

# A conjecture of Gaifman on relative categoricity

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Haim Gaifman in *Proceedings of the Tarski Symposium 1974* (paraphrased):

$L$  a first-order language,

$L(P)$  the result of adding a 1-ary relation symbol  $P$  to  $L$ .

$T$  a complete theory in  $L(P)$ , such that in any model  $A$  of  $T$  the reduct  $A \upharpoonright L$  to  $L$  has a substructure  $A^P$  whose elements are those satisfying  $P$  in  $A$ .

We write  $T^P$  for the theory of  $A^P$ .

We say  $T$  is *relatively categorical* if whenever  $A$  and  $C$  are models of  $T$ , and  $i : A^P \rightarrow C^P$  is an isomorphism, then  $i$  extends to an isomorphism from  $A$  to  $C$ .

General intuition: If  $T$  is relatively categorical then models of  $T$  are in some sense reducible to models of  $T^P$ .

**Reduction Property for relatively categorical theories**  
(a special case of Pillay and Shelah 1985):

*For every formula  $\psi(\bar{y})$  of  $L(P)$   
there is a formula  $(\psi(\bar{y}))^*$  of  $L$  such that  
for all models  $A$  of  $T$  and all  $\bar{b}$  in  $A^P$ ,*

$$A \models \psi(\bar{b}) \iff A^P \models (\psi(\bar{b}))^*.$$

**Gaifman's Conjecture:** If  $T$  is relatively categorical then for every model  $B$  of  $T^P$  there is a model  $A$  of  $T$  with  $B = A^P$ .

Gaifman proved this when  $L$  is countable and  $A$  is rigid over  $A^P$ .

In this case there is a  $\emptyset$ -definable surjection  $f : \text{dom}(A^P)^n \rightarrow \text{dom}(A)$ .

So by the Reduction Property,  $A$  is interpretable in  $A^P$ .

Then given  $B \models T^P$ ,  
we get  $A$  with  $B = A^P$  by interpreting in  $B$ .

This argument doesn't generalise to the non-rigid case. If it did, the map  $f$  would define a homomorphism

$$\sigma : \text{Aut}(A^P) \rightarrow \text{Aut}(P)$$

which splits the natural homomorphism

$$\nu : \text{Aut}(A) \rightarrow \text{Aut}(A^P)$$

(i.e.  $\sigma\nu = 1_{\text{Aut}(A^P)}$ ).

But there are relatively categorical  $A$  where  $\nu$  is non-split.

## Standard example of non-split $A$ :

$$A = \mathbb{Z}(p^2)^{(\omega)}, \quad A^p = pA$$

where  $p$  is any prime.

Evans, Hodges and Hodkinson 1991 has many more examples in abelian groups.

Gaifman's conjecture has a positive answer when  $T$  is countable  $\omega_1$ -categorical and  $P$  picks out a strongly minimal set.

This includes the Standard Example above.  
The  $\omega_1$ -categoricity provides extra structure (theory of covers).

Saharon Shelah, 'Classification over a predicate II', in *Around Classification Theory of Models*, Springer 1986, pp. 47–90.

Shelah and Leo Harrington both told me that Shelah proved Gaifman's conjecture in this paper.

But Gaifman's conjecture is not stated in the paper. The paper contains difficult techniques which might help prove Gaifman's conjecture in specific cases, but (as far as I know) the conjecture is still open.

Another approach: Get some familiarity with concrete cases.

Wilfrid Hodges, 'Relative categoricity in linear orderings', in *Logic and Algebra*, ed. Yi Zhang, AMS 2002, pp. 235–248. (Models of  $T$  are linear orderings with distinguished subordering.)

Wilfrid Hodges and Anatoly Yakovlev, 'Relative categoricity in abelian groups II', *Annals of Pure and Applied Logic* 158 (2009) 203–231. (Models of  $T$  are 'group pairs', i.e. abelian groups with distinguished subgroup.)

Gaifman's conjecture is confirmed in both cases.

The first is not news:  $A$  is rigid over  $A^P$ .

The second rests on the Kaplansky-Mackey proof of Ulm's theorem

(cf. Fuchs *Infinite Abelian Groups II* §77).

Details will appear in the Proceedings of the conference in honour of Şerban Basarab's 70th birthday, Constanta April 2010.

In the abelian group case it was natural to generalise to  $(\kappa, \lambda)$ -categoricity.

Namely  $T$  is  $(\kappa, \lambda)$ -categorical if

1.  $T$  has models  $A$  of cardinality  $\lambda$  with  $A^P$  of cardinality  $\kappa$ , and
2. whenever  $A, B$  are models of  $T$  with these cardinalities, then every isomorphism  $A^P \rightarrow B^P$  extends to an isomorphism  $A \rightarrow B$ .

Hodges and Yakovlev 2009 list the possible spectra of relative categoricity for abelian group pairs.

When  $\kappa$  is infinite, there are four:

- (1)  $T$  is  $(\kappa, \lambda)$ -categorical just when  $\omega \leq \kappa = \lambda$ .
- (2) ... just when  $\omega \leq \kappa < \lambda$  or  $\omega = \kappa = \lambda$ .
- (3) ... just when  $\omega = \kappa = \lambda$ .
- (4) ... just when  $\omega \leq \kappa < \lambda$ .

Case (1) is equivalent to relative categoricity.

It includes the Standard Example,

but also many examples that aren't even  $\omega$ -stable.

In each case (1)–(4), every model  $A$  has the form  $C \oplus^P D$  (direct product of  $L(P)$ -structures) where  $A^P \subseteq C$  and  $C$  is ‘tight’ over  $A^P$ .

Moreover  $C$  is a pushout over  $A^P$  of group pairs  $A_p$  ( $p$  prime) with  $A_p/A^P$  a  $p$ -group.

‘Tight’ can be defined several equivalent ways, e.g. (Villemaire 1990) that the Ulm-Kaplansky invariants of each  $A_p$  over  $A^P$  are all zero.

This allows us to construct each  $A_p$  over  $A^P$  as in Kaplansky-Mackey.

So the proof of Gaifman's conjecture reduces to the case where in each model  $A$  of  $T$ ,  $A/A^P$  is a bounded  $p$ -group with  $A$  tight over  $A^P$ .

We sketch a proof in this case, as far as possible in purely model-theoretic terms — in hopes this will point to a general argument.

**Fact** (deduced from Kaplansky-Mackey).

There is a set  $\Delta$  of formulas  $\delta(\bar{x}, \bar{y})$  of  $L(P)$  such that in every model  $A$  of  $T$ ,

1. for each tuple  $\bar{a}$  in  $A$  there are  $\delta \in \Delta$  and a tuple  $\bar{b}$  in  $A^P$  such that  $A \models \delta(\bar{a}, \bar{b})$  (call  $\bar{b}$  a *support* of  $\bar{a}$ );
2. for any  $\delta \in \Delta$ , any tuples  $\bar{a}, \bar{a}'$  in  $A$  and any tuple  $\bar{b}$  in  $A^P$ , if

$$A \models \delta(\bar{a}, \bar{b}) \wedge \delta(\bar{a}', \bar{b})$$

then

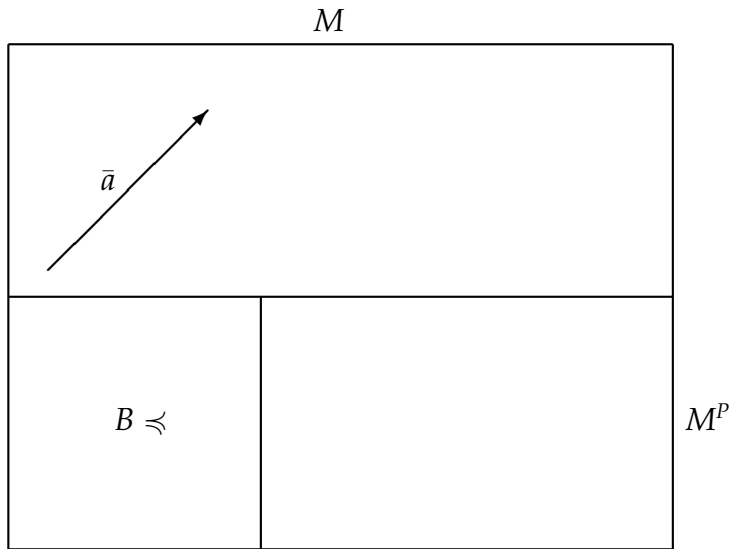
$$\text{tp}(\bar{a}/A^P) = \text{tp}(\bar{a}', A^P).$$

Given  $B \models T^P$ , we construct  $A$  with  $B = A^P$  as follows.

Take  $M$  a big model of  $T$ .

We can assume  $B \preccurlyeq M^P$  (in language  $L$ ).

We inductively build transfinite  $\bar{a}$  disjoint from  $M^P \setminus B$ , so that if  $M \models \exists x \phi(x, \bar{a})$  then  $M \models \phi(c, \bar{a})$  for some  $c$  in  $\bar{a}$ . Then taking  $A$  the set of elements in  $\bar{a} \cup B$ , we will get  $A \models T$  by Tarski-Vaught, and  $B = A^P$ .



Inductive hypothesis: Each subtuple  $\bar{c}$  of  $\bar{a}$  has a support in  $B$ .

A listing  $(\phi_i)$  of formulas is defined in advance.

When  $\bar{a} \upharpoonright i$  has been chosen, if  $M \models \phi_i(a, \bar{c}, \bar{b}_1)$  with  $\bar{c}$  in  $\bar{a} \upharpoonright i$  and  $\bar{b}_1$  in  $B$ , then we will find  $a_i$  so that  $M \models \phi_i(a_i, \bar{c}, \bar{b}_1)$  and  $\bar{a} \upharpoonright i + 1$  meets the inductive hypothesis.

Two cases according as  $a \in M^P$ ,  $a \notin M^P$ .

The more serious is the second; we consider it. Add  $\neg P(x)$  to  $\phi$ .

By IH we have  $M \models \delta(\bar{c}, \bar{b}_2)$  for some  $\delta \in \Delta$  and  $\bar{b}_2$  in  $B$ .

By Fact we have  $M \models \delta'(a, \bar{c}, \bar{b}_3)$  for some  $\delta' \in \Delta$  and  $\bar{b}_3$  in  $M^P$ .

So

$$M \models \exists \bar{y} \exists x (\phi_i(x, \bar{y}, \bar{b}_1) \wedge \delta(\bar{y}, \bar{b}_2) \wedge \delta'(x, \bar{y}, \bar{b}_3)).$$

By Reduction Property rewrite this as

$$M^P \models \theta(\bar{b}_1, \bar{b}_2, \bar{b}_3).$$

Since  $B \preceq M^P$ , there is  $\bar{b}'_3$  in  $B$ ,

$$M^P \models \theta(\bar{b}_1, \bar{b}_2, \bar{b}'_3).$$

By Reduction Property again,

$$M \models \exists \bar{y} \exists x (\phi_i(x, \bar{y}, \bar{b}_1) \wedge \delta(\bar{y}, \bar{b}_2) \wedge \delta'(x, \bar{y}, \bar{b}'_3)).$$

Since  $\delta(\bar{y}, \bar{b}_2)$  determines  $\text{tp}(\bar{c}/M^P)$ ,

$$M \models \exists x (\phi_i(x, \bar{c}, \bar{b}_1) \wedge \delta'(x, \bar{y}, \bar{b}'_3)).$$

Choose  $a_i$  so that

$$M \models \phi_i(a_i, \bar{c}, \bar{b}_1) \wedge \delta'(a_i, \bar{y}, \bar{b}'_3).$$

This doesn't quite establish that every tuple in  $\bar{a} \upharpoonright i + 1$  has a support in  $B$ .

The group-theoretic argument does it by a subtler choice of  $\delta'$  using an infinitary description of  $\bar{a} \upharpoonright i$ .

At present this part of the argument doesn't look like general model theory.

But it's better than what I had in April.

We need (not necessarily in this order):

- (1) more examples of relatively categorical theories where Gaifman's conjecture holds,
- (2) a more model-theoretic analysis of the present case,
- (3) some connection with what Shelah does in the paper cited above.

I'm fairly certain that the conjecture holds also for the other cases of  $(\kappa, \lambda)$ -categoricity. This could help.