

Final Report on the Advanced EPSRC Fellowship, Mirna Džamonja

Background/Context Research of the PI is in set theory and its connections with other fields of mathematics. She has a very interdisciplinary approach and has published papers involving set theory and topology, measure algebra, Boolean algebras, theory of order, functional analysis and model theory. Set theory is a very strong area on the European or worldwide level. It is fundamental in the sense of underpinning the whole of classical mathematics, but also a deep subject in its own right. The proposal for the Fellowship contained a number of topics from many aspects of set theory, connected by the thread of universality and approximation families.

Executive Summary The objectives were to study approximation families and the universality and their connections with other subjects of mathematics, to develop further the theory of iterations of Prikry type forcing and to develop the strict order properties in model theory. All these objectives were met, and in addition, a number of other subjects that grew out of this research were studied. Most of the objectives are mentioned directly as papers in the publications summary below, with the exception of SOP_2 and SOP_1 properties which were in the meantime developed but not published by Shelah and Ustvyasev and combinatorial designs which the PI is working on as part of the preparation for the book mentioned below. The book will also contain a development of SOPs.

Key Advances and Supporting Methodology During the Fellowship the PI produced 21 articles, of which 15 are already published, accepted or submitted. She has a book in preparation. These publications are summarised below in the Publications Summary section. Among them is a solution to an open problem from 1977 in functional analysis which is now solved using the method of approximation families and a solution to a problem in measure theory from 1981 (in [12]). Other papers by the PI were published during the tenure of the Fellowship (9 research papers), but they are not listed here because a significant part of the research that led to them was done before the Fellowship.

Research Plan Review The PI proposed a very ambitious plan at the time of the proposal, but she had not envisioned what it really meant to have a support for five years when one is able to concentrate only on research. It meant that she was able to work much more than expected and to achieve more than stated on the research plan.

Research Impacts and Benefits to Society During the Fellowship the PI has become a very visible and internationally recognised mathematician. Her interdisciplinary approach to mathematics lead to invitations from various mathematical communities. She has developed a large number of international collaborations. In addition to collaborations with her coauthors she has coadvised a number of students from various countries and has been involved in career development of a number of younger mathematicians with whom she collaborated, suggested research projects to and wrote recommendation letters for. These people continue to do research on subjects close to the interests of the PI, for example Natasha Dobrinen and Katherine Thompson at the University of Vienna have become involved in the project of classifying Boolean algebras with finitely additive strictly positive measure (described below). In Britain the PI has had an important share in raising the profile of set theory and attracting young people to start research in the subject. At least three of her Ph.D. students have worked on subjects directly connected to the subject of the Fellowship. Another UEA student, Mark Wong advised by David Evans,

produced a Ph.D. thesis directly influenced by the subject of the Fellowship.

Expenditure The Fellowship came with a very generous Fellowship fund. The PI mostly used this to support a large number of international collaborations, as witnessed by a number of her collaborators, coauthors or students. Her research is very interdisciplinary and she often has to learn new subjects of mathematics in a very short time. This is much easier done in the context of international conferences, or visits by other mathematicians and to them. The fund supported visits by Banach, Halbeisen, Komjáth, Larson, Mitchell, Murtinova, Plebanek, and Vuksanovic and the visits of the PI to various centres of excellence. Without these collaborations the research presented here could not have been done to the same level of excellence, therefore the Fellowship contributed to this research in more than one way. Some of the Fellowship fund was spent to support visits of the PI's graduate students to international centres, since they were working on projects directly associated to the Fellowship. Finally, the PI bought some necessary hardware and software equipment. Detailed expenditure is provided by the University. The PI attended an EPSRC sponsored event of induction for members of the EPSRC College.

Outcomes: Fellow The achievements of the PI have been recognised by her home institution by a promotion to a Reader in 2002. She has been recognised internationally. For example, she was invited to a large number of conferences where she gave about 25 talks per year. Among these were regular invitations to the Oberwölfach and Luminy meetings in set theory, which are both by invitation only meetings with about 50 participants. She is also recognised outside of set theory, for example in September she is invited to a measure theory conference in the Banach Centre in Poland. She was also invited as a Logic Guest Professor at the University of Wisconsin-Madison in 2003, a Visiting Professor for one month at the University of Paris VII in 2004 and the Senior Guest Researcher at the University of Vienna in 2005. She serves in a number of professional bodies, for example she is the Secretary to the British Logic Colloquium since 2002, she was a member of the International Jury for Gödel Prizes in Logic in 2006, and she is a member of the Steering Committee for the European Set Theory Association EST. She is on the Editorial Board of the Sarajevo Journal of Mathematics and a Review Editor for the Bulletin for Symbolic Logic. She served on the Council of the Association for Symbolic Logic ASL 2003-2005, was the chair of the Programme Committee for the 2006 Annual Meeting of ASL and is a member of the Programme Committee for the 2007 European Meeting of ASL. The level of her professional recognition is directly connected with the outcomes of research she did while on the Fellowship. Her research attracted further funding as described on the accompanying form.

Support of the Employing Organisation The institution gave the PI a strong support by completely releasing her from undergraduate teaching duties and administration duties.

Dissemination Activities and Further Research The PI publishes her research in high quality international refereed journals. She has a web site <http://www.mth.uea.ac.uk/~h020/> where her professional activities are recorded, including a complete list of her publications (apart from those in preparation) with downloadable files. As mentioned above, the PI is a frequent speaker at highly visible conferences and she has made a number of longer visits to various centres of excellence where she communicated her research through seminars and personal communication.

Further research will include the project on classifying Boolean algebras that support a finitely additive strictly positive measure, described below. She is in the process of preparing a proposal to EPSRC to support a postdoctoral fellow to work on this subject. One of the ambitions of the PI is to study applications of such Boolean algebras in computer science. Of course, further research will also include finishing the papers in preparation and the book.

Public Awareness Activities The PI has actively engaged in education activities to promote mathematics and set theory. As an invitee of the British Council she gave talks to high school students in France in May 2007. In November 2007 she is invited to present a talk to the Institute of Education in Paris to the audience of high school teachers who teach in English. The PI has also given talks to the Mathematical Association in Norfolk. She has organised professional conferences, such as the series Set Theory and Its Neighbours (coorganised with Charles Morgan and Philip Welch) or a special session on Mathematical Logic at BMC 2007 (coorganised with Arnold Beckmann). Recently, she obtained a Royal Society support through their Scientist MP Pairing Scheme and she will be paired with Dr Gibson, a Labour MP from Norwich, with whom she will exchange views of science funding policies.

Scheme Effectiveness The Fellowship met all the objectives of the scheme.

Summary of the Publications An important theme has been the notion of universality. Here one is given a partially ordered class (\mathcal{K}, \leq) and one looks for subclasses such that all members of \mathcal{K} are \leq than some member of the subclass. Such a subclass is called a universal family and the smallest cardinality of a universal family is called the universality number. If this number is 1 then we say that the class has a universal element. Many questions in mathematics are about universality. At the time of the proposal, the PI was a coauthor on a number of papers which dealt with universality and she proposed some concrete further considerations of this subject for the Fellowship. Publications [3], [5], [7], [10], [18], [Survey 1] and [Book1] deal with various aspects of universality, as did the Ph.D. theses by Katherine Thompson and by Firoz Shaikh.

In [3] we construct a family of 2^{\aleph_1} trees of size \aleph_1 and no uncountable branches that in a certain way codes all ω_1 -sequences of infinite subsets of ω . This allows us to conclude that in the presence of the club guessing between \aleph_1 and \aleph_0 these trees are pairwise very different. Then we can also conclude that the universality number of the ordered class of trees of size \aleph_1 with no uncountable branches under ‘metric-preserving’ reductions must be at least \mathfrak{c} . From the topological point of view these results show that under the same assumptions there are 2^{\aleph_1} pairwise nonisometrically embeddable first countable ω_1 -metric spaces with a strong non-ccc property, with their universality number under isometric embeddings is at least \mathfrak{c} . Without the non-ccc requirement a family of 2^{\aleph_1} pairwise nonisometrically embeddable first countable ω_1 -metric spaces exists in ZFC by an earlier result of Todorćević. The set-theoretic assumptions mentioned above are satisfied in many natural models of set theory. We use a similar method to discuss trees of size any regular $\kappa > \aleph_0$. Paper [14] deals with trees whose height and cardinality is a singular cardinal κ and which have no unbounded branches. We obtain surprising results, including that if $\text{cf}(\kappa) = \aleph_0$ then the universality number of such trees under the usual *reduction* (i.e. a strict order preserving map) is always κ^+ , just in ZFC! Such trees strongly resemble the ordinals. This is not at all the case if $\text{cf}(\kappa) > \aleph_0$. This paper is about combinatorics of singular cardinals, the investigation of which formed part of the

proposal. For $\kappa = \omega_1$ it is known that the universality number under reduction can be \aleph_2 and 2^{\aleph_1} as large as desired, but it is open if this universality number can be 1. In [18] preliminary results indicate that by oracle proper forcing we can obtain one universal tree and 2^{\aleph_1} large.

In [5] it is shown that the universality spectrum of well-founded posets is the same as the spectrum of the class of well-orders. A universality result for a restricted class of well-founded posets under rank and order preserving embeddings is obtained using a method by Kojman.

In [7] we prove that if $\mu^+ < \lambda = \text{cf}(\lambda) < \mu^{\aleph_0}$ for some regular μ then there is no family of less than μ^{\aleph_0} c -algebras of size λ which are jointly universal for c -algebras of size λ . On the other hand, it is consistent to have a cardinal $\lambda \geq \aleph_1$ as large as desired and satisfying $\lambda^{<\lambda} = \lambda$ and $2^{\lambda^+} > \lambda^{++}$, while there are λ^{++} c -algebras of size λ^+ that are jointly universal for c -algebras of size λ^+ . Consequently, by previous results of M. Bell, it is consistent that there is λ as in the last statement and λ^{++} uniform Eberlein compacta of weight λ^+ such that at least one among them maps onto any Eberlein compact of weight λ^+ . The only positive universality results for Eberlein compacta known previously required the relevant instance of *GCH* to hold. These results complete the answer to a question of Benyamini, Rudin and Wage from 1977 who asked if there always was a universal uniform Eberlein compact of a given weight. This result was unexpected from the model-theoretic point because it blends both the positive and the negative behaviours one would expect from the model-theoretic intuition. We solved the problem using the approximation families.

In [10] we showed that the universality number for the class of posets of size a regular cardinal λ omitting chains or antichains of size $\aleph_0 \leq \kappa \leq \lambda$ can be 2^λ for any reasonable size of 2^λ . We also obtained global results where we deal with a class of cardinals λ simultaneously.

In [Survey 1] we survey the use of club guessing and other pcf constructs in the context of showing that a given class does not have a universal element. In [Book 1] we plan to give an extensive and unified exposition of various techniques used in universality results, starting from Cantor's proof of the uniqueness of rationals, to the large cardinals and forcing techniques involved in showing universality results at the successor of a singular cardinal.

Much work during the Fellowship was done on connections between analysis, particularly functional analysis, and set theory. Paper [7] mentioned above is also in this group. In [4] we investigate various kinds of bases in infinite dimensional Banach spaces. We show that in a separable Banach space a Hamel basis cannot be analytic, whereas there are non-separable Hilbert spaces which have a discrete and closed Hamel basis. Further, we investigate the existence of certain complete minimal systems in l_{inf} as well as in separable Banach spaces. This work is continued in [13] where we show that an infinite-dimensional complete linear space X has a dense hereditarily Baire Hamel basis if $|X| \leq \mathfrak{c}^+$ and a dense non-meager Hamel basis if $|X| = \kappa^\omega = 2^\kappa$ for some cardinal κ .

Paper [2] is a comprehensive overview of the method of precalibres and their use in measure algebra, where they have been used to solve a number of interesting problems. Among the new results is an improvement on a specific case of a result of Shelah.

An exciting new long term project was started last year involving measure theory, Boolean algebras, topology and set theory. Maharam famously proved in 1947 that Boolean algebras supporting strongly positive countably additive

measure can be classified in a very simple list, each uniquely corresponding to a countable sequence of cardinals. She asked in 1981 if Boolean algebras supporting strongly positive *finitely* additive measure can be classified. Almost nothing was known about this question. In our paper [12] we started characterising such Boolean algebras where the measure is assumed additional properties. We solved a question of Talagrand from 1981 by showing that a certain characterisation could never be true. In [Survey 2] the PI was invited to write on the present state of the art in her field of science and she described this project.

There is a very interesting combinatorial problem by Fremlin, DU on his list, which may hold the key to the classification problem mentioned above. This problem attracted the attention of a large number of researchers, but it is still open. The PI has made some important advances on this problem in her papers [1] and [17]. The problem can be understood as asking if certain colourings of pairs of countable ordinals always have an infinite homogeneous sets. The main case is to resolve this under $MA + \neg\omega_1$. In [1] we gave a characterisation of the $MA + \neg\omega_1$ in terms of combinatorial properties of the colouring. In [17] we showed that if the colouring is not required to be closed under subsets than there is a counterexample just in ZFC.

Colourings, partitions and large cardinals are a subject of [11] and [15]. In [11] κ is a cardinal with appropriate large cardinal properties and $\mathbb{Q}_\kappa = (Q, \leq_Q)$ is the strongly κ -dense linear order of size κ . We prove that for $m \geq 2$ there is $t_m^+ < \omega$ such that the set m -tuples from Q can be partitioned into classes $\langle C_i : i < t_m^+ \rangle$ such that for any colouring of \mathbb{Q}_κ in fewer than κ colours, there is a copy \mathbb{Q}^* of \mathbb{Q}_κ such that $[\mathbb{Q}^*]^m \cap C_i$ is monochromatic. It follows that for any such colouring of $[\mathbb{Q}_\kappa]^m$ there is a copy $Q' \subseteq Q$ of \mathbb{Q}_κ such that $[Q']^m$ has at most t_m^+ colours. However, there are colourings of \mathbb{Q}_κ such that if $Q' \subseteq Q$ is any copy of \mathbb{Q}_κ then $C_i \cap [Q'] \neq \emptyset$ for all $i < t_m^+$. We characterise t_m^+ combinatorially and obtain bounds on its value. In particular, $t_2^+ = 2$ and t_m^+ for $m > 2$ is larger than the m th tangent number. The condition on κ is the hypothesis for a result of Shelah on which our work relies. A model in which this condition holds for all m can be obtained by forcing from a model with a $\kappa^{+\omega}$ -strong cardinal, but it follows from earlier results of Hajnal and Komjáth that our result, and hence Shelah's theorem, is not implied by any large cardinal assumption. In [15] we obtain analogous results for random graphs.

Paper [6] is also on large cardinals. We show that it is consistent modulo the existence of a strongly unfoldable cardinal that $\diamond(Reg)$ fails at such a cardinal. This improves a previous result of Woodin who proved the analogue for weakly compact cardinals and Hauser who dealt with indescribable cardinals.

Back to combinatorics, in [Notes 1] we refute a conjecture of Moore on colourings of trees dealing with the CAT (Tree colouring axiom) of Todorćević.

Paper [8] gives a generalisation of a hierarchy of posets to uncountable cardinals, continuing the line of research started by Hausdorff in 1908, who characterised scattered linear orders by a hierarchy.

Paper [9] is our take to what is probably the most celebrated problem in set-theoretic topology now, which is the problem of Efimov about a dichotomy in compact spaces. Counterexamples under CH and stronger axioms are known, we use the principle \diamond to give an example of a compact space with no convergent sequences, which in addition has the stronger property that every nonatomic Radon measure on it is uniformly regular. This example refutes a conjecture of Mercourakis from 1996.

Conclusion The PI's research during the Fellowship has been flourishing and has involved a large number of subjects. She has met and went above her research plan. Her career has advanced considerably. She used every opportunity to network, disseminate and promote the subject, including visits to high schools and mathematical associations. The PI is very grateful to EPSRC for the opportunity to pursue her research through such a unique support.

PUBLICATIONS

- [1] Generalisations of ε -density (with G. Plebanek), *Acta Univ. Carolin. Math. Phys.*, vol. 44, no. 2, (2003), pg. 57-64.
- [2] Precalibre pairs of measure algebras (with G. Plebanek), *Topology Appl.*, vol. 144, no. 1-3, (2004), pg. 67-94.
- [3] A family of trees with no uncountable branches (with J. Väänänen), *Topology Proc.* vol. 28, no. 1 (2004), pg. 113-132.
- [4] On Bases in Banach spaces (with T. Bartoszyński, L. Halbeisen, E. Murtinova and A. Plichko), *Studia Math.*, vol. 170, (2005), pg. 147-171.
- [5] Universality results for well-founded posets (with K. Thompson), *Sarajevo J. Math.*, vol. 1, no 2. (2005), pg. 147-160.
- [6] Diamond(Reg) can fail at strongly unfoldable cardinals (with J. D. Hamkins), *Ann. Pure Appl. Logic*, vol. 144, no 1-3. (2006), pg. 83-95.
- [7] Universality of uniform Eberlein compacta, *Proc. Amer. Math. Soc.*, vol. 134, no. 8 (2006), pg. 2427-2435.
- [8] A Poset Hierarchy (with K. Thompson), *Cent. Eur. J. Math.*, vol. 4, no. 2, (2006), pg. 225-241.
- [9] On Efimov spaces and Radon measures (with G. Plebanek), *to appear in Topology Appl.*, DOI <http://dx.doi.org/10.1016/j.topol.2006.04.029>.
- [10] Global Comeplexity Results (with S.-D. Friedman and K. Thompson), *in Quaderni di Matematica: Set Theory: Recent Trends and Applications*, ed. by A. Andretta, vol. 17 (2007), pg. 25-47.
- [11] A partition theorem for a large dense order (with J. Larson and W. Mitchell), *to appear in Israel J. Math.*.
- [12] Strictly Positive Measures on Boolean Algebras (with G. Plebanek), *submitted to London Math. Soc.*.
- [13] Non-separable Banach spaces with non-meager Hamel basis (with T. Banach and L. Halbeisen), *submitted to Studia Math.*.
- [14] Trees of singular cardinality and height (with J. Väänänen), *in preparation*.
- [15] Partitions of large random graphs (with W. Mitchell and J. Larson), *in preparation, to be posted on the site as of 15/09/2007*.
- [16] Spils, *in preparation, an electronic note available on the web site*.
- [17] An almost counterexample, *in preparation*.
- [18] Universal trees at \aleph_1 (with K. Thompson), *in preparation*.

Survey Articles

- [1] Club guessing and the universal models, *Notre Dame J. Formal Logic*, vol. 46, no. 3, (2005), 283-300.
- [2] Measure Recognition Problem, *Phil. Trans. Royal Soc. A*, vol. 364, (2006), 3171-3182.

Books

- [1] The universality problem, *in preparation*.

Notes

- [1] An application of CAT (with J. Larson), *electronic note* May 2006.