VC density and NIP theories

Joint with Aschenbrenner, Dolich, Haskell, Starchenko

Vapnik-Červonenkis: set M, family $\mathcal{F} \subseteq \mathcal{P}(M)$.

Subset $B \subseteq M$ is shattered by $\mathcal{F}$ if

$$\{B \cap S : S \in \mathcal{F}\} = \mathcal{P}(B).$$

$$\text{VC-dim}(\mathcal{F}) = \operatorname{Max}\{d : \text{some } d\text{-subset of } M \text{ is shattered by } \mathcal{F}\}$$

(or ∞).

Rk: Consider ~~circles~~ disks in $\mathbb{R}^2$.

VC-~~density~~ dim $= 3$

Suppose VC-dim $(\mathcal{F}) = d$ and

define $\pi_{\mathcal{F}}(n) = \sup_{\substack{B \subseteq M \\ |B| = n}} |\{B \cap S : S \in \mathcal{F}\}|$.

so if $n > d$ then $\pi_{\mathcal{F}}(n) < 2^n$.

Sauer (1972) If $\mathcal{F}$ has finite VC-dim d, then

$\pi_{\mathcal{F}}(n) \leq \left(\frac{en}{d}\right)^d$ (so poly bounded) if $n > d$

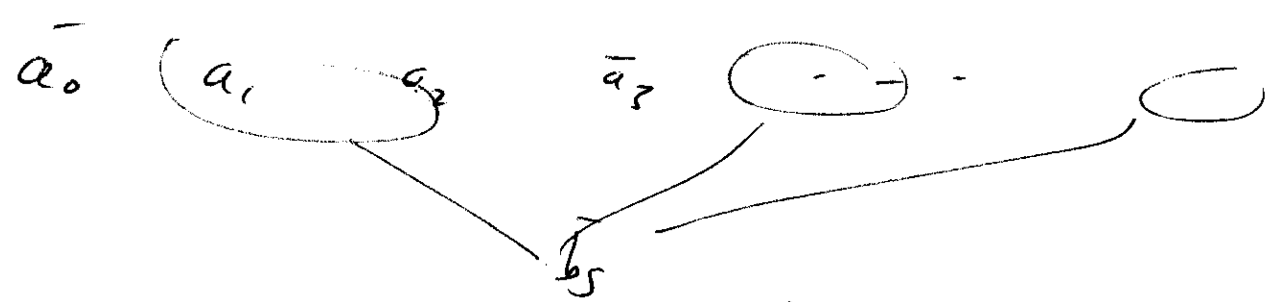
VC-density $(\mathcal{F}) := \inf\{r \in \mathbb{R}^{>0} : \frac{\pi_{\mathcal{F}}(n)}{n^r} \text{ bounded}\}$.

So $r \leq d$.

Complete first order theory T, with infinite models, $\bar{M} \models T$, suff. saturated.

Def T has <u>independence property</u> (IP) if there is a formula $\phi(\bar{x}, \bar{y})$ and $M \models T$ and $\{\bar{a}_i : i \in \omega\} \subseteq M^{l(\bar{x})}$ s.t. for any $S \subset \omega$ there is $\bar{b}_S \in M^{l(\bar{y})}$ s.t.

$$M \models \phi(\bar{a}_i, \bar{b}_S) \iff i \in S.$$



T is <u>NIP</u> (or <u>dependent</u>) otherwise.

<u>Examples:</u> <u>NIP</u>
- stable theories
- o-minimal theories
- weakly o-minimal theories

(in all models M of T, any def $X \subset M$ is a finite union of convex sets)
- ordered abelian groups
- $\mathbb{Q}_p$, ACVF
- structures interpretable in these.

<u>Non-examples</u>: Random graph, n.p. ultraproducts of finite fields

Given a theory T and $M \models T$, consider $\phi(\bar{x}, \bar{y})$ $l(\bar{x}) = m$, $l(\bar{y}) = n$.

Let $\mathcal{F} = \mathcal{F}_\phi = \{\phi(\bar{a}, M^n) : \bar{a} \in M^m\}$, a family of subsets of M^n.

Observation (Laskowski) T is NIP

$\Longleftrightarrow$ all such families $\mathcal{F}$ of sets (over all m, n) have finite VC-dim.

Goal: Compute VC density of families $\mathcal{F}_\phi$, in familiar theories with NIP.

Unpublished (Wilkie early 1990s). If T is o-minimal, then for any formula $\phi(\bar{x}, \bar{y})$, VC density of $\mathcal{F}_\phi$ is at most $l(\bar{x})$

Special case (Karpinski–Macintyre) Above, for o-minimal expansions of $\mathbb{R}$.

Johnson, Laskowski

Guingona.

Translate to counting types.

Suppose $B \subseteq \bar{M}^n$ and

$$\mathcal{F} = \{ \phi(\bar{a}, M^n) : \bar{a} \in M^m \}.$$

Then $|\{ S \cap B : S \in \mathcal{F} \}| = |S^{\phi}_{\bar{x}}(B)|$

for $\phi(\bar{a}, M^n) \cap B = \phi(\bar{a}', M^n) \cap B$

$\iff tp^{\phi}(\bar{a}/B) = tp^{\phi}(\bar{a}'/B)$

So we aim to bound $|S^{\phi}_{\bar{x}}(B)|$.

For this, we replace ϕ by a finite set Δ of formulas of form $\phi(\bar{x}, \bar{y})$

Bound $|S^{\Delta}_{\bar{x}}(B)|$

$S^{\Delta}_{\bar{x}}(B)$ = set of Δ-types $/B$

maximal consistent subsets of

$$\{ \phi(\bar{x}, \bar{b}) : \bar{b} \in B, \phi \in \Delta \} \cup \{ \neg\phi(\bar{x}, \bar{b}) : \bar{b} \in B, \phi \in \Delta \}$$

New Results

1) If T is weakly o-minimal,
then $\phi(\bar{x}, \bar{y})$ has VC density $\leq l(\bar{x})$

e.g. RCVF $\equiv_{es}$ $({}^*\mathbb{R}, <, +, \cdot, V)$

Hence in $ACVF_{(0,0)}$, $\phi(x, \bar{y})$ has VC density $\leq 2l(\bar{x})$

Nothing known for $ACVF_{(0,p)}$ or $ACVF_{(p,p)}$.

2) $T = Th(\mathbb{Q}_p)$
density of $\phi(\bar{x}, \bar{y}) \leq 2\,l(\bar{x}) - 1$

(Likewise $\mathbb{Q}_{p,an}$)

3) T "quasi ominimal ~~theory~~ with definable bounds"
$\phi(\bar{x}, \bar{y})$ has density $\leq l(\bar{x})$

e.g. $(\mathbb{Z}, +, <)$, $(\mathbb{R}, <, \mathbb{Q})$, $(\mathbb{Z} \times \mathbb{Z}, <, +)$

4) T finite Morley rank with $RM(x=x) = t$,
and assume rank, degree are definable.

Then $\phi(\bar{x}, \bar{y})$ has density $\leq t\, l(\bar{x})$

Defn: Fix $\Delta = \{ \phi(x, \bar{y}) \} : \phi \}$

(1) A definable Δ-family of definable types is

a set $p(x, \bar{u})$ of partial 1-types indexed by $l(\bar{u})$-tuples (i.e. set of types of form $p(x, \bar{c})$) s.t.

i) for any $\bar{c} \in \bar{M}^{l(\bar{u})}$, $p(x, \bar{c})$ is consistent

ii) for any $\phi(x, \bar{y}) \in \Delta$, the relation

$\phi_p(\bar{u}, \bar{y}) : \quad p(x, \bar{u}) \vdash \phi(x, \bar{y})$

is definable

(2) T has VC_t property if:

for any finite set Δ of flas $\phi(x, \bar{y})$,

there is a finite set

$\cancel{\mathcal{P}} \; P_\Delta = \{ p_i(x; \bar{y}_1, \dots, \bar{y}_t) : i \in I \}$ of

definable Δ-families of definable types

s.t. for any finite $B \subseteq \bar{M}^{l(\bar{y})}$, and any

$q \in S_x^\Delta(B)$, there are

$\bar{b}_1, \dots, \bar{b}_t \in B$ and $p_i \in P_\Delta$ s.t.

$p_i(x, \bar{b}_1, \dots, \bar{b}_t) \vdash q$.

Thm: NIP Suppose T has VC_t. Then for any finite $\Delta(\bar{x}, \bar{y})$ there is a constant K s.t. for any finite $B \subseteq \bar{M}^{l(\bar{y})}$,

$$|S_{\bar{x}}(B)| \leq K \cdot |B|^{t \cdot l(\bar{x})}.$$

(So $\phi(\bar{x}, \bar{y})$ has density $\leq t \cdot l(\bar{x})$).

RL: $\mathbb{Q}_p$ has VC_2,
weakly o-minimal theories have VC_1.

On pf of Thm: Induction on $l(\bar{x})$

For $m = l(\bar{x}) = 1$, immediate.

Put $K = |I|$

There are $\leq |B|^t \cdot K$ many types $q_i(x, \bar{b}_1, \ldots, \bar{b}_t)$,

and each q is determined by one of these

Inductive step.

Rewrite $\phi(x_1, \ldots, x_m; \bar{y})$ as

$\phi(x_1 ; x_2 \ldots x_m \bar{y})$