} = 2^{\beth_n}$.

Theorem [Shelah 2007; Göbel Shelah 2009]
For every $n < \omega$, $TDU\aleph_n, \beth_n$ holds.

The proof appeals to a new simplified black box.

Under GCH, $TDU\aleph_n, \beth_n$ is exactly $TDU\aleph_n, \aleph_n$.

Thus the assertion $TDU\aleph_1, \aleph_1$ is ambiguous: generalised as $TDU\aleph_n, \aleph_n$, it is independent of ZFC, while as $TDU\aleph_n, \beth_n$, it is now provable in ordinary set theory.

The conjecture $TDU\aleph_\omega$ remains open.

However, there is a recent advance in [Shelah 898], where another black box principle, BB(λ, μ, θ, κ), is introduced and proved using hypotheses of pcf theory.

Instances of BB(λ, μ, θ, κ) are sufficient to establish $TDU\aleph_\omega$.

It is also shown that the failure of $TDU\aleph_\omega$ implies restrictions on cardinal arithmetic, and the consistency of the failure is large.

<u>The Trivial Endormorphism Conjecture</u>

The *endomorphism ring* End(G) of G is the ring of endomorphisms of G: End(G) = Hom(G, G).

An abelian group G is *endo-rigid* if End(G) ≅ $\mathbb{Z}$.

If G is endo-rigid, torsion-free and uncountable, then Hom(G, $\mathbb{Z}$) = 0.

The Trivial Endormorphism Conjecture for μ :
TED$_\mu$
There exists a μ*-free abelian group G with trivial endomorphism ring, i.e.,* End(G) = $\mathbb{Z}$.

The Trivial Endormorphism Conjecture for μ at λ:
TED$_{\mu,\lambda}$
There exists a μ*-free abelian group G of cardinality* λ *with trivial endomorphism ring, i.e.,* End(G) = $\mathbb{Z}$.

The pattern of results parallels that of the *TDU*$_\mu$ conjectures.

Theorem [Dugas Göbel 1982]
Suppose that $V = L$ *(*$\diamondsuit_\mu$ *is sufficient) and* μ *is uncountable regular not weakly compact. Then there exists a strongly* μ*-free abelian group G of cardinality* μ *such that* $End(G) \cong \mathbb{Z}$.

Theorem [Shelah 1984; Corner, Göbel 1985]
Suppose that $\mu^{\aleph_0} = \mu$. *Then there exists an* $\aleph_1$*-free abelian group G of cardinality* $\lambda = \mu^+$ *such that* $End(G) \cong \mathbb{Z}$.

The case $TED\aleph_n, \beth_n$ has been studied by Göbel, Herden and Shelah (not yet published).

However, the conjecture $TED\aleph_\omega$ appears open.

Absolute Notions: Indecomposability, Rigid Systems, *E*-rings

An abelian group G is *indecomposable* if *G* is not expressible as a non-trivial direct sum.

[Fuchs 1958] Problem 21
Do indecomposable groups of power μ *exist for every cardinal* μ*? If they do, how many of them?*

A *rigid system* is a collection $\mathscr{F}$ of torsion-free abelian groups such that for distinct $G, H \in \mathscr{F}$:

- Hom(G, G) is isomorphic to a subgroup of $\mathbb{Q}$
- Hom(G, H) = 0.

Every group in a rigid system is indecomposable, since non-trivial direct summands give endomorphisms (viz., projections) that are not multiplication by a rational.

Theorem [Fuchs 1973 (up to first strongly inaccessible); Fuchs 1974 (up to first measurable); Shelah 1984 (the whole lot)]
For every cardinal κ*, there exists a rigid system of* 2^κ *abelian groups, each of cardinality* κ.

[Nadel 1994]
Does there exist a proper class of pairwise absolutely non-isomorphic abelian groups?

Two abelian groups A and B are *absolutely non-isomorphic* if they do not become isomorphic in any generic extesion of the universe.

An abelian group G is *absolutely indecomposable* if G is indecomposable in every generic extension of the universe.

An *absolutely rigid system* is a collection $\mathscr{F}$ of torsion-free abelian groups such that for distinct $G, H \in \mathscr{F}$:

- Hom(G, G) is isomorphic to a subgroup of $\mathbb{Q}$ in every generic extension
- Hom(G, H) = 0 in every generic extension.

Example

If $\mathscr{F}$ is a rigid family of $\aleph_1$-free groups, each of cardinality $\aleph_1$, then $\mathscr{F}$ is not absolutely rigid, since by any forcing that collapses $\aleph_1$, the groups become countable and hence free (by Scott's theorem on countable $L_{\infty\,\omega}$-equivalent structures).

Theorem [Nadel 1994]

For every κ *less than the first strongly inaccessible cardinal, there exists an absolutely rigid system of* 2^κ *abelian groups, each of cardinality* κ.

Why the first strongly inaccessible cardinal?

Nadel observed that the proofs of [Fuchs 1974] and [Shelah 1984] were not “absolute”: the former used the concept of slenderness and infinite products, the latter rested on the stationary black box. Both slenderness and stationarity are non-absolute.

Nevertheless, there could be “absolute proofs”.

However, there is a precise large cardinal bound, up to which absolutely indecomposable groups and absolutely rigid systems exist and above which they cease to exist.

The first ω-*Erdős* cardinal, denoted $\kappa(\omega)$, is the least cardinal κ such that $\kappa \to (\omega)^{<\omega}$, i.e., for every function f from the finite subsets of κ to 2 there exists an infinite subset $X \subseteq \kappa$ and a function $g : \omega \to 2$ such that $f(Y) = g(|Y|)$ for all finite subsets Y of X.

If $\kappa(\omega)$ exists, then $\kappa(\omega)$ is strongly inaccessible and remains the first ω-*Erdős* cardinal in L.

Above $\kappa(\omega)$, there is a Vopenka effect:

Theorem [Eklof Shelah 1999]
Suppose $\lambda \geq \kappa(\omega)$.

1. *If* $\{G_\alpha : \alpha < \lambda\}$ *is a family of non-trivial abelian groups, then there exist distinct ordinals* α *and* β *such that in some generic extension of the universe, there is an injective homomorphism* $\eta : G_\alpha \rightarrow G_\beta$.

2. *If G is an abelian group of cardinality* λ, *then there exists a generic extension of the universe such that* G *has an endomorphism which is not multiplication by an integer.*

Below $\kappa(\omega)$, positive assertions about absoluteness hold:

Theorem [Göbel Shelah 2007; Fuchs Göbel 2008] *Suppose* $\lambda < \kappa(\omega)$.

1. *There exists an absolutely indecomposable group of cardinality* λ.

2. *There exists an absolutely* $\mathbb{Z}$*-rigid family of torsion-free abelian groups* G_U *(*$U \subseteq \lambda$*), each of cardinality* λ.

Below $\kappa(\omega)$, positive assertions about absoluteness hold:

Theorem [Göbel Shelah 2007; Fuchs Göbel 2008] *Suppose* $\lambda < \kappa(\omega)$.

1. *There exists an absolutely indecomposable group of cardinality* λ.

2. *There exists an absolutely* $\mathbb{Z}$*-rigid family of torsion-free abelian groups* G_U *(*$U \subseteq \lambda$*), each of cardinality* λ.

The proofs proceed by encoding rigid systems of valuated trees (from [Shelah 1982]) into a family of groups.

A ring R with 1 is called an *E-ring* if $\mathrm{End}_{\mathbb{Z}}(R)$ is ring-isomorphic to R under the canonical homomorphism taking the value $\eta(1)$ for any $\eta \in \mathrm{End}_{\mathbb{Z}}(R)$.

An E-ring R is an *absolute E-ring* if it remains an E-ring in every generic extension.

Theorem [Göbel Herden Shelah 2011]
There exist E-rings in every cardinal λ.
If $\lambda < \kappa(\omega)$, *then the E-rings constructed are absolute.*

Conclusion

Some conjectures can be phrased so as to minimise set-theoretic independence.

Diamonds can be removed.

Duals can be made to vanish from black boxes.

However, absolute disappearance is rarer and may be bounded by the presence of large cardinals.